# Conway and The 3x+1 Problem Continued

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**Abstract**

We present a conjecture concerning a family of iterative functions that includes the most well-known example ($3k+1 \to 4k+1$, $3k \to 2k$, $3k-1 \to 4k-1$) associated with Conway's generalization of The 3x+1 Problem.

I don't think Martin Gardner's Mathematical Games column devoted to The 3x+1 Problem [2] made any lasting impression on me. I only became interested in the problem after reading the "stopping times" paper by Riho Terras [8]. Dan Drucker and I then began kicking around some ideas and fumbling with it a bit, but never got anywhere. When we mentioned this to Erdös during a mid-70s visit he made to Wayne State University he repeated a phrase we had previously heard attributed to him:

> "Mathematics may not yet be ready for such problems."

and followed that up with:

> "We will have to find something else for you to do."

We did indeed find other things to do, but over the years thanks to well-publicized teasers such as Richard Guy's [4] students continually re-discover the problem and it has always remained at the fore.

Until Conway's related paper [1] was reprinted in Lagarias' book [7, pp. 219–224] it was relatively obscure. I can't remember how I first learned of it (perhaps from the references in the preprint of [6] Lagarias sent me) but in the late-90s I secured a copy of Conway's paper via Interlibrary Loan. Therefore, when prospective University of Richmond Honor's student, Robin Givens, expressed interest in The 3x+1 Problem, although reluctant, I gave her the go-ahead to look at Conway's paper and see what she could learn about generalizations of The 3x+1 Problem. Obtaining significant or noteworthy results was not possible — after all, the problem IS hard — but a conjecture we formulated concerning a generalization [3] may be of interest and I will briefly cover it below.

With $p > 1$ fixed, Conway considers the generalization

$$g(n) = a_i n + b_i$$

where $i = n \bmod p$, and $a_0, b_0, \ldots, a_{p-1}, b_{p-1}$ are *rational* constants chosen such that $g(n)$ is always *integral*. Conway proves that it is undecidable whether given a function $g$ and positive integer $n$ such that $g(n)/n$ is periodic there exists an integer $k$ such that the $k$-fold iterate $g^k(n) = 1$.

Although it is not mentioned in Conway's paper, according to Guy [5] the motivating example was:

$$t(n) = \begin{cases} \frac{2}{3}n + 0 & \text{if } n \bmod 3 = 0 \\ \frac{4}{3}n - \frac{1}{3} & \text{if } n \bmod 3 = 1 \\ \frac{4}{3}n + \frac{1}{3} & \text{if } n \bmod 3 = 2 \end{cases}$$

and according to Lagarias [6] this example was found in a 1932 journal of Collatz. The function $t$ is a permutation on the positive integers with known finite cycles (1), (2 3), (4 5 7 9 6) and (44 59 79 105 70 93 62 83 111 74 99 66). It is not known if the iterates of $t$ starting with 8 form an "infinite" cycle or, assuming it does, if there is more than one infinite cycle.

Because $t$ can be viewed as the mapping:

$$\begin{array}{lcl} 3k+1 & \rightarrow & 4k+1 \\ 3k & \rightarrow & 2k \\ 3k-1 & \rightarrow & 4k-1 \end{array}$$

the generalization of The 3x+1 Problem that Robin and I considered was the family of functions $c_q$ defined for *odd* $q > 1$ by:

$$c_q(n) = \begin{cases} \frac{q+1}{2}k & \text{if } n = qk \\ (q+1)k + \ell & \text{if } n = qk + \ell, \text{ with } 0 < |\ell| \leq \frac{q-1}{2}. \end{cases}$$

Note that $t$ is the function $c_3$. There are several elementary facts about this family that the reader is invited to discover on his or her own (e.g., only $c_3$ has a 2-cycle). What I wish to highlight here is the following conjecture:

**Remainder Conjecture:** For any finite sequence $r_0, r_1, \ldots, r_m$ such that $0 \leq |r_i| < (q+1)/2$, there exist an $n$ such that the $i$-fold iterate ${c_q}^i(n)$ satisfies ${c_q}^i(n) \bmod q = r_i$.

The conjecture says there exist $k_0, k_1, \ldots, k_m$ such that if we set $n = qk_0 + r_0$, then ${c_q}^i(n) = qk_i + r_i$ for $0 < i \leq m$. The conjecture is true when all the $r_i$ are identically zero. Its significance is that it reveals there are effectively $q^m$ cases one must consider in order to decide on a case by case basis whether or not $c_q$ has a cycle of length $m$.

## References

[1] J. H. Conway, Unpredictable iterations, Proc. 1972 Number Theory Conference, University of Colorado, Boulder, Colorado (1972) 49–52.

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[3] R. Givens, On Conway's Generalization of the 3x+1 Problem, Honors Thesis, University of Richmond, 2006.

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[6] J. Lagarias, The 3x+1 problem and its generalizations, Amer. Math. Monthly, 92 (1985) 3–23.

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