Fitting Shapes Inside Shapes (and the Snugohedron)

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Abstract

I got curious about a basic question: if you have a unit square, what is the largest equilateral triangle you can fit inside it? Then I wondered the same thing in 3D (a tetrahedron in a cube). In the last section I talk about some special dimensions where you can fit a regular simplex snugly inside the corners of a hypercube (a *snugohedron*).

1 Fitting an equilateral triangle inside a square

A few years ago, I became curious about the largest equilateral triangle I could fit in a unit square. At first glance, this problem seems to require trigonometry, but with a few tricks we can find the answer using only a geometric proof and the Pythagorean theorem.

First, imagine that we put a small equilateral triangle inside a square. We can keep expanding it until at least one corner of the triangle lands exactly in one corner of the square. (If no corner lines up, we can rotate a tiny bit and still expand.)

What about two corners? If two triangle corners land in two corners of the square, that *still* can't be the biggest possible: you can dislodge one corner, pivot the triangle a little, and keep expanding. And if two triangle corners land in *opposite* corners of the square, then the triangle is too "wide" and part of it will stick out of the square. (See Figure 1.)

So for the largest equilateral triangle, the triangle and the square overlap in *exactly one* corner.

Now suppose the triangle shares the bottom-left corner of the square. We can pivot the triangle around that corner. As we pivot, the triangle is constrained by whichever wall of the square it hits first. The biggest triangle happens in the "balanced" position where it fits equally between the top wall and the right wall. That makes it symmetric around the main diagonal of the square. (See Figure 2.)

In the final picture, the square has side length 1 and the triangle has side length x. On the top edge and right edge, the triangle hits at a short distance y from the nearby corner, leaving a leftover segment of length $(1 - y)$. The two right triangles labeled A and B are the only things we need.

Using triangle A, the legs are 1 and y, so

$$x^2 = 1^2 + y^2 = 1 + y^2.$$

Using triangle B, the legs are $(1 - y)$ and $(1 - y)$, so

$$x^2 = (1 - y)^2 + (1 - y)^2 = 2(1 - y)^2.$$

Setting those equal gives

$$1 + y^2 = 2(1 - y)^2.$$

Expanding,

$$1 + y^2 = 2(1 - 2y + y^2) = 2 - 4y + 2y^2,$$

so

$$0 = 1 - 4y + y^2.$$

That quadratic has solutions

$$y = 2 \pm \sqrt{3}.$$

Since y is a length inside a unit segment, we know $0 < y < 1$, so we take

$$y = 2 - \sqrt{3}.$$

Now plug back in. For example, using $x^2 = 1 + y^2$:

$$x = \sqrt{1 + (2 - \sqrt{3})^2} = 2\sqrt{2 - \sqrt{3}}.$$

It turns out this simplifies nicely (square both sides to check):

$$x = \sqrt{6} - \sqrt{2}.$$

Numerically, $x \approx 1.0353$, so it is just a bit larger than the side of the square.

2 Fitting a tetrahedron inside a cube

OK, so in 2D we placed 3 points in a square as far apart as possible (the vertices of an equilateral triangle). What happens if we increase the dimension?

In 3D, the analogous question is: what is the biggest regular tetrahedron you can fit inside a unit cube?

It turns out there is a very clean answer. Label the corners of the unit cube as $(0,0,0), (0,0,1), \ldots, (1,1,1)$. Now look at these four corners:

$$(0,0,0), \quad (0,1,1), \quad (1,0,1), \quad (1,1,0).$$

Any pair of these points differs in exactly two coordinates, so every edge has length $\sqrt{2}$. That means these four corners form a regular tetrahedron of side length $\sqrt{2}$. (See Figure 3.)

Why is $\sqrt{2}$ the largest possible side length? If you look at a cube and consider its cross section, the biggest equilateral triangular cross section occurs between $(0,1,1), (1,0,1), (1,1,0)$. Our tetrahedron already has this as a face, so any larger tetrahedron wouldn't fit inside the cube.

3 Snugohedra

Building a simplex using only corners of the hypercube feels extra satisfying. With that in mind, we define a *snugohedron*:

Definition. A *snugohedron* is a choice of $D + 1$ corners of the D-dimensional unit hypercube $\{0,1\}^D$ such that all pairwise distances are equal. Equivalently: the $D + 1$ corners form a regular D-simplex.

Warm-up: $D = 2, 3, 4$

$D = 2$ **(no).** In a square, the only possible corner-to-corner distances are 1 (adjacent) and $\sqrt{2}$ (diagonal). You cannot pick three corners where all three distances match, so there is no 2D snugohedron.

$D = 3$ **(yes).** The tetrahedron from the previous section *is* a 3D snugohedron: four corners, all distances $\sqrt{2}$.

$D = 4$ **(no).** If we try to pick 5 corners of a 4D hypercube with equal distances, the only possible squared distances are

$$1,\ 2,\ 3,\ 4$$

(because we're counting the number of steps in the shortest path between two corners).

- Distance$^2 = 1$ is impossible because a corner only has 4 neighbors at distance 1, and those neighbors are not all distance 1 from each other (so you can't get a 5-clique).
- Distance$^2 = 4$ is impossible because if we start at one corner, there is exactly one other corner at distance$^2 = 4$ from it (the opposite corner). But we need 5 corners total, not just two.
- Distance$^2 = 3$ fails quickly: take one corner as 0000. Any point distance$^2 = 3$ from it must have exactly three 1's, like 1110. But any two distinct "three-ones" points differ in *two* coordinates, not three.
- Distance$^2 = 2$ also fails: again start at 0000. Points at distance$^2 = 2$ have exactly two 1's. Two such points have distance$^2 = 2$ exactly when the corresponding edges share a vertex. You can pick at most 3 edges that all share a vertex, not 4 of them, so you cannot get 5 corners total.

So there is no 4D snugohedron.

Some Snug Dimensions

OK, but there *are* more snugohedra. They happen when D is one less than a power of two:

$$D = 3, 7, 15, 31, \ldots$$

Let k be a positive integer, and let $D = 2^k - 1$. We will build $D + 1 = 2^k$ corners of $\{0, 1\}^D$ that are all the same distance apart.

Step 1: list all k-bit strings. Write all 2^k binary strings of length k in order. This gives us 2^k strings.For example, for $k = 3$, we get:

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

Think of this as a table with $2^k = 8$ rows and $k = 3$ columns.

Step 2: combine columns mod 2. Given columns C_i and C_j, define $C_i + C_j$ to mean: add entry-by-entry, and keep only whether the sum is even or odd. (So this is just XOR.) For example, if we add the first and third columns, we get:

$$C_1 \oplus C_3 = \begin{matrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{matrix}$$

If you add the same column twice, it cancels:

$$C_i \oplus C_i = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Also, order does not matter: $C_i \oplus C_j = C_j \oplus C_i$.

So, if we take the k original columns and form all *nonempty* XOR-combinations, we get $2^k - 1$ new columns. We might think that some of these combinations might be the same, but it turns out that they are all different. It's easiest to see why with a little example. Suppose $k = 5$ and $C_1 \oplus C_2 \oplus C_3 = C_1 \oplus C_4 \oplus C_5$. Then we can cancel out any common terms to get $C_2 \oplus C_3 = C_4 \oplus C_5$. We can put all but one term on the same side to get $C_2 \oplus C_3 \oplus C_5 = C_4$. If that were true, then when we were listing all possible k-bit strings, column 5 would be a function of the other columns. But then there would be at most 2^{k-1} distinct k-bit strings, which we know is not true. Therefore, all $2^k - 1$ XOR-combinations are different.

Step 3: each combined column has half 1s. Pick any nonzero XOR-combination of the original columns. As you run through all 2^k rows, that combined bit is equally likely to be 0 or 1. So it has exactly 2^{k-1} ones.

Even better: the XOR of two different combined columns is another nonzero combined column. So for any two distinct columns, their XOR has exactly 2^{k-1} ones. That means *the two columns differ in exactly* 2^{k-1} *rows.*

Step 4: turn columns into corners of a cube. Each column is a length-2^k bitstring. But if you look at the first row, the entries are all zero. So the first row is always 0 in every combined column. So we can delete that first entry from every column without changing the distance between any two corners. Now each column becomes a bitstring of length $2^k - 1 = D$, i.e. a corner of the D-dimensional hypercube.

Also include the all-zero corner. That gives us $2^k = D + 1$ corners total.

What is the distance? We just proved that any two of our nonzero columns differ in exactly 2^{k-1} positions, and the all-zero corner also differs from each nonzero column in exactly 2^{k-1} positions.

So every pair of corners has squared distance 2^{k-1}, meaning the snugohedron edge length is

$$\boxed{\sqrt{2^{k-1}} = 2^{(k-1)/2}}.$$

Example: $k = 3$ gives $D = 7$

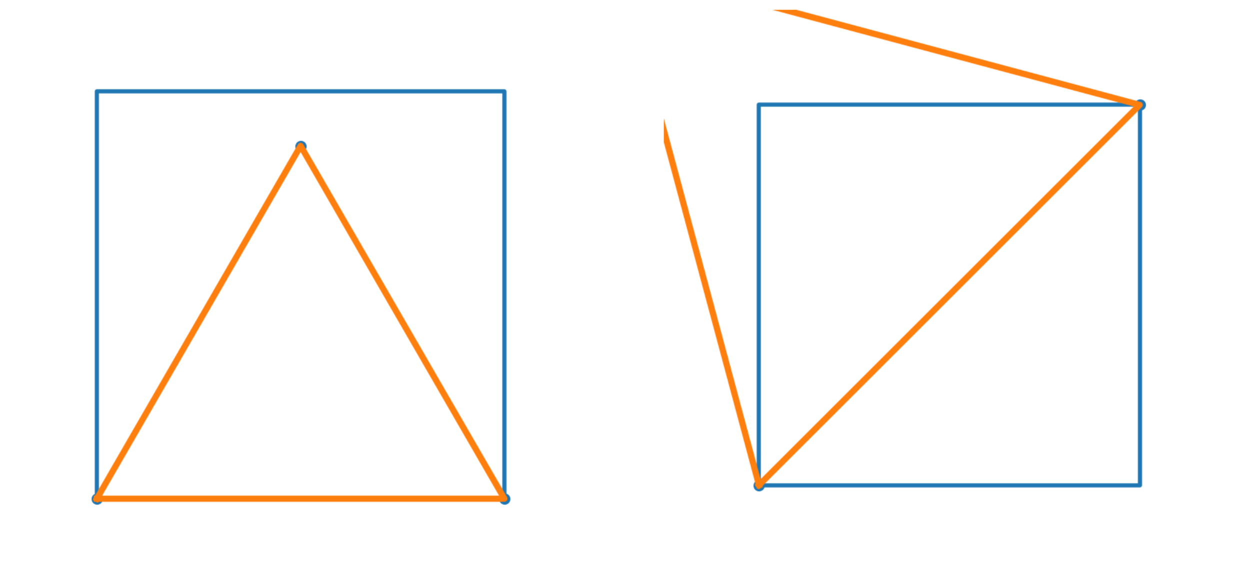
When $k = 3$, we get $D = 7$, and this construction produces 8 corners of $\{0,1\}^7$. Every pair is distance $\sqrt{4} = 2$ apart, so this is a 7D snugohedron. (See Figure 4.)

Conclusion

Here is the story in one sentence:

- The largest equilateral triangle in a unit square has side length $x = \sqrt{6} - \sqrt{2}$.
- The largest regular tetrahedron in a unit cube has side length $\sqrt{2}$.
- In higher dimensions, corner-snugohedra exist when $D = 2^k - 1$, with simplex edge length $\sqrt{2^{k-1}}$.

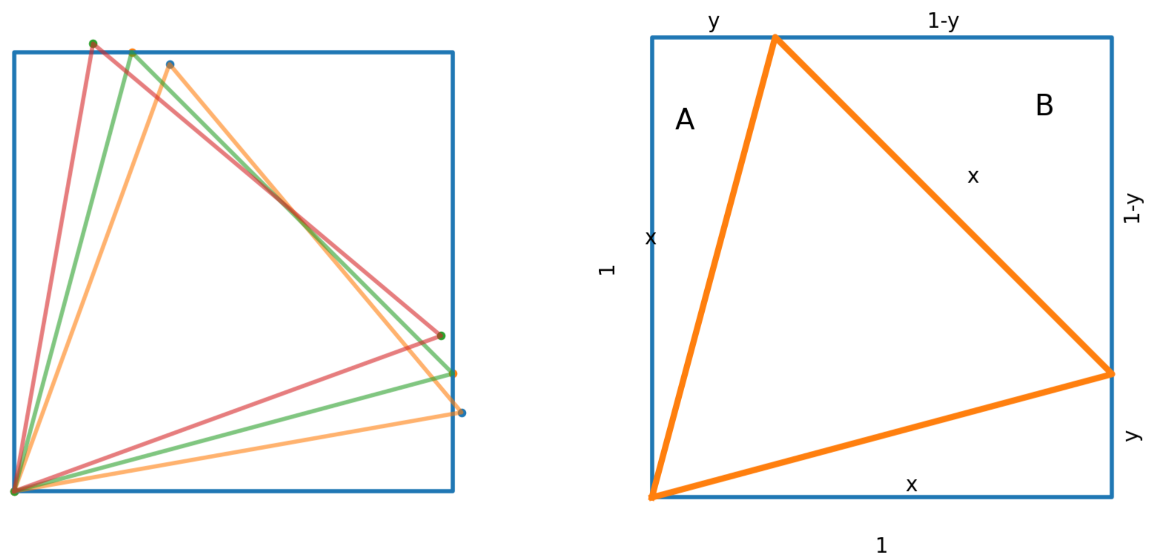
I still think it would be cool to classify *all* dimensions where snugohedra exist, or at least find more families. But the $D = 3, 7, 15, \ldots$ family is already a nice infinite stash of examples.



(a) Two corners can happen, but it isn't maximal.

(b) Opposite corners forces the triangle to stick out.

Figure 1: Why the largest configuration overlaps in exactly one corner.



(a) Pivoting around the shared corner.

(b) The final (maximal) configuration and labels.

Figure 2: The largest triangle is symmetric about the main diagonal.

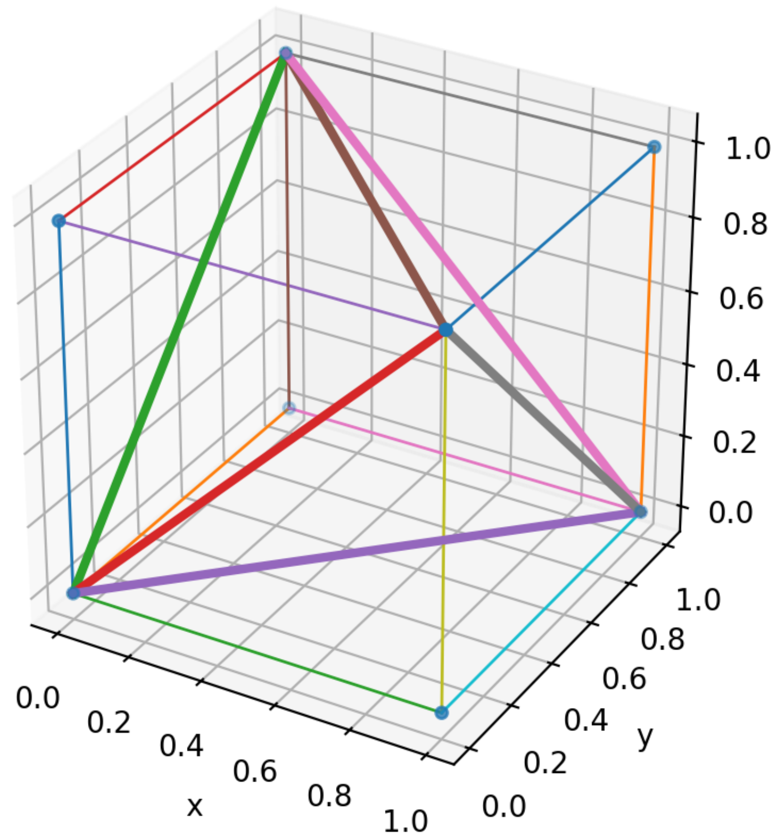
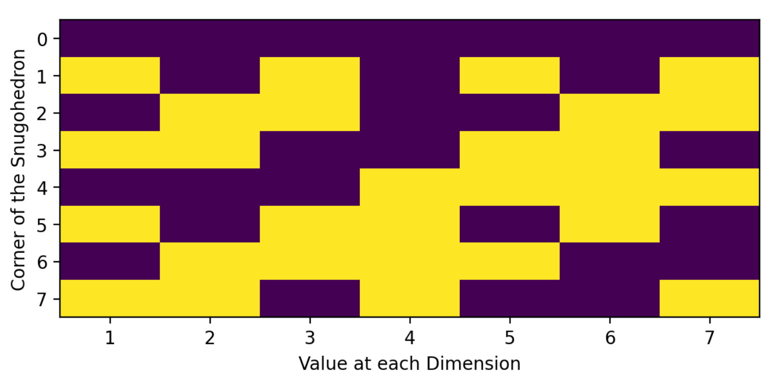


Figure 3: A regular tetrahedron with vertices at alternating corners of the cube.



(a) The 8 corners (rows) in $\{0, 1\}^7$.

Figure 4: A concrete snugohedron in $D = 7$.