# ROOTS OF POLYNOMIALS WITH FIBONACCI COEFFICIENTS

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In our class for humanities majors students complete an activity where they construct a golden rectangle and then consider powers of the golden ratio $\varphi$. The primary idea behind the activity is to provide a geometric foundation for the idea of the golden ratio. As an extension, as the students are computing powers of $\varphi$ they see they can be written as linear combinations of $\varphi$ and 1 according to the following pattern

$$\varphi^2 = \varphi + 1, \quad \varphi^3 = 2\varphi + 1, \quad \varphi^4 = 3\varphi + 2, \quad \varphi^5 = 5\varphi + 3, \quad \cdots \tag{1}$$

and notice that the coefficients are Fibonacci numbers. This also reinforces the connection between this famous sequence and this equally famous number. (Hooray for math!) An unexpected consequence for the authors was that this activity led to an interesting question about sequences. In particular, one day after completing this activity in class we were discussing the equations in (1) and wondered what real zeros these equations might have besides the golden ratio.

**The sequence of polynomials.** Taking each equation in (1) and rewriting as a monic polynomial gives us a sequence of polynomials of the form

$$f_n(x) = x^n - F_n x - F_{n-1}.$$

The process by which the students generate the equations in (1) uses the fact that $\varphi^2 = \varphi + 1$, so for $n \geq 2$ each $f_n$ will have $f_2(x) = x^2 - x - 1$ as a factor. This allows us to rewrite $f_n$ as shown below.

$$f_n(x) = f_2(x) \cdot \sum_{i=1}^{n-1} F_i x^{n-1-i}$$

Because we will need it later, we will call the second factor $g_{n-2}$. For example $f_7(x) = f_2(x)(x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8) = f_2(x) \cdot g_5(x)$. The general form of $g_n$ is given by

$$g_n(x) = \sum_{i=1}^{n+1} F_i \, x^{n+1-i}. \tag{2}$$

Interestingly, each $g_n$ looks like a partial sum of the generating function of the Fibonacci sequence, but in this case the coefficients increase while

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the exponents decrease rather than having the coefficients and exponents increasing concomitantly.

Since the roots of $f_2$ are $\varphi$ and $-\frac{1}{\varphi}$, we know that each $f_n$ will also have these two numbers as roots. Hence our original question about roots can be answered by looking for the roots of $g_n$.

**Finding roots of $\mathbf{g_n}$.** Notice that the first and second derivatives of $f_n$ are given by

$$f_n'(x) = nx^{n-1} - F_n \quad \text{and} \quad f_n''(x) = n(n-1)x^{n-2}.$$

Then we can see that $f_n$ will have one critical value when $n$ is even, and two critical values when $n$ is odd.

*The Even Degree Case.* In the case where the degree is even we see that the one critical value corresponds to a relative minimum since the second derivative is positive everywhere except $x = 0$. Hence $f_{2n}$ is convex and can have at most two real roots. So the corresponding $g_{2n-2}$ will have no real roots.

*The Odd Degree Case.* In the case where $n \geq 3$ is odd, the second derivative test tells us that the critical values correspond to a relative minimum at $m_n = (F_n/n)^{1/(n-1)}$ and a relative maximum at $M_n = -(F_n/n)^{1/(n-1)}$. Then, since $f_n'(-1/\varphi) < 0$ and $f_n(-\varphi) < 0$, we know that $f_n$ has a single real root in the interval $\left[-\varphi, -\frac{1}{\varphi}\right]$. That is $g_{n-2}$ has one real root. If we define the sequence $(z_n)$ by letting $z_k$ be the real root of $g_{2k-1}$ then this sequence has an interesting limit.

**The limit of $(\mathbf{z_n})$.** To calculate the limit of $(z_n)$ we need three basic ideas. The first two of these are well known.

**Property 1.** $\lim\limits_{n\to\infty} n^{1/(n-1)} = 1$ **Property 2.** $\varphi^{n-2} \leq F_n \leq \varphi^{n-1}$

The third property was one which we knew, but could not find a reference for, so we verify it here. Notice that if we take the $(n-1)$-st root of all three sides of the identity in Property 2 we get the inequality

$$\varphi^{(n-2)/(n-1)} \leq (F_n)^{1/(n-1)} \leq \varphi^{(n-1)/(n-1)}.$$

Then if we take the limit as $n \to \infty$ on all three sides we get

$$\varphi = \lim_{n\to\infty} \varphi^{(n-2)/(n-1)} \leq \lim_{n\to\infty} (F_n)^{1/(n-1)} \leq \lim_{n\to\infty} \varphi = \varphi.$$

This gives us the third property.

**Property 3.** $\lim\limits_{n\to\infty} (F_n)^{1/(n-1)} = \varphi$

Applying these three ideas to the consequences of the derivative tests from before gives us the limit of $(z_n)$.

**Proposition.** *For each $n$ let $z_n$ be the real root of the polynomial $g_{2n-1}$ defined in Equation 2. Then $\lim_{n\to\infty} z_n = -\varphi$.*

*Proof.* Let $M_n$ be the value at which $f_{2n+1}$ has a local maximum. That is $M_n = -\left(\frac{F_{2n+1}}{2n+1}\right)^{1/(2n)}$. Then it follows that $-\varphi \le z_n \le M_n$.

By the properties above we have the following limit.

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} -\left(\frac{F_{2n+1}}{2n+1}\right)^{1/(2n)} = -\frac{\lim_{n\to\infty}(F_{2n+1})^{1/(2n)}}{\lim_{n\to\infty}(2n+1)^{1/(2n)}} = -\frac{\varphi}{1}$$

Hence $-\varphi \le \lim_{n\to\infty} z_n \le \lim_{n\to\infty} M_n = -\varphi$. That is $\lim_{n\to\infty} z_n = -\varphi$.

□

**Laurent polynomials.** If we divide both sides of the equation $\varphi^2 = \varphi + 1$ by $\varphi^2$ then we get $1 = \varphi^{-1} + \varphi^{-2}$. Then we can use this fact to write negative integer powers of $\varphi$ as a linear combination of $\varphi^{-1}$ and 1 in an analogous manner to the way we did in (1) as follows:

$$\varphi^{-2} = -\varphi^{-1} + 1, \quad \varphi^{-3} = 2\varphi^{-1} - 1, \quad \varphi^{-4} = -3\varphi^{-1} + 2, \quad \varphi^{-5} = 5\varphi^{-1} - 3, \ldots$$

Using the same ideas as before, we can write a sequence of Laurent polynomials of the form

$$\ell_n(x) = x^{-n} + (-1)^n F_n x^{-1} + (-1)^{n+1} F_{n-1}.$$

Each of these will have $\ell_2(x) = x^{-2} + x^{-1} - 1$ as a factor and so we can express them as

$$\ell_n(x) = \ell_2(x) \cdot \sum_{i=0}^{n-2} (-1)^i F_{i+1}\, x^{i+2-n}.$$

We call the second factor $h_{n-2}$ whose general form looks like

$$h_n(x) = \sum_{i=0}^{n} (-1)^i F_{i+1}\, x^{i-n}.$$

Then we can connect the Laurent polynomials to the polynomials from before using a function transformation. Notice that we can realize $h_n$ from $g_n$ as follows.

$$(-1)^n g_n\left(-\frac{1}{x}\right) = (-1)^n \sum_{i=1}^{n+1} F_i \left(-\frac{1}{x}\right)^{n+1-i} = \sum_{i=0}^{n} (-1)^i F_{i+1}\, x^{i-n} = h_n(x)$$

So the roots of $h_n$ will correspond to the roots of $g_n$ using the same transformation. Hence $h_n$ will have no roots when $n$ is even. When $n$ is odd, the real root $y_n$ of $h_n$ will correspond to $z_n$ according to $y_n = -1/z_n$. Note that the factor of $(-1)^n$ is not necessary since the function values are zero. Thus we can see that the sequence $(y_n)$ of real roots of the odd degree Laurent polynomials $h_{2n+1}$ converges to $\varphi - 1$ as follows.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(-\frac{1}{z_n}\right) = \frac{-1}{\lim_{n\to\infty} z_n} = \frac{-1}{-\varphi} = \varphi - 1.$$

**Summary.** We define a sequence of polynomials and a sequence of Laurent polynomials by considering powers of the golden ratio $\varphi$. A subsequence of each polynomial sequence corresponds to a sequence of real numbers. The numerical sequences have interesting limits.

## References

[1] E. Burger & M. Starbird, *The Heart of Mathematics: An Invitation to Effective Thinking*, 3rd. Ed., Wiley, 2009.

[2] S Vajda, *Fibonacci & Lucas numbers, and the Golden Section: Theory and Applications*, John Wiley & Sons, New York, 1989.