Let's play a concentration game with a few cards

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I believe that many of you will want to play a concentration game with a few cards after you read this article—because it shows nice strategies to win the game with a high probability. In fact, when you play a two-player concentration game described below, the probability of your winning will be more than twice that of the opponent's if she is a normal clever person with a perfect memory, whether you are the first player or the second player!

The game

You can apply the central idea of this article to a concentration game with a different number of cards. But, for convenience, we concentrate on the following concentration game:

Number of players: Two.

Cards used: Four pairs of cards (e.g. ♢ A, J, Q, K; ♡ A, J, Q, K).

Rule: The usual rule as follows. Shuffle the cards and lay them on the table, face down, in a pattern (e.g. 2 cards $\times$ 4 cards). In turn each player turns over two cards (one at a time). If they are of the same rank and color, that player wins the pair and plays again. If they don't match, they are turned face down again and it becomes the other player's turn. The game ends when the last pair has been picked up. The winner is the person with more pairs, and there may be a tie.

Normal players

To consider strategies, suppose that

> your opponent has a perfect memory and is such a *normal* person that she never gives up a chance to make a pair in any turn of hers (more specifically, she never chooses a known card unless she is sure she can then make a pair using that card).

At least in my observation, we are actually allowed to suppose this for a regular person when we play a concentration game with a few cards.

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We don't need any advanced knowledge to compute the probabilities of winning/losing this game. The calculation is, however, quite cumbersome because we need to consider separately so many cases although there are only eight cards. So I show here only the results of calculation:

If both players are *normal* in our sense,

The probability of the first player's winning $= 3/5 = 60\%$,
The probability of the second player's winning $= 4/15 \fallingdotseq 26.7\%$,
The probability of a tie $= 2/15 \fallingdotseq 13.3\%$.

It means that the probability of the first player's winning is 2.25 times that of the opponent's. Therefore, this game may look very unfair to the second player. But the truth is that there are nice strategies for the second player.

The best strategy against a *normal* opponent

If you are the first player, being *normal* is good enough as we've seen it. Indeed, it is the best strategy against a *normal* opponent.

How about when you are the second player? The following is the best strategy against a *normal* opponent:

> If the very first two cards opened by the first player don't match, you turn over the same two cards.

If you take this strategy and the opponent is still *normal*,

The probability of your winning $= 62/105 \fallingdotseq 59.0\%$,
The probability of the opponent's winning $= 29/105 \fallingdotseq 27.6\%$,
The probability of a tie $= 2/15 \fallingdotseq 13.3\%$.

It means that the probability of your winning is more than 2.1 times that of the opponent's. Wow, whether you are the first player or not, you win, in a long run, more than twice as many as you lose.

The practically best strategy

You may say, "OK. But, as the suggested best strategy gives the opponent a strange impression, she may become cautious and copy your play to disable your strategy."

Well, you're so clever as I thought. But don't worry. You can take the following quieter strategy, which can be taken also after your opponent copies your first play:

> Choose a known card (if possible) for a second card in your each turn if there remains a pair neither of which has appeared.

If you take this strategy and the opponent remains *normal* as I hope,

The probability of your winning $= 10/21 \fallingdotseq 47.6\%$,
The probability of the opponent's winning $= 43/105 \fallingdotseq 41.0\%$,
The probability of a tie $= 4/35 \fallingdotseq 11.4\%$.

So you still have advantage over the opponent!

What happens if both players do the best?

By the way, what is the first player's best strategy when the opponent plays best? A trick appears, for example, in the following situation:

> There still remain all the eight cards. Three of them were turned over before and no two of the three match.

When you encounter this situation in your turn, don't you turn over a new card? —That's not good! You should turn over two of the known cards, mathematics says.

So, at such a tricky point, the game stops in effect as neither player will turn over any new card if they play best. And it's not rare—it occurs at the rate of $4/7 (\fallingdotseq 57.1\%)$.

However, I think it rarely happens actually unless, say, both players have read this article. So, don't hesitate to play this concentration game with a regular person. —Good luck!

Appendix: The best strategy for the standard two-player concentration game

If you have a very good memory, you may want to use more cards. Here is the best strategy for up to 26 pairs (the full deck) optimizing (not the probability of your winning but) the expected number of cards you get.

Let (m, n) express a situation in which there remain m pairs on the table and there are n known cards among them.

The best strategy:

1. When your turn comes,
   (a) if there is a pair among the known cards, take them;
   (b) otherwise, choose one of the known cards if and only if the situation (m, n) is such that $n \geq 3$ and $m - n = 1$ or the situation (m, n) is equal to either of $(4, 2), (5, 3), (7, 2), (7, 3), (8, 4), (9, 2), (9, 5), (10, 3), (10, 6), (11, 4), (12, 2), (13, 3), (15, 3), (17, 2), (18, 3), (20, 2)$.

2. After you choose your first card,
   (a) if there is a known card that matches your first card, choose it;
   (b) otherwise, choose a new card if and only if the situation (m, n) is such that $n \leq 1$ or $m = n$ or the situation (m, n) is equal to either of
   $(5, 2), (6, 3), (7, 4), (8, 5), (8, 2), (9, 3), (10, 2), (11, 3), (13, 2), (14, 3), (16, 2), (17, 3), (18, 2), (21, 2), (22, 20), (23, 21), (24, 22), (24, 2), (25, 23), (26, 24)$.

If you take this strategy against a normal person, the probabilities of your winning, losing and a tie are 0.70458, 0.23895 and 0.05647, respectively, when you're the first player; and they're 0.70452, 0.23907 and 0.05640 when the second player.

You might feel the best strategy is too complicated for practice. Then you may simplify 1(b) to:

> otherwise, choose a new card unless the situation (m, n) is such that $n \geq 3$ and $m - n = 1$

and 2(b) to:

> otherwise, choose a known card (if possible) unless the situation (m, n) is such that $m = n$.

If you take this simpler strategy against a normal person, the probabilities are 0.70395, 0.23950 and 0.05656 when you're the first player; and they're 0.70390, 0.23962 and 0.05649 when the second player. Practically speaking, the effect is almost the same as the best strategy.

What is the best strategy in the sense that both players do the best? There is a nice paper on this topic.[1] I've confirmed their results also in my calculation up to the 26 pairs' case. In my representation of the best strategy in this sense, 1(a) and 2(a) are the same as above but 1(b) is:

> otherwise, choose a new card unless the situation (m, n) is such that $m - n$ is odd and $n \geq 2(m + 1)/3$

and 2(b) is:

> otherwise, choose a known card (if possible) unless the situation (m, n) is such that $m - n$ is odd or $(m, n) = (6, 2)$.

If you wish, you may take this strategy also against a normal person. Then the probabilities are 0.64140, 0.29478 and 0.06382 when you're the first player; they're 0.63960, 0.29635 and 0.06405 when the second player.

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[1] U. Zwick, M.S. Paterson, "The Memory Game," *Theoretical Computer Science*, Vol.110, No.1, 1993.