Nicomachus’s Theorem

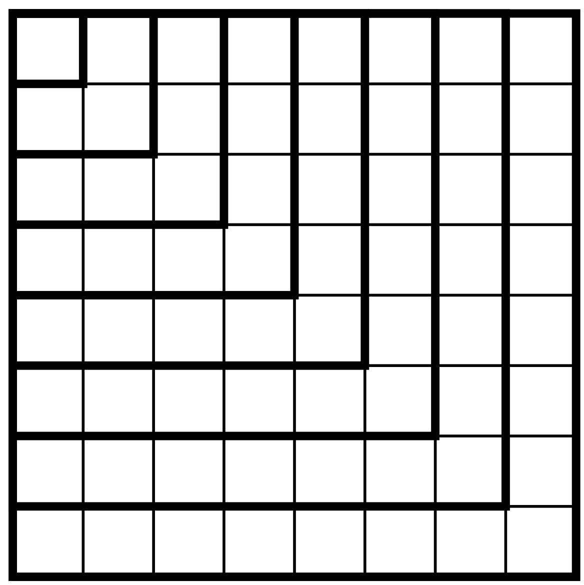
A contribution to the 2026 G4G Gift Exchange from Joseph DeVincentis

We didn’t have a Gathering in 2025, but I wanted to point out one of the wonderful properties that number has. It’s the sum of the cubes of the numbers 1 to 9, and also the square of the sum of the numbers 1 to 9.

$$1^3+2^3+3^3+4^3+5^3+6^3+7^3+8^3+9^3=(1+2+3+4+5+6+7+8+9)^2=2025$$

This equality holds in general for numbers other than 9, and it is called Nicomachus’s Theorem after Nicomachus of Gerasa, a Greek philosopher and mathematician who lived c. 60-120 AD in what is today Jerash, Jordan. However, the last year with the property was 1296 and the next is 3025 so celebrate it now!

Nicomachus’s Version



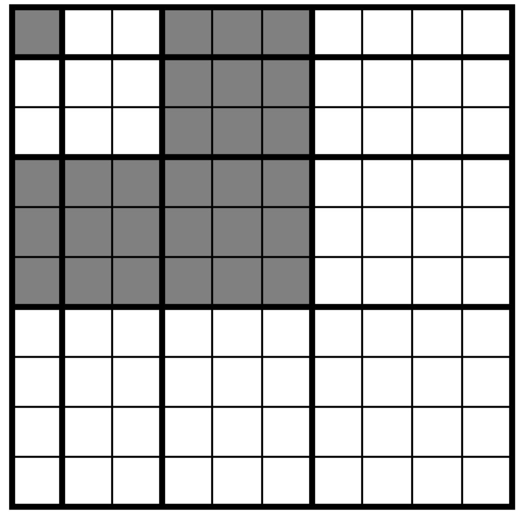
What Nicomachus wrote in *Introduction to Arithmetic* is that the first odd number (1) is the cube of 1, the sum of the next two odd numbers (3+5) is the cube of 2, the sum of the next three odd numbers (7+9+11) is the cube of 3, and so on.

Some of you are no doubt aware that the sum of the first n odd numbers is n^2. This follows from each odd number being the difference between consecutive square numbers, as can be seen geometrically in the diagram at the right. When you sum Nicomachus’s version from 1 to n, you get the sum of the cubes and the sum of $1+2+...+n$ odd numbers which is the square of $1+2+...+n$, the form in which the theorem is usually stated today. Of course, that sum from 1 to n is the nth triangular number, and the squares of the triangular numbers are the numbers with this wonderful property.

Proofs

Nicomachus did not prove the theorem, but there are many proofs.

In 1854, Charles Wheatstone (the physicist known for the Wheatstone bridge used to measure electrical resistance) gave a simple proof. His proof involves writing n^3 as the sum of n consecutive odd numbers, which naturally have n^2 as their average and median. These are the groups of odd numbers Nicomachus described. These numbers span from n^2-n+1 to n^2+n-1. Wheatstone proceeds to show that these groups of odd numbers follow directly after one another to form the sequence of odd numbers as Nicomachus defined the groups.



T. Sundara Row in 1893 provided a proof based on summing the numbers in a square multiplication table. For an $n \times n$ table, the sum of the *i*th row is *i* times a triangular number, *i*(1+2+...+*n*) and so the sum of the entire table is the square of the triangular number.

Alternatively, the table can be decomposed into a set of nested gnomons, each consisting of all the products for which the larger factor is the same number. This can be represented graphically as shown at the right. Each gnomon's largest number is n^2 and the other numbers can be paired so they sum to n^2 (for example, $(4\times1)+(4\times3)$ is one of the pairs in the 4th gnomon). There are *n* such numbers/pairs so the gnomon sums to n^3.

Generalization

Faulhaber polynomials, named for German mathematician Johann Faulhaber (1580-1635), express the sum of the *p*th powers of the numbers 1 to *n* as a polynomial in the *n*th triangular number. These exist for odd *p*, but only for the *p*=3 case is the polynomial a square.

Let $a=T(n)=\frac{n(n+1)}{2}$. Then:

$$\sum_{k=1}^{n} k^1 = a$$

$$\sum_{k=1}^{n} k^3 = a^2$$

$$\sum_{k=1}^{n} k^5 = \frac{4a^3-a^2}{3}$$

$$\sum_{k=1}^{n} k^7 = \frac{6a^4-4a^3+a^2}{3}$$

$$\sum_{k=1}^{n} k^9 = \frac{16a^5-20a^4+12a^3-3a^2}{5}$$

$$\sum_{k=1}^{n} k^{11} = \frac{16a^6-32a^5+34a^4-20a^3+5a^2}{3}$$

Faulhaber provided 17 examples of these formulas. Jacob Bernoulli generalized the formulas in 1713 using what have become known as the Bernoulli numbers. They were first proven by Carl Jacobi in 1834.