Tiling Tetris Boards

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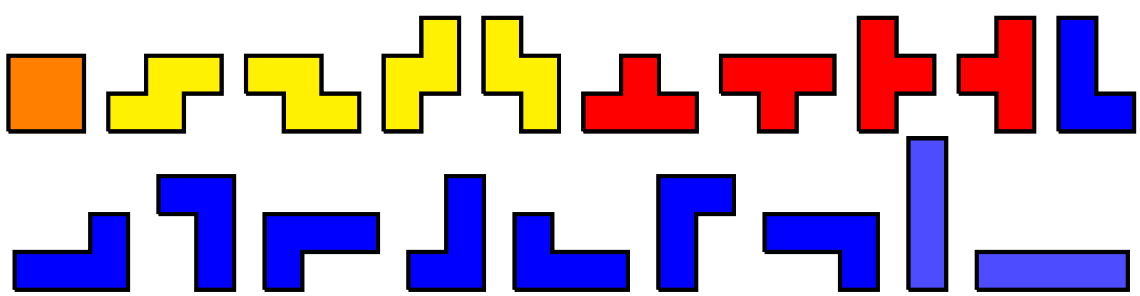
Introduction

One of the most popular topics in recreational mathematics are questions related to tiling. These come in various shapes and forms from Penrose tiles, to Golomb's polyominoes, to Marjorie Rice's discovery of pentagons that tile the plane (which she was motivated to find after reading Martin's column in Scientific American).

One of the simplest classes of polyominoes are the tetronimoes, i.e., made from four squares. These have become well known through the game Tetris, wherein these pieces continue to fall from the top into a 10×20 field and the players must arrange them so that whole rows are filled up (thus freeing up space as pieces continue to fall). While the game Tetris has been well studied, for example Erik Demaine has recently shown that the game is hard, the question of how many ways there are to tile the 10×20 board using Tetris pieces has not previously been considered.

To be more precise we want to consider how many different ways that there are to tile a 10×20 board using the following possible sets of pieces (i.e., all possible Tetris pieces in all possible orientations):



We want to cover the board using some or all of these pieces. For instance we can easily cover the board using 50 of the 2×2 Tetris piece, so we know the number of possible ways to tile is at least one. Below we give another tiling that bears

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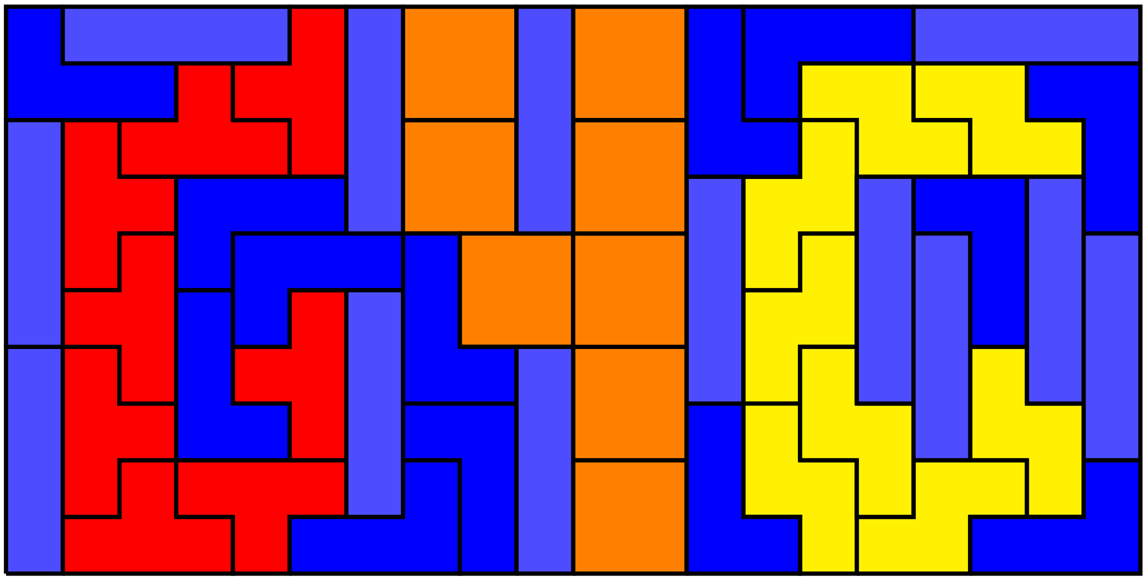
some passing connection to this conference, showing that the number of ways is at least two.



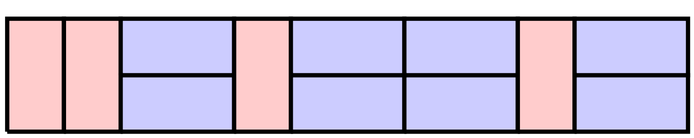
In fact the number of ways to tile is much more than 2. To be precise the number of ways to tile is

$$291,053,238,120,184,913,211,835,376,456,587,574.$$

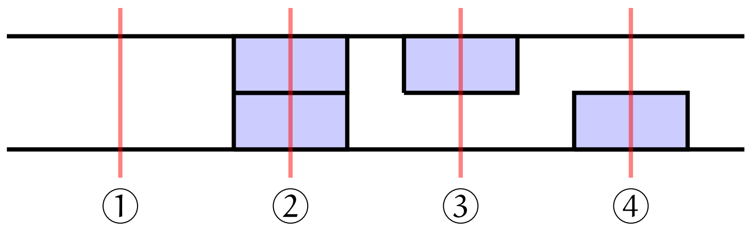
In other words, a little bit more than 291 decillion. This is a ridiculously large number, so large that all of the world's supercomputers working together for a billion years could barely begin to make a complete list of all of these. Nevertheless, we know exactly how many ways there are to tile, and this uses some simple ideas and a little bit of work with linear algebra (matrices). We will outline the basic approach by a simpler example involving tiling the $2 \times n$ board with dominoes.

Counting tilings on a $2 \times n$ board with dominoes

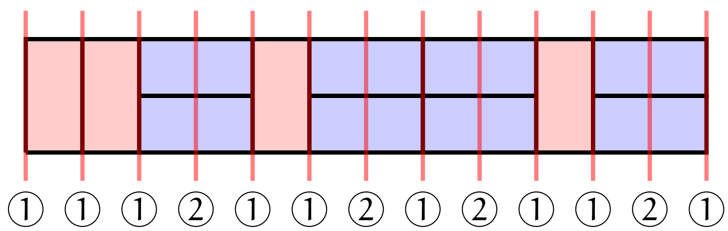
We consider a classical problem. Namely, the number of ways to tile the $2 \times n$ board using dominoes which can be either horizontal or vertical. An example of this is shown below for a 2×12 board.



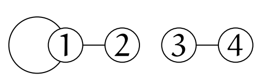
The usual way that people approach counting these is to find a recurrence relationship and then work with that recurrence relationship, ultimately leading to the Fibonacci numbers. We will take a slightly different approach and start by focusing on what happens in the transitions between columns (this can be thought of as sort of the zen approach to counting tiling, wherein it is not the tiles that are important but what happens between the tiles). To be precise there are four possible ways we can cover a transition between columns and these are drawn and labelled below.



If we take these labellings and apply them to the tiling given above we get the following:



The next thing to note is that there are limitations on what order things can occur. For instance a ① can either be followed by another ① (i.e., a vertical domino in between the two columns) or by a ②; but it can't be followed by a ③ or a ④ since that would result in uncovered parts of the board. We can represent our situation by drawing a graph wherein we connect any two crossings which can occur consecutively.



Finally, note that we always start and end our tiling with a ①, the tiling then corresponds to moving in this graph from vertex to vertex along edges connecting legal crossings. In fact, for every such way there is to start at ① then take n steps and have returned back to ① there is a tiling and vice-verse. Such a sequence of moves in the graph is called a walk that starts and stops at ①. So we now have the following.

Observation. *The number of tilings of our board is equal to the number of walks of appropriate length that start and stop at ① in this associated graph.*

We have transformed our problem from counting tilings to counting walks. The great news is that there is a mathematical tool which is built for counting walks. Namely, matrices and matrix multiplication. So first we can observe that ③ and ④ can never be reached so we only need to worry about ① and ②. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We are indexing this matrix by letting the first row and column correspond to ① and the second row and column corresponding to ②. This matrix keeps track of where we are allowed to move, i.e., we can move from ② to ① but we cannot move from ② back to ②. This is known as the *adjacency matrix*. Another way to view this matrix, is that it records how many ways we can get from one node to another node in the graph by taking one step.

Suppose

$$B_k = \begin{pmatrix} b_{11}^{(k)} & b_{12}^{(k)} \\ b_{21}^{(k)} & b_{22}^{(k)} \end{pmatrix}$$

is such that $b_{ij}^{(k)}$ corresponds to the number of walks of length k that start at i and stops at j (so in terms of our problem we want the b_{11} entry in the matrix B_n). We have already noted that $B_1 = A$.

The next thing to observe is that the walks of length $k+1$ are closely connected to the walks of length k. For example if we want to end at ② then at the previous step we had to be at ①; on the other hand if we wanted to end at ① then at the previous step we could have been at either ① or ②. So this gives us the following:

$$B_{k+1} = \begin{pmatrix} b_{11}^{(k+1)} & b_{12}^{(k+1)} \\ b_{21}^{(k+1)} & b_{22}^{(k+1)} \end{pmatrix} = \begin{pmatrix} b_{11}^{(k)} + b_{12}^{(k)} & b_{11}^{(k)} \\ b_{21}^{(k)} + b_{22}^{(k)} & b_{21}^{(k)} \end{pmatrix} = \underbrace{\begin{pmatrix} b_{11}^{(k)} & b_{12}^{(k)} \\ b_{21}^{(k)} & b_{22}^{(k)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{Matrix multiplication}} = B_k A$$

In particular, we can now see us that $B_k = A^k$. So we have the following.

Observation. *The number of tilings of our board is equal to an entry in A^n where A is the adjacency matrix of our associated graph.*

So this means that counting walks in the end comes down to some simple matrix algebra.

The advantage of this approach is that as we multiply matrices together we don't have to keep track of the past history of all of our walks, i.e., this allows us to enumerate without having to individually generate each one. This is what allows for these ridiculously huge numbers to be computed.

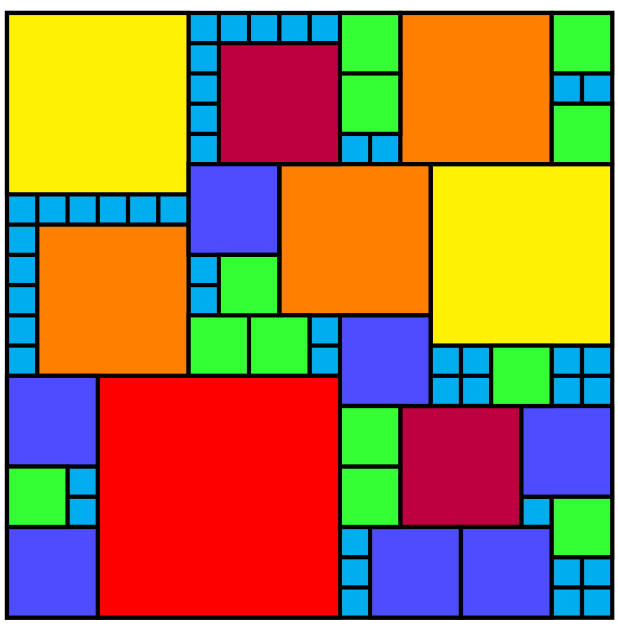
Counting tilings on the Tetris board with Tetris pieces

The count for tilings on the Tetris board is based on the same principles of turning it into a matrix multiplication problem. Namely, we started by looking at all the possible ways that Tetris pieces could cross column transitions of height 10. It turns out that there are about half a billion such crossings, and then we figured out which can occur consecutively. This gave us the graph which in turn gave us the matrix A. This is a **huge** matrix, but it is also fairly sparse in that most column transitions cannot occur consecutively. This allows us to easily pull an entry out of the matrix A^{20}, which is how we derived our count.

This technique works for tilings of the board with any collection of polyominoes, i.e., dominoes, triominoes, tetronimoes, and so on. This has also been extended to other problems, including the counting the number of ways to subdivide a 20×20 square into smaller squares of whole length. There are a little over five sexdecillion ways to subdivide the square, or to be more precise:

$$5,107,719,132,342,188,248,003,492,889,317,730,068,733,587,913,765,138$$

An example of one of these is shown below.



This goes to show that a little bit of linear algebra can go a long way!