Hexaflexagons and the Other Feynman Diagrams

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Abstract

In my G4G12 gift, I describe what I think must be (or *should* be) well-understood properties of the hexaflexagon structure diagrams that did *not* appear in the *Scientific American* article that spawned Martin Gardner's "Mathematical Games" column. I also share the embarrassing story of how I came to know who originated these diagrams.

A Mortifying Incident

There are some embarrassments that are hard to forget.

I grew up reading Martin Gardner's collected *Scientific American* columns, and like so many before me, I developed a passionate interest in hexaflexagons. By the time I was an undergraduate, I had worked out the classification of possible hexaflexagon structures, made dozens of color-coded hexaflexagons, and even organized a January-term course on hexaflexagons for some of my classmates. So, as an enthusiastic but—in retrospect—young, naïve, and awkward graduate student in the mid-nineties, I decided to give a hexaflexagon talk in our graduate student seminar.

The seminar in question was freewheeling, though the topics were usually less recreational and more connected to our graduate work. The broad focus and lighthearted ethos stemmed from the seminar's unusual origin story. This was in the Harvard Mathematics Department, where at some point in the murky past, the faculty had established a seminar series for the benefit of the grad students called "Basic Notions." In theory, this was a lovely educational opportunity. In *practice*, visitors would accept invitations to speak and, in an effort to impress the world-famous Harvard math professors, give talks that were completely incomprehensible to students. By way of revenge, the graduate students started their own seminar, called "Trivial Notions."

When I was getting my Ph.D., Trivial Notions was well established as a friendly series of talks by and for graduate students in which we would share things we were working on or had stumbled across that we thought our classmates would enjoy. It was understood that the faculty were *not* to attend, although from time to time a visiting faculty member, unaware of the unwritten rule, would sit in on our meetings. What was unprecedented was for someone *else* to attend the seminar. But when I gave my hexaflexagon talk, there was an elderly couple in the audience that none of us recognized.

I began my talk with a brief account of the discovery of hexaflexagons by a group of graduate students: Tuckerman, Tukey, Feynman, and this other guy whose name I could never remember. Then I showed the trihexaflexagon and the standard hexahexaflexagon, as in the original Martin Gardner article, and ran through the general theory that produces arbitrary hexaflexagons as I (and countless predecessors) had worked it out.

After my talk, the man of the unknown couple came up and started commenting on what I had written and drawn on the blackboard. It quickly became apparent from his remarks that he must have known at least some of the original Flexagon Committee, making me even more confused about his mysterious presence. I took in what he said in bemused wonder, but it was one of my lingering classmates who had the presence of mind to ask who he was.

"Arthur Stone."

"The Arthur Stone who invented hexaflexagons?"

"Yes."

I don't recall much after that, though apparently I managed not to drop dead of embarrassment. I can't even remember if I had the good grace to apologize to Stone, who had seen the talk listed as one of the events in the Harvard Mathematics Department and decided he had to check it out, for not crediting him for his discovery, though given my social skills at the time I'm inclined to doubt it. But I have never forgotten Arthur Stone's name since.

Mapping Hexaflexagon Structure

What Stone remarked upon in particular was a set of diagrams I had drawn on the board, the first few of which are shown in Figure 1. These diagrams represent the structures of the tri-, tetra-, and pentahexaflexagons in the first row, and the three different hexahexaflexagons in the second row.

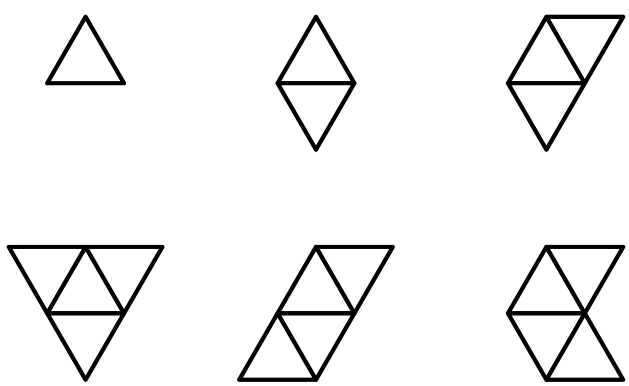

Figure 1: *Structure diagrams for the six smallest hexaflexagons as I prefer to draw them.*

Stone told us that Richard Feynman had diagrammed hexaflexagons in the same way, except that he had drawn the diagrams as polygons (I presumed regular ones) triangulated by diagonals, as in Figure 2. This connects the question of counting the number of different hexaflexagons with *n* faces to combinatorial questions about triangulations of polygons.

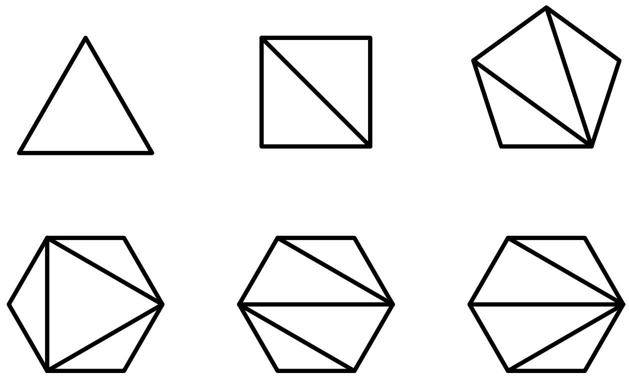

Figure 2: *Richard Feynman's hexaflexagon structure diagrams.*

Readers who are familiar with Martin Gardner's original hexaflexagon article [1] will note that diagrams of the types in Figures 1 and 2 do not appear there. There is a structure diagram for the standard hexahexaflexagon, but it is of a different type pegged to the Tuckerman traverse method of sequentially reaching every face in the hexahexaflexagon. Figure 3 shows a simplified version of this diagram, in which a Tuckerman traverse corresponds to traveling counterclockwise (or clockwise) around the outside of the diagram.

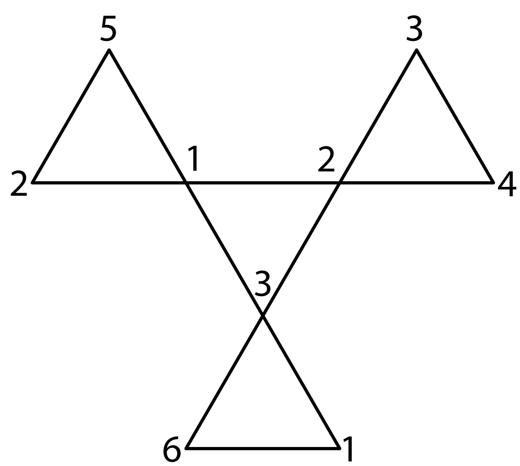

Figure 3: *A sketch of the Tuckerman traverse diagram of the hexahexaflexagon from Martin Gardner's* Scientific American *article [1].*

I am sharing this account with Gathering for Gardner for two reasons: first, because I think the Arthur Stone story is pretty funny, and second, because there are wonderful properties of the

Feynman-style[1] structure maps that I am sure must have been discovered many times over, but that I have not found written up in any concise and unified way. I will now briefly describe those properties in the hope that they may be of interest, but more importantly because if there is a nice account of these readily available, surely one of you can point me to it. If there is none, I might consider expanding these remarks so that the ideas are more widely accessible.

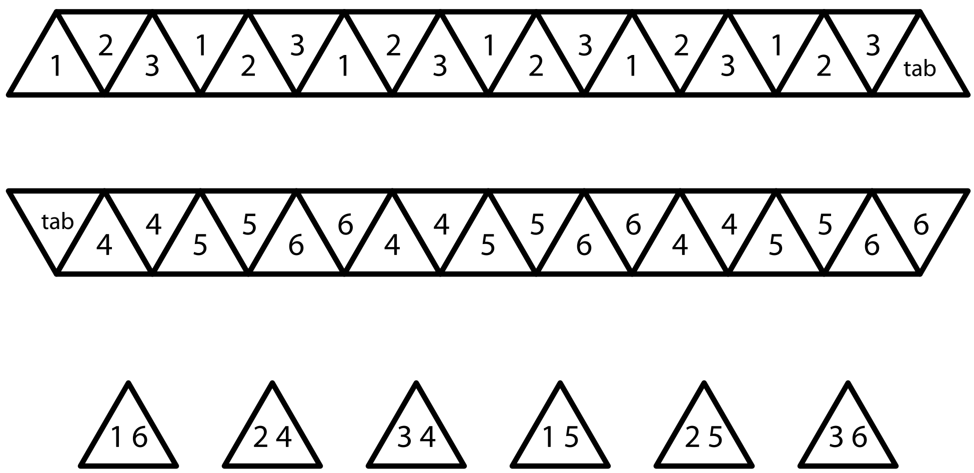

Figure 4: *The net for the hexahexaflexagon, and the six panels that comprise it.*

The net for the standard hexahexaflexagon is shown in the top and middle rows of Figure 4. When the net is folded and glued, the resulting hexaflexagon contains 18 triangles of paper, each with a number on either side. We use the terminology *panel* to describe each distinct numbered triangle that appears; in this case, the panels (pictured at the bottom of Figure 4) bear the number pairs 1 6, 2 4, 3 4, 1 5, 2 5, and 3 6. We denote these panels as (1 6), (2 4), and so on. In the hexaflexagon, these panels are stacked into what Oakley and Wisner [2] termed *pats*; in each possible position, there are two distinct pats that alternate around the hexaflexagon, each appearing three times.

Figure 5 shows the numbered diagram for this hexahexaflexagon. The six vertices naturally represent the six numbered faces of the hexaflexagon. But the key to reading these maps is in the edges. The position of a hexaflexagon—and the positions accessible via a single flex—are determined by which faces are currently on the top and on the bottom, which is essentially a directed edge of the diagram. For instance, at the top of Figure 3, the marked edge indicates the position (1,2) in which 1 is the bottom face and 2 is the top face, represented by an arrow pointing from 1 (the bottom face) to 2 (the top face). A single flex will move the arrow to another edge of one of the triangles containing (1,2); moreover, since the flex pushes face 2 from the top to the bottom, this arrow will point from 2 to another face. Therefore, as shown in the bottom row of Figure 5, the position after a flex will be either (2,3) or (2,5).

[1] Admittedly, Feynman may also have used the type of diagram in Figure 3, but if so, Stone did not mention it. As I hadn't drawn any such diagrams, there's not much to be deduced from this either way.

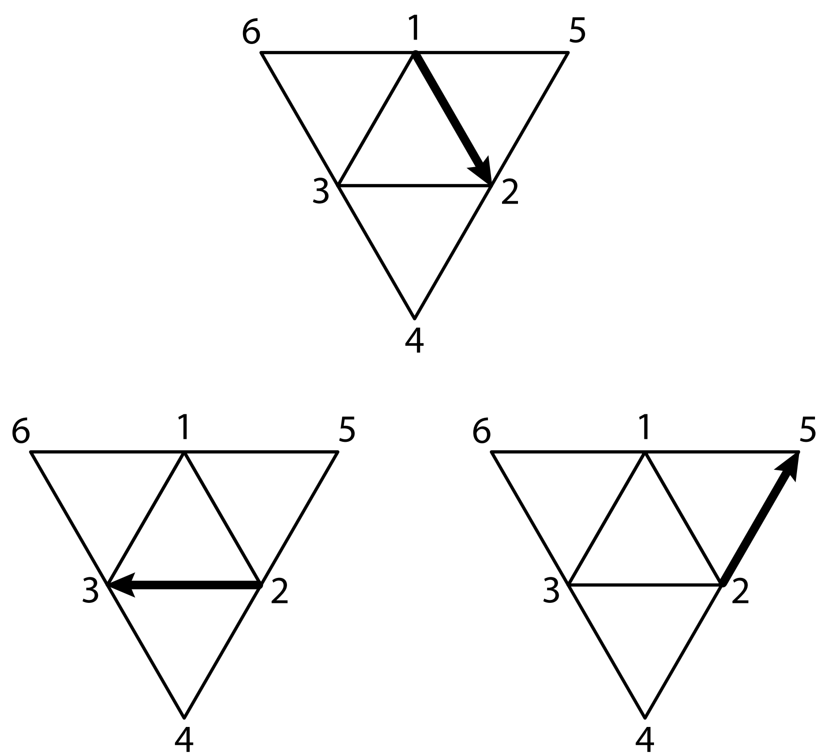

Figure 5: *The diagram for the standard hexahexaflexagon, and the two possible flexes from position (1,2)*

We can further deduce from the diagram that there are two positions accessible from (2,3), namely (3,1) and (3,4), but that there is only one position accessible from (2,5), namely (5,1). We call positions such as (5,1) from which only one flex is possible *terminal positions*.

The essential observations about how the "Feynman" map for any hexaflexagon reflects its structure are as follows.

1. A hexaflexagon with *n* faces has exactly *n* terminal positions, not counting which of the faces is on the bottom or the top. (If you count the order, there are *2n* terminal positions.)

2. Each of the *n* terminal positions corresponds to one of the *n* panels in the hexaflexagon. In particular, up to the order of the faces, the 6 terminal positions in the hexaflexagon in Figures 4 and 5 are (1,6), (2,4), (3,4), (1,5), (2,5), and (3,6). Thus, we can read the panels off of the exterior edges of the diagram.

3. In each position, the sets of exterior edges on either side of the edge marking that position correspond to the panels in each of the two distinct pats. For instance, when the hexaflexagon in Figures 4 and 5 is in the position (1,2), one pat contains panels (1 5) and (2 5), and the other contains panels (1 6), (3 6), (3 4), and (2 4). When we flex to position (2,5), the panel (1 5) moves from the thinner pat to the thicker one.

Notice that this final rule explains why the external edges correspond to terminal positions: a flex passes one or more panels from one pat to another, and in the terminal position there is a pat with only one panel in it that cannot be subdivided. With a little thought and experimentation, we can use these observations to determine the effects of "deleting" a face that is in two terminal positions (effectively gluing it shut by collapsing the net), and to discover a method to reverse this process, grafting a new face into any terminal position of a hexaflexagon.

If you know of a particularly nice source of this theory (or, for that matter, any source less technical than [2], which is so dense and sparsely diagrammed that I find it difficult to discern if the authors were aware of these structure maps), or of a straightforward account of how to construct an arbitrary hexaflexagon, please email me at sgoldstine@smcm.edu and tell me about it. And if you don't, then I hope you have learned something new that will encourage you to pull out paper, scissors, and glue and start playing!

References

[1] M. Gardner, Hexaflexagons, *Probability Paradoxes, and the Tower of Hanoi: Martin Gardner's First Book of Mathematical Puzzles and Games*, Cambridge University Press, 2008.

[2] C. O. Oakley and R. J. Wisner, *Flexagons*, The American Mathematical Monthly, Vol. 64, No. 3 (Mar., 1957), pp. 143-154.