SymChaos: Symmetrically-Constrained Chaos

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Abstract

Symmetry is traditionally associated with perfect order, whereas chaos is often perceived as a form of destructive disorder. We propose a straightforward and general framework for integrating these two concepts. By constraining the dynamics of a chaotic system to a finite region — specifically, the orbifold of a corresponding symmetry group — we can generate chaos with predefined symmetric properties. Tiling the plane with the resulting orbifold images produces visually compelling structures that simultaneously exhibit both symmetry and chaotic complexity.

Strange Attractors

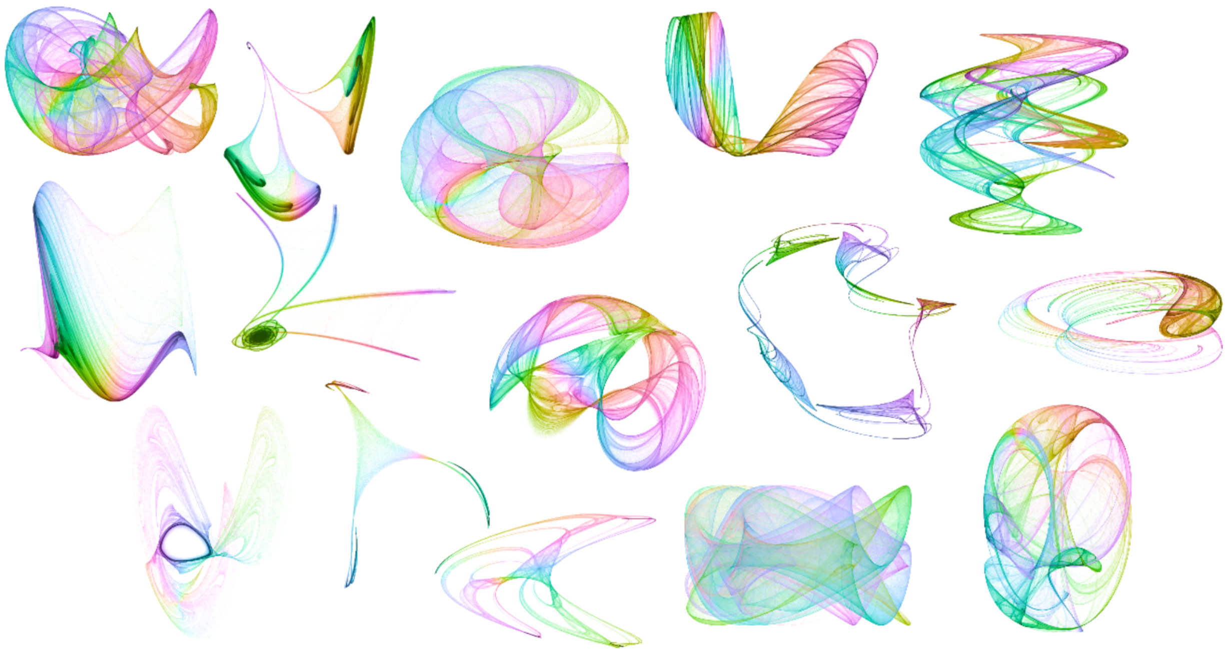
Strange attractors are a very common feature of dynamic systems. These are states where the system never settles to any static form, but evolves in the vicinity of a region with complex often fractal shape. Often the evolution of such systems can be described by ordinary differential equations or by partial differential equations. However a much simpler system – iterated function can also exhibit chaotic behavior with strange attractors. One of the simplest examples is Clifford Attractor. It is defined by iterations of function F(x,y) on a pair of point coordinates $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$ where $F(x, y)$ is defined by the following equations:

$$x_{n+1} = sin\,(a\,y_n) + c\,cos\,(a\,x_n)$$

$$y_{n+1} = sin\,(b\,x_n) + d\,cos\,(b\,y_n)$$

where $a,\ b,\ c,\ d$ are arbitrary parameters.

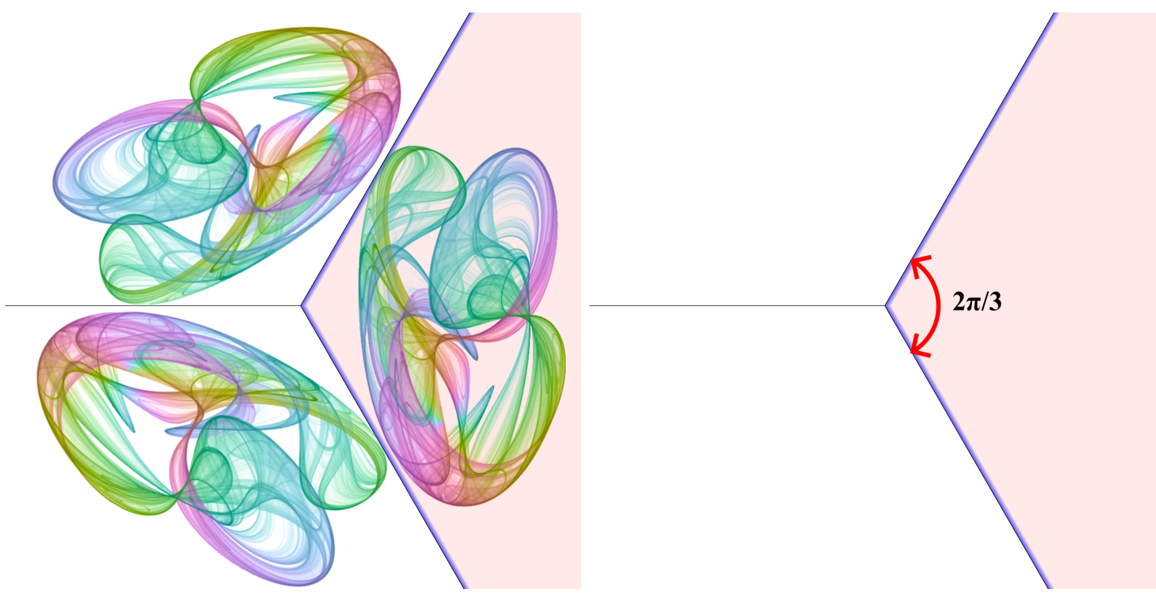
Starting from arbitrary point (x_0, y_0) we apply iterations many times (typically 10^9) and plot the 2D histogram of the points density using some appropriate coloring. Clifford attractor produces incredibly intricate, smoke-like textures. Below are few examples of the resulting shapes.



Dynamics on orbifold

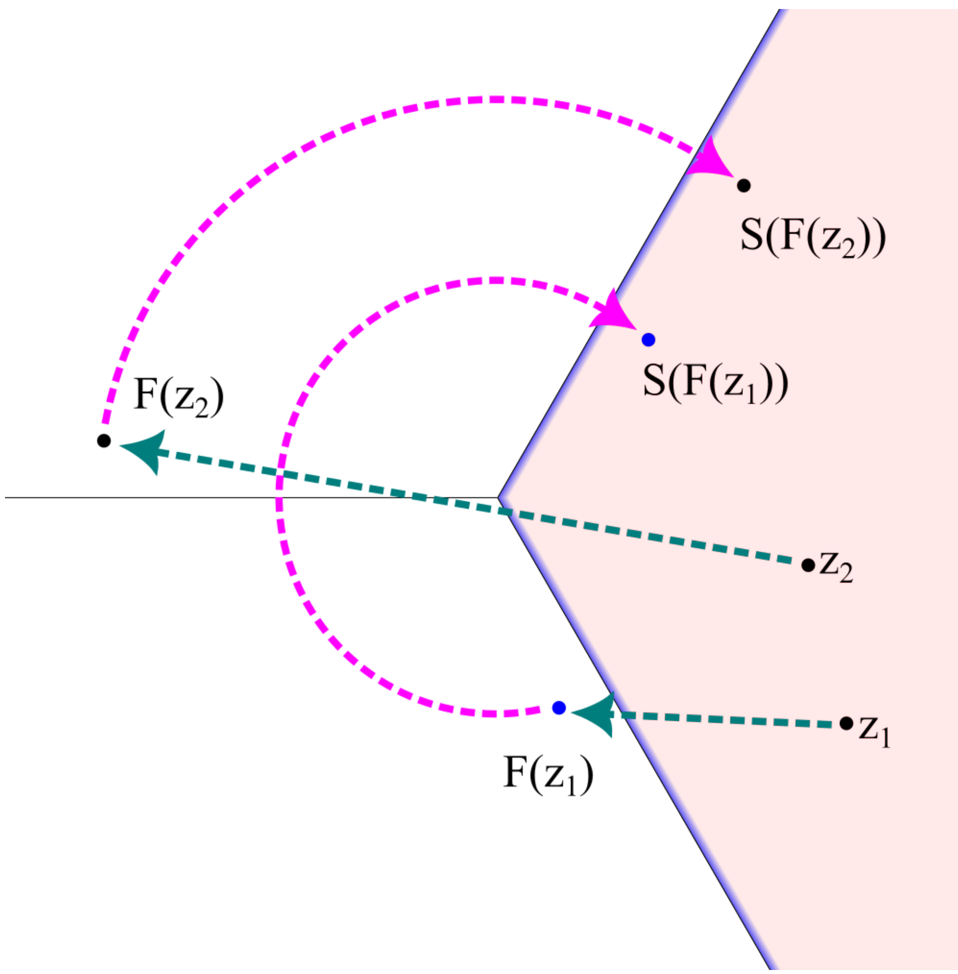
How to make such interesting shapes even more beautiful by making them symmetrical? The idea of creating strange attractors with symmetry was pioneered by Michael Field and Martin Goplibitsky in early 1990s [1]. They approached the problem by carefully constructing the iteration function $F(p)$ to make it obey the desired symmetry properties. In general it involves special treatment in cases of different symmetry types and results in rather complicated expressions for iterated functions.

We propose a simple and universal solution for any symmetry type and geometry. The core of the idea is to restrict the whole dynamics of the system to the **orbifold** of the corresponding symmetry group. The orbifold of the symmetry group is an abstract topological representation of all **non-equivalent** points of the symmetrical set (see [2] for good introduction into the orbifold). Below is an example of a symmetry group with a tri-fold axis of rotation. The orbifold of that group is a topological infinite cone.

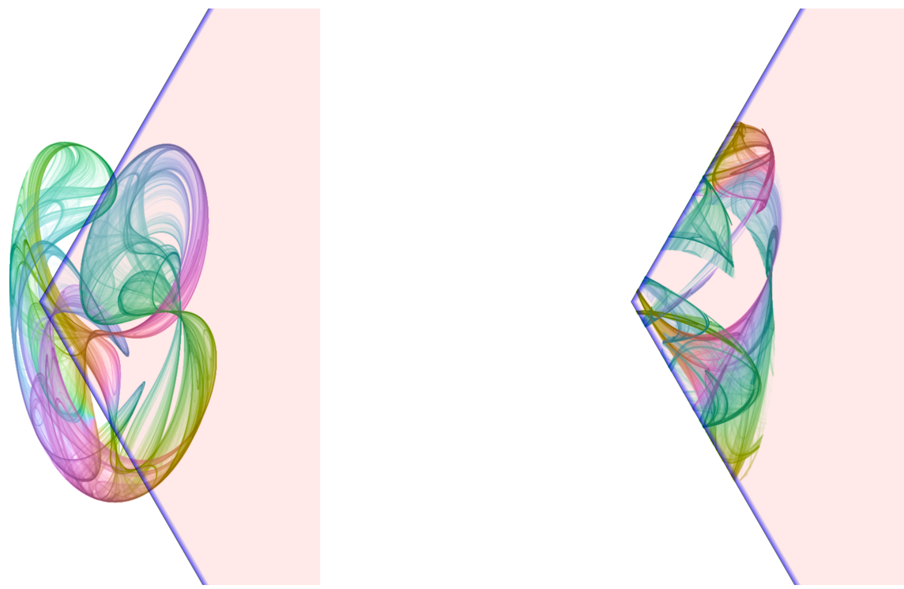


When the cone is cut along strain line and flattened it becomes infinite wedge with angle $2\pi/3$. The flattened orbifold is also called **Fundamental Domain**. The interior of the fundamental domain represents all symmetrically non-equivalent points of the shape on the left.

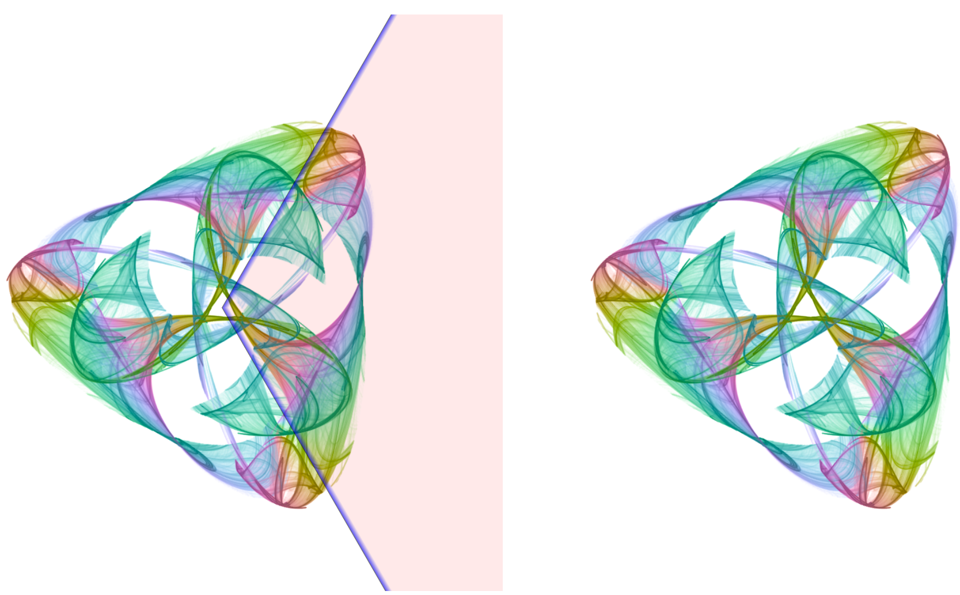
We propose to restrict dynamics of iterations to the interior of the orbifold. For that we replace iterative function $F(x, y)$ with composition of iteration and symmetrization $F(x, y) \rightarrow F_S(x, y) = S(F(x, y))$. New symmetrization function $S(x, y)$ act on any point in the plane by transforming it into an equivalent point inside of the fundamental domain. Below we illustrate the action of the function new iteration function $F_S()$ on two different points z_1 and z_2. The action of the symmetrization function $S(x, y)$ are performed using reverse pixel mapping algorithm [3].



Below is an illustration of how the new iteration function behaves when the rotation center is located inside the area of the original Clifford Attractor. On the left is the image or original attractor and overlaid with the image of the fundamental domain. On the right is the result of the iterations using $F_S(x, y)$. All the points are now located inside of the fundamental domain but the image looks not quite symmetric yet.



The important last step is to use the image of the fundamental domain as a tile to fill the entire plane. This step is performed once again using symmetrization function $S(x, y)$. Now we scan all the pixels of the entire visible area of the plane and using $S(x, y)$ map those pixels inside of the fundamental domain and use the color of that pixel to color the original pixel. The image below illustrates the result of those operations.



The wonderful result of the operation is a symmetrical and completely **continuous** and **seamless** image on the right. Important property of this procedure: the symmetrical shapes significantly depend on the relative positioning of the attractor and the symmetry group elements. This creates a wide array of new attractor shapes. The algorithm works equally well for any other attractors of iterated functions.

SymChaos and SymmHub

SymChaos is one of the applications of open source package SymmHub [4]. Vladimir Bulatov created SymmHub in collaboration with Chaim Goodman-Strauss and Scott Vorthmann.

SymmHub can be used in a wide range of geometries: Euclidean, Hyperbolic, Spherical, Inversive. It can be used in any number of dimensions: 2,3,4. It is implemented as an online application using HTML, JavaScript and WebGL. Therefore it runs on any platform which supports these technologies (desktop, tablet, cellphone).

References

[1] Field M. Golubitsky M. Symmetry and Chaos 1992.

[2] Conway J., Goodman-Strauss C., Burguel H. The Symmetries of Things. AK Peters 2008.

[3] Von Gagern M., Richter-Gebert J. "Hyperbolization of Euclidean Ornaments". *Electronic Journal of Combinatorics* Vol.16, no.2 (2009)

[4] SymmHub pages https://symmhub.github.io/SymmHub/

Gallery

Below is small selection of examples images created using SymChaos