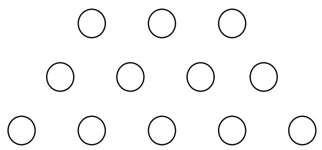


t w e l v e

Trapezoids, Primes, and Cousins

John J. Watkins, Colorado College
jwatkins@coloradocollege.edu

Here is a simple story that contains three nice ideas about numbers, two are in fact somewhat old, but one I think is new. I am sure all three of these ideas would have tickled Martin, especially because they were discovered (or rediscovered) by an amateur mathematician.

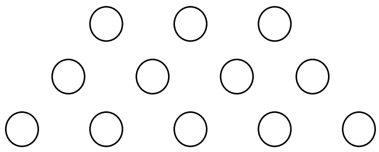
My story begins more than thirty years ago when a good friend of mine, Carlton Gamer, came to me with a conjecture about sums of consecutive integers. Carlton is a prominent American composer and music theorist who often works with *equal tempered systems* — these are musical systems, such as the familiar twelve-tone system, but that may have a modulus other than 12 in order to allow for differing combinatorial structures to emerge in the music. In this particular instance Carlton was exploring the idea of generalizing the way Schoenberg in his Variations for Orchestra, Op. 31 partitioned its 12-tone set into subsets containing three, four, and five tones. Schoenberg then used this partition $12 = 3 + 4 + 5$ to determine such musical details as the number of pitches in a motive, the number of notes in a chord, and the number of measures in a phrase.

So, the mathematical question Carlton asked himself was: Which numbers can (like 12) be written as a sum of consecutive positive integers? He then proceeded to check all numbers up to 100 by hand. Having done this he came to me with a remarkable conjecture: *The only numbers that cannot be written as a consecutive sum are the powers of* 2. This struck me as a very nice conjecture; it was certainly one that was entirely new to me. You may want to try to prove it for yourself before I divulge a proof below.

My first thought was that the idea of a sum of consecutive positive integers is only a very slight variation on the familiar Greek notion of *triangular numbers* — that is, the numbers

$$t_1 = 1, \quad t_2 = 1 + 2 = 3, \quad t_3 = 1 + 2 + 3 = 6, \quad \ldots \quad , \quad t_n = 1 + 2 + 3 + \cdots + n, \quad \ldots$$

that are the sum of the first n integers. So, I decided to coin the geometric term *trapezoidal numbers* for these numbers since the sum of (at least two) consecutive positive integers can be represented in the form of a trapezoid. For example, here is 12 represented as a trapezoid:



$12 = 3 + 4 + 5$

Note that the difference of any two (non-consecutive) triangular numbers is automatically a trapezoidal number. For example, it is clear from the figure above that if we subtract the triangular number t_2 from the triangular number t_5 we are left with the trapezoidal number 12; or, in other words,

$$t_5 - t_2 = (1 + 2 + 3 + 4 + 5) - (1 + 2) = 3 + 4 + 5 = 12.$$

Here is Carlton Gamer's first *nice* idea now stated as a theorem about trapezoidal numbers:

All positive integers except the powers of 2 *are trapezoidal.*

Let us first show that any trapezoidal number cannot be a power of 2. Suppose that n is a trapezoidal number; then, as we saw above, we can write n as the difference of two non-consecutive triangular numbers: $n = t_{k+s} - t_k$, where $s > 1$. Fortunately, there is a convenient formula — known to the Pythagoreans — for the kth triangular number: $t_k = \frac{k(k+1)}{2}$. Using this formula we can write

$$n = t_{k+s} - t_k = \frac{(k+s)(k+s+1)}{2} - \frac{k(k+1)}{2} = \frac{s(2k+s+1)}{2}.$$

Now, one of s or $2k + s + 1$ must be odd (and greater than 1), and the other even, so n cannot be a power of 2.

Next, let us show that conversely any number not a power of 2 is trapezoidal. This is obvious for odd numbers since an odd number $2k + 1$ can be written trapezoidally as $2k + 1 = k + (k + 1)$. Now suppose n is an even positive number, but not a power of 2. Then we can write $n = 2^m \cdot k$, where k is an odd number ($k \geq 3$). In this case we simply express n as the sum of k consecutive integers with 2^m in the middle. Several examples will make this constuction clear.

As a first example, let $n = 12$. We write $12 = 2^2 \cdot 3$, so we put $2^2 = 4$ in the middle and express 12 as the sum of 3 consecutive integers: $3 + 4 + 5$.

As a more illustrative example, let $n = 112$. We write $112 = 2^4 \cdot 7$, so we put $2^4 = 16$ in the middle and express 112 as the sum of 7 consecutive integers: $13 + 14 + 15 + 16 + 17 + 18 + 19$.

We need to give one more example to show that even if this construction produces negative integers we still have a trapezoidal number! So, let $n = 18$, and we write $18 = 2 \cdot 9$ and dutifully put 2 in the middle and express 18 as the sum of 9 consecutive integers:

$$(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + (5) + (6).$$

This may not look very trapezoidal, but once you realize that $(-2)+(-1)+0+1+2=0$, we have in fact written

$$18 = 3 + 4 + 5 + 6,$$

which is clearly trapezoidal.

Quite recently Gamer (the amateur mathematician) discovered two rather surprising connections between trapezoidal numbers and prime numbers. The connections are surprising because prime numbers are the *multiplicative* building blocks for the integers — that is, they are the irreducible "atoms" from which the integers are composed. So, for example, the number 2016 is composed of the atoms $2, 3, 7$ in the form $2016 = 2^5 \cdot 3^2 \cdot 7$, where 2, 3, and 7 are prime numbers. On the other hand, as we have seen, trapezoidal numbers represent an *additive* property as in $12 = 3 + 4 + 5$.

Now, of course, all prime numbers, except 2, are odd. We have already noted that any odd number has a trivial representation as a trapezoidal number. For example, $29 = 14 + 15$ and $27 = 13 + 14$. But there is an interesting difference here as Carlton Gamer discovered just by doing lots and lots of examples by hand. The prime number 29 has *only* this trivial representation as a trapezoidal number, whereas 27 can also be represented trapezoidally as $27 = 8+9+10$ (and, incidentally, also as $2+3+4+5+6+7$).

So, here is Gamer's second *nice* idea:

An odd number $n = 2k + 1$ $(k \geq 1)$ is prime
if and only if
its only trapezoidal representation is the trivial one: $n = k + (k + 1)$.

What is remarkable about this idea is that it provides an *additive* characterization for prime numbers. Here is the proof.

First, let $n = 2k + 1$ be an odd prime and suppose, by way of contradiction, that it has a nontrivial trapezoidal representation

$$n = r + (r + 1) + \cdots + (r + d - 1)$$

as a sum of d consecutive positive integers where $d > 2$. Then, since n is the difference of two triangular numbers we can write

$$n = \frac{(r + d - 1)(r + d)}{2} - \frac{(r - 1)(r)}{2} = \frac{d(2r + d - 1)}{2}.$$

Now, both d and $2r + d - 1$ are greater than 2 and one is odd and the other is even; therefore we have a factorization of n, which is a contradiction since n is prime. Thus, the only trapezoidal representation for n is $n = k + (k + 1)$.

Next, let $n = 2k + 1$ $(k \geq 1)$ be an odd number that has only $n = k + (k + 1)$ as a trapezoidal representation. Assume, by way of contradiction, that n is composite and has a factorization $n = ab$ where $a \leq b$. Note that a and b are both odd and $a \geq 3$. Then we can write n as a sum of a consecutive positive integers as follows:

$$n = \Big(b - \frac{a-1}{2}\Big) + \cdots + (b - 1) + b + (b + 1) + \cdots + \Big(b + \frac{a-1}{2}\Big),$$

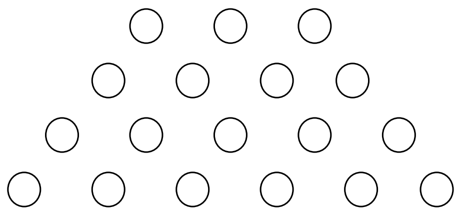
contrary to the assumption. Thus, n is prime, and this completes the proof.

The other connection that Carlton Gamer noticed between prime numbers and trapezoidal numbers has to do with the famous *Twin Prime Conjecture*, one of the most elusive conjectures in number theory. Two primes are called *twin* primes if they differ by 2, for example, 17 and 19, or, for a really large example:

$$65,516,468,355 \cdot 2^{333,333} - 1 \quad \text{and} \quad 65,516,468,355 \cdot 2^{333,333} + 1.$$

The Twin Prime Conjecture is that there are infinitely many pairs of twin primes. The evidence is overwhelming, but yet there is still no proof.

Since twin primes differ by 2, it is entirely appropriate that pairs of primes such as 7 and 11, or 43 and 47, that differ by 4 are called *cousin* primes (and it is highly likely that there are infinitely many cousin primes). Carlton Gamer spotted an interesting connection between cousin primes and trapezoidal numbers. For example, here is a *trapezoidal* representation of the cousin primes 7 and 11:



7 and 11

Gamer's third *nice* idea is that

Cousin primes have a natural representation as trapezoidal numbers of height 4.

Let p and $p+4$ be two cousin primes. If we write $p = 2k+1$, then $p+4 = 2k+5$ and their sum

$$p + (p+4) = k + (k+1) + (k+2) + (k+3)$$

is a trapezoidal number of *height* 4. Now, of course, this statement in and of itself doesn't tell us very much about cousin primes since it depends only upon p being odd. However it does lead to a general method for finding cousin primes.

We will soon see that this method for finding cousin primes is actually very similar to the well-known method for finding primes called the *sieve of Eratosthenes*. In what follows I will omit most of the details in order to make the sieving process as clear as possible.

At this point we have observed that if p and $p+4$ are two cousin primes then p is not divisible by the prime 2, and so the sum of the two cousin primes must be in the sequence

$$10,\ 14,\ 18,\ 22,\ 26,\ 30,\ 34,\ 38,\ 42,\ 46,\ \ldots \tag{1}$$

since, as we saw above, the sum of p and $p+4$ is $4k+6$.

Similarly we can conclude that since the prime 3 does not divide p or $p+4$ (except when $p=3$) the sum of two cousin primes must also be in the sequence

$$10,\ 12,\ 18,\ 24,\ 30,\ 36,\ 42,\ 48,\ 54,\ 60,\ \ldots\ . \tag{2}$$

Since the sum of cousin primes must be in sequence (1) *and* in sequence (2) the sum of cousin primes must be in the intersection of these two sequences; that is, the sum must be in the sequence

$$10,\ 18,\ 30,\ 42,\ 54,\ 66,\ 78,\ 90,\ 102,\ 114,\ \ldots\ . \tag{3}$$

Did you notice how the sieve just removed potential sums such as $12, 14, 22, 24, 26, \ldots$ from further consideration?

Next, by considering the prime 5, we can conclude that the sum of cousin primes must be in the sequence

$$10,\ 12,\ 18,\ 20,\ 22,\ 28,\ 30,\ 32,\ 38,\ 40,\ 42,\ 48,\ 50,\ \ldots\ . \tag{4}$$

Now we know the sum of cousin primes must be in sequence (3) *and* in sequence (4), so the sum of cousin primes must be in the intersection of these two sequences, that is, in the sequence

$$10,\ 18,\ 30,\ 42,\ 78,\ 90,\ 102,\ 138,\ 150,\ 162,\ 198,\ 210,\ 222,\ 258,\ \ldots\ . \tag{5}$$

By doing one more stage of this sieving process (for the prime 7) we remove 102 and 150 as potential sums, leaving us with the first <u>TWELVE</u> pairs of cousin primes:

$$3+7=10, \qquad 7+11=18, \qquad 13+17=30, \qquad 19+23=42,$$

$$37+41=78, \qquad 43+47=90, \qquad 67+71=138, \qquad 79+83=162,$$

$$97+101=198, \qquad 103+107=210, \qquad 109+113=222, \qquad 127+131=258.$$

This Eratosthenean sieving process could of course be continued ad infinitum.