

PUZZLES THAT WILL HAVE YOU RUNNING IN CIRCLES



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Introduction: These Puzzles are prepared for the Gathering for Gardner 16 held in San Francisco, California from 18 to 22 February 2026. In each puzzle, runners start out at the same point and time and run in the same direction on a track in the shape of a circle or equilateral triangle. You are asked to find when they form certain arrangements on the track. For each workout the runners have constant identified running speeds expressed as an integer number of laps per hour (LPH) or seconds per lap (SPL). Puzzles begin with two, three or four runners on a circular track and then move to two or three runners on an equilateral triangular track.

Each workout is defined by the speeds of the runners as in (3 LPH, 5 SPL, 8 LPH). The runners are idealized in that they never fatigue and are modeled as points, each moving at a constant speed on the track. Each workout starts at $t = 0$ and ends when all runners arrive simultaneously at the starting point. The starting point on the equilateral triangular track is at one of its vertices. You will be asked to find certain defined time points, in minutes, for each workout, depending on the shape of the track and number of runners. Requested time points for workouts on the circular track will all be rational quantities; some of those on the equilateral triangle will be solutions to quadratic equations.

PUZZLES WITH TWO RUNNERS ON A CIRCULAR TRACK

In each of the given workouts involving two runners, find time points when they are

- (a) first at the ends of a diameter of the circle (t_d),
- (b) next together at the same point on the track (t_i),
- (c) next together at the starting point (t_s).

Workout 2A: (1 LPH, 7 LPH)
Workout 2B: (1 LPH, 7 SPL)
Workout 2C: (1 SPL, 7 LPH)
Workout 2D: (1 SPL, 7 SPL)

PUZZLES WITH TWO RUNNERS ON A CIRCULAR TRACK

Workout 2M: (52 LPH, 68 LPH)
Workout 2N: (52 LPH, 68 SPL)
Workout 2O: (52 SPL, 68 LPH)
Workout 2P: (52 SPL, 68 SPL)

Workout 2Q: (57 LPH, 76 LPH)
Workout 2R: (57 LPH, 76 SPL)
Workout 2S: (57 SPL, 76 LPH)
Workout 2T: (57 SPL, 76 SPL)

PUZZLES WITH TWO RUNNERS ON A CIRCULAR TRACK

Workout 2E: (59 LPH, 61 LPH)
Workout 2F: (59 LPH, 61 SPL)
Workout 2G: (59 SPL, 61 LPH)
Workout 2H: (59 SPL, 61 SPL)

Workout 2I: (58 LPH, 62 LPH)
Workout 2J: (58 LPH, 62 SPL)
Workout 2K: (58 SPL, 62 LPH)
Workout 2L: (58 SPL, 62 SPL)

PUZZLES WITH TWO RUNNERS ON A CIRCULAR TRACK

Workout 2U: (39 LPH, 52 LPH)
Workout 2V: (39 LPH, 52 SPL)
Workout 2W: (39 SPL, 52 LPH)
Workout 2X: (39 SPL, 52 SPL)

PUZZLES WITH THREE RUNNERS ON A CIRCULAR TRACK

In each of the given workouts involving three runners, find time points when they first

- (a) form a right triangle (t_r),
- (b) form an equilateral triangle (t_e),
- (c) are again together at the same point on the track (t_t),
- (d) are again together at the starting point (t_s).

Workout 3A: (1 LPH, 3 LPH, 5 LPH)

Workout 3B: (2 SPL, 3 LPH, 6 SPL)

Workout 3C: (2 SPL, 3 SPL, 6 SPL)

Workout 3D: (3 LPH, 10 LPH, 17 LPH)

PUZZLES WITH THREE RUNNERS ON A CIRCULAR TRACK

Workout 3M: (51 LPH, 62 SPL, 66 LPH)

Workout 3N: (51 LPH, 62 SPL, 66 SPL)

Workout 3O: (51 LPH, 62 SPL, 75 LPH)

Workout 3P: (51 LPH, 62 SPL, 75 SPL)

Workout 3Q: (51 LPH, 63 LPH, 93 LPH)

Workout 3R: (51 SPL, 63 LPH, 93 SPL)

Workout 3S: (51 LPH, 69 LPH, 87 LPH)

Workout 3T: (51 SPL, 69 SPL, 87 LPH)

PUZZLES WITH FOUR RUNNERS ON A CIRCULAR TRACK

In each of the given workouts involving four runners, find the first time ($t > 0$) when

- (a) a trio of runners forms a right triangle (t_r),
- (b) a trio forms an equilateral triangle (t_{e1}),
- (c) a different trio later forms an equilateral triangle (t_{e2}),
- (d) the four runners form a square (t_{sq}),
- (e) the four are together at the same point on the track (t_t),
- (f) the four are again all at the starting point (t_s).

PUZZLES WITH FOUR RUNNERS ON A CIRCULAR TRACK

Workout 4A: (1 LPH, 2 LPH, 7 LPH, 12 LPH).

Workout 4B: (1 LPH, 4 LPH, 6 LPH, 11 LPH).

Workout 4C: (1 LPH, 3 LPH, 15 LPH, 29 LPH).

Workout 4D: (1 SPL, 3 SPL, 15 SPL, 29 SPL).

Workout 4E: (1 LPH, 11 LPH, 15 LPH, 21 LPH).

Workout 4F: (1 SPL, 11 SPL, 15 SPL, 21 SPL).

Workout 4G: (2 LPH, 6 LPH, 30 LPH, 58 LPH).

Workout 4H: (2 SPL, 6 SPL, 30 SPL, 58 SPL).

Workout 4I: (5 LPH, 47 LPH, 59 LPH, 89 LPH).

Workout 4J: (5 SPL, 47 SPL, 59 SPL, 89 SPL).

PUZZLES WITH THREE RUNNERS ON A CIRCULAR TRACK

Workout 3E: (6 LPH, 42 LPH, 78 LPH)

Workout 3F: (6 SPL, 42 LPH, 78 LPH)

Workout 3G: (6 LPH, 42 SPL, 78 SPL)

Workout 3H: (6 SPL, 42 SPL, 78 SPL)

Workout 3I: (11 LPH, 23 LPH, 26 LPH)

Workout 3J: (11 SPL, 23 SPL, 26 SPL)

Workout 3k: (39 LPH, 48 LPH, 57 LPH)

Workout 3L: (39 SPL, 48 SPL, 57 SPL)

PUZZLES WITH THREE RUNNERS ON A CIRCULAR TRACK

Workout 3U: (52 LPH, 58 LPH, 64 LPH)

Workout 3V: (52 SPL, 58 SPL, 64 SPL)

Workout 3W: (54 LPH, 57 LPH, 78 LPH)

Workout 3X: (54 LPH, 57 SPL, 78 SPL)

Workout 3Y: (55 LPH, 70 LPH, 76 LPH)

Workout 3Z: (55 SPL, 70 SPL, 76 SPL)

Workout 3AA: (57 LPH, 79 SPL, 87 SPL)

Workout 3BB: (81 LPH, 101 LPH, 103 SPL)

Note that when three runners form a right triangle, the fourth runner will also form a right triangle with some pair of runners. The integer running speeds, whether in LPH or SPL, need to be chosen carefully to make it possible for all requested time points to occur. A few smaller cases are presented followed by some more rare cases where the same integer speeds in LPH and then SPL allow solutions for all requested time points.

PUZZLES WITH FOUR RUNNERS ON A CIRCULAR TRACK

Workout 4K: (6 LPH, 10 LPH, 30 LPH, 50 LPH).

Workout 4L: (6 SPL, 10 SPL, 30 SPL, 50 SPL).

Workout 4M: (11 LPH, 31 LPH, 39 LPH, 51 LPH).

Workout 4N: (11 SPL, 31 SPL, 39 SPL, 51 SPL).

Workout 4O: (19 LPH, 61 LPH, 97 LPH, 103 LPH).

Workout 4P: (19 SPL, 61 SPL, 97 SPL, 103 SPL).

Workout 4Q: (37 LPH, 67 LPH, 97 LPH, 103 LPH).

Workout 4R: (37 SPL, 67 SPL, 97 SPL, 103 SPL).

Workout 4S: (42 LPH, 46 LPH, 66 LPH, 86 LPH).

Workout 4T: (42 SPL, 46 SPL, 66 SPL, 86 SPL).

**PUZZLES WITH TWO RUNNERS ON
AN EQUILATERAL TRIANGLE**

In each of the given workouts involving two runners, find time points when they are first

- (a) on a line passing through the triangle's center (t_c),
- (b) together again at the same point on the track (t_t),
- (c) at different midpoints of the triangle's sides (t_m),
- (d) at different vertices of the triangle (t_v),
- (e) at the starting point again (t_s).

**PUZZLES WITH TWO RUNNERS
ON AN EQUILATERAL TRIANGLE**

Workout T2L: (28 LPH, 44 LPH)
Workout T2M: (28 LPH, 44 SPL)
Workout T2N: (28 SPL, 44 LPH)
Workout T2O: (28 SPL, 44 SPL)
Workout T2P: (52 SPL, 108 LPH)
Workout T2Q: (29 SPL, 58 LPH)

**PUZZLES WITH THREE RUNNERS
ON AN EQUILATERAL TRIANGLE**

Workout T3A : (44 LPH, 76 LPH, 84 LPH)
Workout T3B : (44 LPH, 76 LPH, 84 SPL)
Workout T3C : (36 LPH, 76 LPH, 92 LPH)
Workout T3D : (36 SPL, 76 SPL, 92 LPH)
Workout T3E : (68 LPH, 76 LPH, 108 LPH)
Workout T3F : (68 SPL, 76 SPL, 108 LPH)
Workout T3G : (21 SPL, 25 SPL, 105 SPL)
Workout T3H : (21 SPL, 25 SPL, 87 SPL)

**PUZZLES WITH TWO RUNNERS ON
AN EQUILATERAL TRIANGLE**

Workout T2A: (1 LPH, 2 LPH)
Workout T2B: (1 LPH, 4 LPH)
Workout T2C: (1 LPH, 3 LPH)
Workout T2D: (20 LPH, 20 SPL)
Workout T2E: (52 LPH, 52 SPL)
Workout T2F: (108 LPH, 108 SPL)
Workout T2G: (116 LPH, 116 SPL)
Workout T2H: (4 LPH, 44 LPH)
Workout T2I: (4 LPH, 44 SPL)
Workout T2J: (4 SPL, 44 LPH)
Workout T2K: (4 SPL, 44 SPL)

**PUZZLES WITH THREE RUNNERS
ON AN EQUILATERAL TRIANGLE**

In each of the given workouts

involving three runners, first find when

- (a) each pair of runners is first on a line passing through the center of the triangle (t_{c1} , t_{c2} , t_{c3}),
- and then, find when the runners are first**
- (b) at the vertices of an equilateral triangle (t_e),
- (c) at different midpoints of the triangle's sides (t_m),
- (d) at different vertices of the triangle (t_v),
- (e) together again at the same point on the track (t_t),
- (f) at the starting point again (t_s).

**PUZZLES WITH THREE RUNNERS
ON AN EQUILATERAL TRIANGLE**

Workout T3I : (15 SPL, 21 SPL, 35 SPL)
Workout T3J : (15 SPL, 21 SPL, 83 SPL)
Workout T3K : (51 SPL, 75 SPL, 85 SPL)
Workout T3L : (51 SPL, 75 SPL, 95 SPL)
Workout T3M : (16 SPL, 93 LPH, 105 LPH)
Workout T3N : (16 LPH, 93 SPL, 99 SPL)

SOLUTIONS

Workout 2A: (1 LPH and 7 LPH) $t_d = 5 \text{ min.}, t_l = 10 \text{ min.}, t_s = 60 \text{ min.}$ **Workout 2B:** (1 LPH and 7 SPL) $t_d = 210/3593 \approx 0.0584469802393543 \text{ min.},$ $t_l = 420/3593 \approx 0.116893960478709 \text{ min.},$ $t_s = 420 \text{ min.}$ **Workout 2C:** (1 SPL and 7 LPH) $t_d = 30/3593 \approx 0.0083495686056204 \text{ min.},$ $t_l = 60/3593 \approx 0.0166991372112441 \text{ min.},$ $t_s = 60 \text{ min.}$ **Workout 2D:** (1 SPL and 7 SPL) $t_d = 7/720 \approx 0.0097222222222222 \text{ min.},$ $t_l = 7/360 \approx 0.0194444444444444 \text{ min.},$ $t_s = 7/60 \approx 0.11666666666666667 \text{ min.}$ **Workout 2I:** (58 LPH and 62 LPH) $t_d = 7.5 \text{ min.}, t_l = 15 \text{ min.}, t_s = 30 \text{ min.}$ **Workout 2J:** (58 LPH and 62 SPL) $t_d = 465 \text{ min.}, t_l = 930 \text{ min.}, t_s = 930 \text{ min.}$ **Workout 2K:** (58 SPL and 62 LPH) $t_d = 435 \text{ min.}, t_l = 870 \text{ min.}, t_s = 870 \text{ min.}$ **Workout 2L:** (58 SPL and 62 SPL) $t_d = 2697/360 \approx 7.491666666666667 \text{ min.},$ $t_l = 2697/180 \approx 14.983333333333333 \text{ min.},$ $t_s = 2697/30 \approx 29.966666666666667 \text{ min.}$ **Workout 2Q:** (57 LPH and 76 LPH) $t_d = 30/19 \approx 1.578947368421053 \text{ min.},$ $t_l = 60/19 \approx 3.1578947368421053 \text{ min.},$ $t_s = 60/19 \approx 3.1578947368421053 \text{ min.}$ **Workout 2R:** (57 LPH and 76 SPL) $t_d = 190/61 \approx 3.114754098360656 \text{ min.},$ $t_l = 120/61 \approx 6.229508196721311 \text{ min.},$ $t_s = 380 \text{ min.}$ **Workout 2S:** (57 SPL and 76 LPH) $t_d = 285/122 \approx 2.336065573770492 \text{ min.},$ $t_l = 285/61 \approx 4.672131147540984 \text{ min.},$ $t_s = 285 \text{ min.}$ **Workout 2T:** (57 SPL and 76 SPL) $t_d = 1.9 \text{ min.}, t_l = 3.8 \text{ min.}, t_s = 3.8 \text{ min.}$ **Workout 3A:** (1 LPH, 3 LPH, and 5 LPH) $t_r = 7.5 \text{ min.}, t_e = 10 \text{ min.},$ $t_l = 30 \text{ min.}, t_s = 60 \text{ min.}$ **Workout 3B:** (2 SPL, 3 LPH, and 6 SPL) $t_r = 30/1797 \approx 0.0166944908180301 \text{ min.},$ $t_e = 20/3 \approx 6.666666666666667 \text{ min.},$ $t_l = t_s = 20 \text{ min.}$ **Workout 3C:** (2 SPL, 3 SPL, and 6 SPL) $t_r = 1/40 \text{ min.}, t_e = 1/30 \text{ min.},$ $t_l = t_s = 0.1 \text{ min.}$ **Workout 3D:** (3 LPH, 10 LPH, and 17 LPH) $t_r = 15/7 \approx 2.1428571428571428 \text{ min.},$ $t_e = 20/7 \approx 2.857142857142857 \text{ min.},$ $t_l = 60/7 \approx 8.571428571428571 \text{ min.},$ $t_s = 60 \text{ min.}$ **Workout 2E:** (59 LPH and 61 LPH) $t_d = 15 \text{ min.}, t_l = 30 \text{ min.}, t_s = 60 \text{ min.}$ **Workout 2F:** (59 LPH and 61 SPL) $t_d = 1830 \text{ min.}, t_l = 3660 \text{ min.}, t_s = 3660 \text{ min.}$ **Workout 2G:** (59 SPL and 61 LPH) $t_d = 1770 \text{ min.}, t_l = 3540 \text{ min.}, t_s = 3540 \text{ min.}$ **Workout 2H:** (59 SPL and 61 SPL) $t_d = 10797/720 \approx 14.995833333333333 \text{ min.},$ $t_l = 10797/360 \approx 29.991666666666667 \text{ min.},$ $t_s = 10797/180 \approx 59.983333333333333 \text{ min.}$ **Workout 2M:** (52 LPH and 68 LPH) $t_d = 1.875 \text{ min.}, t_l = 3.75 \text{ min.}, t_s = 15 \text{ min.}$ **Workout 2N:** (52 LPH and 68 SPL) $t_d = 31.875 \text{ min.}, t_l = 63.75 \text{ min.}, t_s = 255 \text{ min.}$ **Workout 2O:** (52 SPL and 68 LPH) $t_d = 24.375 \text{ min.}, t_l = 48.75 \text{ min.}, t_s = 195 \text{ min.}$ **Workout 2P:** (52 SPL and 68 SPL) $t_d = 221/120 \approx 1.8416666666666667 \text{ min.},$ $t_l = 221/60 \approx 3.6833333333333333 \text{ min.},$ $t_s = 221/15 \approx 14.733333333333333 \text{ min.}$ **Workout 2U:** (39 LPH and 52 LPH) $t_d = 30/13 \approx 2.307692307692307 \text{ min.},$ $t_l = 60/13 \approx 4.615384615384615 \text{ min.},$ $t_s = 60/13 \approx 4.615384615384615 \text{ min.}$ **Workout 2V:** (39 LPH and 52 SPL) $t_d = 130/131 \approx 0.99236641221374 \text{ min.},$ $t_l = 260/131 \approx 1.984732824427481 \text{ min.},$ $t_s = 260 \text{ min.}$ **Workout 2W:** (39 SPL and 52 LPH) $t_d = 195/262 \approx 0.744274809160305 \text{ min.},$ $t_l = 195/131 \approx 1.4885496183320611 \text{ min.},$ $t_s = 195 \text{ min.}$ **Workout 2X:** (39 SPL and 52 SPL) $t_d = 1.3 \text{ min.}, t_l = 2.6 \text{ min.}, t_s = 2.6 \text{ min.}$ **Workout 3E:** (6 LPH, 42 LPH, and 78 LPH) $t_r = 5/12 \approx 0.416666666666667 \text{ min.},$ $t_e = 5/9 \approx 0.5555555555555555 \text{ min.},$ $t_l = 5/3 \approx 1.666666666666667 \text{ min.}, t_s = 10 \text{ min.}$ **Workout 3F:** (6 SPL, 42 LPH, and 78 LPH) $t_r = 5/93 \approx 0.0537634408602151 \text{ min.},$ $t_e = 10/9 \approx 1.1111111111111111 \text{ min.},$ $t_l = 10/3 \approx 3.3333333333333333 \text{ min.}, t_s = 10 \text{ min.}$ **Workout 3G:** (6 LPH, 42 SPL, and 78 SPL) $t_r = 35/93 \approx 0.376344086021505 \text{ min.},$ $t_e = 910/9 \approx 101.11111111111111 \text{ min.},$ $t_l = 910/3 \approx 303.3333333333333 \text{ min.}, t_s = 910 \text{ min.}$ **Workout 3H:** (6 SPL, 42 SPL, and 78 SPL) $t_r = 13/240 \approx 0.05416666666666667 \text{ min.},$ $t_e = 91/180 \approx 0.5055555555555555 \text{ min.},$ $t_l = 91/60 \approx 1.5166666666666667 \text{ min.}, t_s = 9.1 \text{ min.}$

Workout 3I: (11 LPH, 23 LPH, and 26 LPH)
 $t_r = 2 \text{ min.}$, $t_e = 20/3 \approx 6.666666666666667 \text{ min.}$,
 $t_l = 20 \text{ min.}$, $t_s = 60 \text{ min.}$
Workout 3J: (11 SPL, 23 SPL, and 26 SPL)
 $t_r = 143/900 \approx 0.1588888888888889 \text{ min.}$,
 $t_e = 3289/270 \approx 12.18148148148148 \text{ min.}$,
 $t_l = 3289/90 \approx 36.54444444444444 \text{ min.}$,
 $t_s = 3289/30 \approx 109.63333333333333 \text{ min.}$
Workout 3K: (39 LPH, 48 LPH, and 57 LPH)
 $t_r = 5/3 \approx 1.666666666666667 \text{ min.}$,
 $t_e = 20/9 \approx 2.222222222222222 \text{ min.}$,
 $t_l = 20/3 \approx 6.666666666666667 \text{ min.}$, $t_s = 20 \text{ min.}$
Workout 3L: (39 SPL, 48 SPL, and 57 SPL)
 $t_r = 247/240 \approx 1.029166666666667 \text{ min.}$,
 $t_e = 988/45 \approx 21.955555555555555 \text{ min.}$,
 $t_l = 988/15 \approx 65.86666666666667 \text{ min.}$, $t_s = 197.6 \text{ min.}$

Workout 3Q: (51 LPH, 63 LPH, and 93 LPH)
 $t_r = 5/7 \approx 0.7142857142857143 \text{ min.}$,
 $t_e = 10/3 \approx 3.333333333333333 \text{ min.}$,
 $t_l = 10 \text{ min.}$, $t_s = 20 \text{ min.}$
Workout 3R: (51 SPL, 63 LPH, and 93 SPL)
 $t_r = 527/560 \approx 0.9410714185714286 \text{ min.}$,
 $t_e = 10540/3 \approx 3513.3333333333333 \text{ min.}$,
 $t_l = t_s = 10540 \text{ min.}$
Workout 3S: (51 LPH, 69 LPH, and 87 LPH)
 $t_r = 5/6 \approx 0.8333333333333333 \text{ min.}$,
 $t_e = 10/9 \approx 1.111111111111111 \text{ min.}$,
 $t_l = 10/3 \approx 3.333333333333333 \text{ min.}$, $t_s = 20 \text{ min.}$
Workout 3T: (51 SPL, 69 SPL, and 87 LPH)
 $t_r = 230/267 \approx 0.861423220973783 \text{ min.}$,
 $t_e = 7820/9 \approx 868.8888888888889 \text{ min.}$,
 $t_l = 7820/3 \approx 2606.6666666666667 \text{ min.}$, $t_s = 7820 \text{ min.}$

Workout 3Y: (55 LPH, 70 LPH, and 76 LPH)
 $t_r = 10/7 \approx 1.4285714285714286 \text{ min.}$,
 $t_e = 20/3 \approx 6.666666666666667 \text{ min.}$,
 $t_l = 20 \text{ min.}$, $t_s = 60 \text{ min.}$
Workout 3Z: (55 SPL, 70 SPL, and 76 SPL)
 $t_r = 209/126 \approx 1.6587301587301587 \text{ min.}$,
 $t_e = 1463/27 \approx 54.185185185185 \text{ min.}$,
 $t_l = 1463/9 \approx 162.55555555555555 \text{ min.}$,
 $t_s = 1463/3 \approx 487.6666666666667 \text{ min.}$
Workout 3AA: (57 LPH, 79 SPL, and 87 SPL)
 $t_r = 290/151 \approx 1.920529801324503 \text{ min.}$,
 $t_e = 45820/3 \approx 15273.333333333333 \text{ min.}$,
 $t_l = t_s = 45820 \text{ min.}$
Workout 3BB: (81 LPH, 101 SPL, and 103 SPL)
 $t_r = 1030/1581 \approx 0.651486401012018 \text{ min.}$,
 $t_e = 208060/9 \approx 23117.777777777777 \text{ min.}$,
 $t_l = t_s = 208060/3 \approx 69353.33333333333 \text{ min.}$

Workout 4C: (1 LPH, 3 LPH, 15 LPH, 29 LPH).
For (1, 3 or 15, 29 LPH), $t_r = 15/14 \approx 1.0714285714286 \text{ min.}$,
For (1, 5, 29 LPH), $t_{e1} = 10/7 \approx 1.4285714285714286 \text{ min.}$,
For (1, 3, 29 LPH), $t_{e2} = 10 \text{ min.}$,
 $t_{sq} = 7.5 \text{ min.}$, $t_l = 30 \text{ min.}$, $t_s = 60 \text{ min.}$

Workout 4D: (1 SPL, 3 SPL, 15 SPL, 29 SPL).
For (1, 3 or 15, 29 SPL), $t_r = 29/3360 \approx 0.00863095238 \text{ min.}$,
For (1, 3, 15 SPL), $t_{e1} = 5/12 \approx 0.4166666666666667 \text{ min.}$,
For (3, 15, 29 SPL), $t_{e2} = 29/24 \approx 1.2083333333333333 \text{ min.}$,
 $t_{sq} = 0.90625 \text{ min.}$, $t_l = 3.625 \text{ min.}$, $t_s = 7.25 \text{ min.}$

Workout 3M: (51 LPH, 62 SPL, and 66 LPH)
 $t_r = 2 \text{ min.}$, $t_e = 620/3 \approx 206.6666666666667 \text{ min.}$,
 $t_l = t_s = 620 \text{ min.}$
Workout 3N: (51 LPH, 62 SPL, and 66 SPL)
 $t_r = 310/73 \approx 4.246575342465753 \text{ min.}$,
 $t_e = 6820/3 \approx 2273.3333333333333 \text{ min.}$,
 $t_l = t_s = 6820 \text{ min.}$
Workout 3O: (51 LPH, 62 SPL, and 75 LPH)
 $t_r = 1.25 \text{ min.}$, $t_e = 620/3 \approx 206.6666666666667 \text{ min.}$,
 $t_l = t_s = 620 \text{ min.}$
Workout 3P: (51 LPH, 62 SPL, and 75 SPL)
 $t_r = 155/52 \approx 2.980769230768231 \text{ min.}$,
 $t_e = 620/3 \approx 206.6666666666667 \text{ min.}$,
 $t_l = t_s = 620 \text{ min.}$

Workout 3U: (52 LPH, 58 LPH, and 64 LPH)
 $t_r = 5/2 \text{ min.}$, $t_e = 10/3 \text{ min.}$,
 $t_l = 10 \text{ min.}$, $t_s = 30 \text{ min.}$
Workout 3V: (52 SPL, 58 SPL, and 64 SPL)
 $t_r = 104/45 \approx 2.311111111111111 \text{ min.}$,
 $t_e = 6032/135 \approx 44.6814814814 \text{ min.}$,
 $t_l = 6032/45 \approx 134.0444444444 \text{ min.}$,
 $t_s = 6032/15 \approx 402.1333333333 \text{ min.}$
Workout 3W: (54 LPH, 57 LPH, and 78 LPH)
 $t_r = 1.25 \text{ min.}$, $t_e = 20/3 \approx 6.666666666666667 \text{ min.}$,
 $t_l = 20 \text{ min.}$, $t_s = 20 \text{ min.}$
Workout 3X: (54 LPH, 57 SPL, and 78 SPL)
 $t_r = 247/140 \approx 1.7642857142857143 \text{ min.}$,
 $t_e = 2470/3 \approx 823.3333333333333 \text{ min.}$,
 $t_l = t_s = 2470 \text{ min.}$

Workout 4A: (1 LPH, 2 LPH, 7 LPH, 12 LPH).
For (1, 2 or 7, 12 LPH), $t_r = 30/11 \approx 2.72727272727 \text{ min.}$,
For (2, 7, 12 LPH), $t_{e1} = 4 \text{ min.}$,
For (1, 2, 12 LPH), $t_{e2} = 20 \text{ min.}$,
 $t_{sq} = 15 \text{ min.}$, $t_l = 60 \text{ min.}$, $t_s = 60 \text{ min.}$

Workout 4B: (1 LPH, 4 LPH, 6 LPH, 11 LPH).
For (1, 4 or 6, 11 LPH), $t_r = 3 \text{ min.}$,
For (1, 6, 11 LPH), $t_{e1} = 4 \text{ min.}$,
For (4, 6, 11 LPH), $t_{e2} = 20 \text{ min.}$,
 $t_{sq} = 15 \text{ min.}$, $t_l = 60 \text{ min.}$, $t_s = 60 \text{ min.}$

Workout 4E: (1 LPH, 11 LPH, 15 LPH, 21 LPH).
For (1, 11 or 15, 21 LPH), $t_r = 1.5 \text{ min.}$,
For (1, 11, 21 LPH), $t_{e1} = 2 \text{ min.}$,
For (1, 11, 15 LPH), $t_{e2} = 10 \text{ min.}$,
 $t_{sq} = 7.5 \text{ min.}$, $t_l = 30 \text{ min.}$, $t_s = 60 \text{ min.}$

Workout 4F: (1 SPL, 11 SPL, 15 SPL, 21 SPL).
For (1, 11 or 15, 21 SPL), $t_r = 0.00875 \text{ min.}$,
For (1, 15, 21 SPL), $t_{e1} = 7/24 \approx 0.2916666666666667 \text{ min.}$,
For (11, 15, 21 SPL), $t_{e2} = 77/24 \approx 3.2083333333333333 \text{ min.}$,
 $t_{sq} = 2.40625 \text{ min.}$, $t_l = 9.625 \text{ min.}$, $t_s = 19.25 \text{ min.}$

Workout T2D: (20 LPH and 20 SPL)

$$t_c = (11 - 2\sqrt{10})/27 \approx 0.173164617765305 \text{ min.},$$

$$t_l = 0.375 \text{ min.}, t_m = 0.5 \text{ min.},$$

$$t_v = 1 \text{ minute}, t_s = 3 \text{ min.}$$

Workout T2E: (52 LPH and 52 SPL)

$$t_c = (2927 - 2\sqrt{2926})/1755 \approx 1.60616237954195 \text{ min.},$$

$$t_l = 195/56 \approx 3.482142857 \text{ min.},$$

$$t_m = 32.5 \text{ min.}, t_v = 65 \text{ min.}, t_s = 195 \text{ min.}$$

Workout T2F: (108 LPH and 108 SPL)

$$t_c = (281 + \sqrt{6061})/810 \approx 0.443027684563713 \text{ min.},$$

$$t_l = 45/56 \approx 0.803571428571428 \text{ min.},$$

$$t_m = 7.5 \text{ min.}, t_v = 15 \text{ min.}, t_s = 45 \text{ min.}$$

Workout T2J: (4 SPL and 44 LPH)

$$t_c = (247 - \sqrt{38734})/1485 \approx 0.03379828106822 \text{ min.},$$

$$t_l = 15/214 \approx 0.0700934579439252 \text{ min.},$$

$$t_m = 2.5 \text{ min.}, t_v = 5 \text{ min.}, t_s = 15 \text{ min.}$$

Workout T2K : (4 SPL and 44 SPL)

$$t_c = (13 - \sqrt{70})/135 \approx 0.0343214795159944 \text{ min.},$$

$$t_l = 11/150 \approx 0.0733333333333333 \text{ min.},$$

$$t_m = 11/90 \approx 0.1222222222222222 \text{ min.},$$

$$t_v = 11/45 \approx 0.2444444444444444 \text{ min.},$$

$$t_s = 11/15 \approx 0.7333333333333333 \text{ min.}$$

Workout T2L: (28 LPH and 44 LPH)

$$t_c = 5/3 \approx 1.66666666666666666667 \text{ min.},$$

$$t_l = 3.75 \text{ min.}, t_m = 2.5 \text{ min.}, t_v = 5 \text{ min.}, t_s = 15 \text{ min.}$$

Workout T2P : (52 SPL and 108 LPH)

$$t_c = (377 - \sqrt{1729})/405 \approx 0.82819445488251 \text{ min.},$$

$$t_l = 65/42 \approx 1.547619047619048 \text{ min.},$$

$$t_m = 65/18 \approx 3.6111111111111111 \text{ min.},$$

$$t_v = 65/9 \approx 7.222222222222222 \text{ min.},$$

$$t_s = 65/9 \approx 21.666666666666667 \text{ min.}$$

Workout T2Q : (29 SPL and 58 LPH)

$$t_c = (14887 - \sqrt{3635569})/15660 \approx 0.828881429649112 \text{ min.},$$

$$t_l = 1740/959 \approx 1.814389989572471 \text{ min.},$$

$$t_m = \text{never}, t_v = 580 \text{ min.}, t_s = 1740 \text{ min.}$$

Note that t_c in Workout T2P is shorter than t_c in Workout T2Q by only about 41.21848596127413 milliseconds, while the other time points are quite different.

Workout T3C: (36 LPH, 76 LPH, and 92 LPH) $\equiv$ (a, b, c)

$$\text{For (a, c), } t_c = (475 + 5\sqrt{1573})/1242 \approx 0.542113784340193 \text{ min.}$$

$$\text{For (a, b), } t_c = (395 - 5\sqrt{85})/513 \approx 0.680121399051726 \text{ min.}$$

$$\text{For (b, c), } t_c = 5/3 \approx 1.666666666666667 \text{ min.}$$

$$t_e = 2.5 \text{ min.}, t_m = 2.5 \text{ min.}, t_v = 5 \text{ min.},$$

$$t_l = 7.5 \text{ min.}, t_s = 15 \text{ min.}$$

Workout T3D: (36 SPL, 76 SPL, and 92 LPH) $\equiv$ (a, b, c)

$$\text{For (a, c), } t_c = (95 + \sqrt{1573})/270 \approx 0.498744681592977 \text{ min.}$$

$$\text{For (a, b), } t_c = (79 - \sqrt{85})/135 \approx 0.516892263279312 \text{ min.}$$

$$\text{For (b, c), } t_c = 437/135 \approx 3.2370370370370 \text{ min.}$$

$$t_e = 47.5 \text{ min.}, t_m = 47.5 \text{ min.}, t_v = 95 \text{ min.},$$

$$t_l = 142.5 \text{ min.}, t_s = 285 \text{ min.}$$

Workout T2G: (116 LPH and 116 SPL)

$$t_c = (2641 + \sqrt{162781})/7830 \approx 0.38882008613969 \text{ min.},$$

$$t_l = 435/616 \approx 0.706168831168831 \text{ min.},$$

$$t_m = 72.5 \text{ min.}, t_v = 145 \text{ min.}, t_s = 435 \text{ min.}$$

Workout T2H: (4 LPH and 44 LPH)

$$t_c = (65 - 5\sqrt{70})/33 \approx 0.702030262827158 \text{ min.},$$

$$t_l = 1.5 \text{ min.}, t_m = 2.5 \text{ min.}, t_v = 5 \text{ min.}, t_s = 15 \text{ min.}$$

Workout T2I: (4 LPH and 44 SPL)

$$t_c = (247 - \sqrt{38734})/135 \approx 0.37178109175046 \text{ min.},$$

$$t_l = 165/214 \approx 0.771028037383178 \text{ min.},$$

$$t_m = 27.5 \text{ min.}, t_v = 55 \text{ min.}, t_s = 165 \text{ min.}$$

Workout T2M: (28 LPH and 44 SPL)

$$t_c = (841 + \sqrt{83581})/1890 \approx 0.597938511953402 \text{ min.},$$

$$t_l = 165/148 \approx 1.11486486486486486 \text{ min.},$$

$$t_m = 27.5 \text{ min.}, t_v = 55 \text{ min.}, t_s = 165 \text{ min.}$$

Workout T2N: (28 SPL and 44 LPH)

$$t_c = (841 + \sqrt{83581})/2970 \approx 0.380506325788529 \text{ min.},$$

$$t_l = 105/148 \approx 0.709459459459459 \text{ min.},$$

$$t_m = 17.5 \text{ min.}, t_v = 35 \text{ min.}, t_s = 105 \text{ min.}$$

Workout T2O: (28 SPL and 44 SPL)

$$t_c = 77/135 \approx 0.570370370370370 \text{ min.},$$

$$t_l = 77/60 \approx 1.2833333333333333 \text{ min.},$$

$$t_m = 77/90 \approx 0.8555555555555555 \text{ min.},$$

$$t_v = 77/45 \approx 1.7111111111111111 \text{ min.},$$

$$t_s = 77/15 \approx 5.133333333333333 \text{ min.}$$

Workout T3A: (44 LPH, 76 LPH, and 84 LPH) $\equiv$ (a, b, c)

$$\text{For (a, c), } t_c = (695 - 5\sqrt{610})/693 \approx 0.824688469910123 \text{ min.}$$

$$\text{For (a, b), } t_c = (655 - 10\sqrt{58})/627 \approx 0.92319341138973 \text{ min.}$$

$$\text{For (b, c), } t_c = (4195 - 5\sqrt{85})/1197 \approx 3.4660837741968 \text{ min.}$$

$$t_e = 2.5 \text{ min.}, t_m = 2.5 \text{ min.}, t_v = 5 \text{ min.},$$

$$t_l = 7.5 \text{ min.}, t_s = 15 \text{ min.}$$

Workout T3B: (44 LPH, 76 LPH, and 84 SPL) $\equiv$ (a, b, c)

$$\text{For (b, c), } t_c = (907 - \sqrt{14674})/855 \approx 0.919138845749914 \text{ min.}$$

$$\text{For (a, b), } t_c = (655 - 10\sqrt{58})/627 \approx 0.92319341138973 \text{ min.}$$

$$\text{For (a, c), } t_c = (11627 - \sqrt{154})/495 \approx 23.463818841119 \text{ min.}$$

$$t_e = 17.5 \text{ min.}, t_m = 17.5 \text{ min.}, t_v = 35 \text{ min.},$$

$$t_l = 52.5 \text{ min.}, t_s = 105 \text{ min.}$$

Workout T3E: (68 LPH, 76 LPH, and 108 LPH) $\equiv$ (a, b, c)

$$\text{For (a, c), } t_c = (965 - 5\sqrt{70})/1377 \approx 0.670419025906533 \text{ min.}$$

$$\text{For (b, c), } t_c = (1360 - 20\sqrt{7})/1539 \approx 0.849307975164853 \text{ min.}$$

$$\text{For (a, b), } t_c = 10/3 \approx 3.3333333333333333 \text{ min.}$$

$$t_e = 2.5 \text{ min.}, t_m = 2.5 \text{ min.}, t_v = 5 \text{ min.},$$

$$t_l = 7.5 \text{ min.}, t_s = 15 \text{ min.}$$

Workout T3F: (68 SPL, 76 SPL, and 108 LPH) $\equiv$ (a, b, c)

$$\text{For (b, c), } t_c = 38/81 \approx 0.469135802469136 \text{ min.}$$

$$\text{For (a, c), } t_c = (43 - \sqrt{13})/81 \approx 0.486351218821432 \text{ min.}$$

$$\text{For (a, b), } t_c = 646/135 \approx 4.78518518518518 \text{ min.}$$

$$t_e = t_m = 1615/18 \approx 89.72222222222222 \text{ min.},$$

$$t_v = 1615/9 \approx 179.44444444444444 \text{ min.},$$

$$t_l = 1615/6 \approx 269.16666666666666 \text{ min.},$$

$$t_s = 1615/3 \approx 538.33333333333333 \text{ min.}$$

Workout T3G: (21 SPL, 25 SPL, and 105 SPL) $\equiv$ (a, b, c)
 For (a, c), $t_c = 7/36 \approx 0.19444444444444444444$ min.
 For (b, c), $t_c = 0.25$ min. For (a, b), $t_c = 1$ min.
 $t_e = 35/48 \approx 0.79216666666666666667$ min.
 $t_m = 35/24 \approx 1.45833333333333333333$
 $t_v = 35/12 \approx 2.91666666666666666667$ min.,
 $t_l = 2.1875$ min., $t_s = 8.75$ min.

Workout T3H: (21 SPL, 25 SPL, and 87 SPL) $\equiv$ (a, b, c)
 For (a, c), $t_c = (43 - \sqrt{22})/180 \approx 0.212831023556537$ min.
 For (b, c), $t_c = (287 + \sqrt{4069})/1080 \approx 0.32480436456970$ min.
 For (a, b), $t_c = 1$ min.
 $t_e = t_m = 1015/24 \approx 42.2916666666666666667$ min.,
 $t_v = 1015/12 \approx 84.58333333333333333333$ min.,
 $t_l = 126.875$ min., $t_s = 253.75$ min.

Workout T3K: (51 SPL, 75 SPL, and 85 SPL) $\equiv$ (a, b, c)
 For (a, c), $t_c = (595 - 17\sqrt{10})/45 \approx 1.002298666253958$ min.
 For (a, b), $t_c = (62 - \sqrt{19})/180 \approx 1.280913356810207$ min.
 For (b, c), $t_c = (527 - \sqrt{34})/108 \approx 4.825639334306988$ min.
 $t_e = t_m = 85/24 \approx 3.5416666666666666667$ min.,
 $t_v = 85/12 \approx 7.08333333333333333333$ min.,
 $t_l = 10.625$ min., $t_s = 21.25$ min.

Workout T3L: (51 SPL, 75 SPL, and 95 SPL) $\equiv$ (a, b, c)
 For (a, c), $t_c = (635 - 7\sqrt{220})/540 \approx 0.98365411329752$ min.
 For (a, b), $t_c = (62 - \sqrt{19})/180 \approx 1.280913356810207$ min.
 For (b, c), $t_c = (152 - \sqrt{19})/54 \approx 2.734094464008506$ min.
 $t_e = 1615/48 \approx 33.64583333333333333333$ min.,
 $t_m = 1615/24 \approx 67.2916666666666666667$ min.,
 $t_v = 1615/12 \approx 134.58333333333333333333$ min.,
 $t_l = 100.9375$ min., $t_s = 403.75$ min.

Workout T3I: (15 SPL, 21 SPL, and 35 SPL) $\equiv$ (a, b, c)
 For (a, c), $t_c = (14 - \sqrt{7})/54 \approx 0.210263864609915$ min.
 For (a, b), $t_c = 7/18 \approx 0.38888888888888888889$ min.
 For (b, c), $t_c = (245 - 7\sqrt{10})/540 \approx 0.412711215516336$ min.
 $t_e = t_m = 7/24 \approx 0.29166666666666666667$ min.,
 $t_v = 7/12 \approx 0.58333333333333333333$ min.,
 $t_l = .875$ min., $t_s = 1.75$ min.

Workout T3J: (15 SPL, 21 SPL, and 83 SPL) $\equiv$ (a, b, c)
 For (a, c), $t_c = (113 - 2\sqrt{391})/540 \approx 0.13602325950635$ min.
 For (b, c), $t_c = (251 + \sqrt{253})/1080 \approx 0.24713516085240$ min.
 For (a, b), $t_c = 7/18 \approx 0.38888888888888888889$ min.
 $t_e = t_m = 581/24 \approx 24.20833333333333333333$ min.,
 $t_v = 581/12 \approx 48.1666666666666666667$ min.,
 $t_l = 72.625$ min., $t_s = 145.25$ min.

Workout T3M: (16 SPL, 93 LPH, and 105 LPH) $\equiv$ (a, b, c)
 For (a, b), $t_c = (1184 - 8\sqrt{979})/4185 \approx 0.2231035112175$ min.
 For (a, c), $t_c = 8/35 \approx 0.228571428571428$ min.
 For (b, c), $t_c = 20/9 \approx 2.2222222222222222$ min.
 $t_e = 5/3 \approx 1.6666666666666666667$ min.,
 $t_m = 10/3 \approx 3.33333333333333333333$ min.,
 $t_v = 20/3 \approx 6.6666666666666666667$ min.,
 $t_l = 5$ min., $t_s = 20$ min.

Workout T3N: (16 LPH, 93 SPL, and 99 SPL) $\equiv$ (a, b, c)
 For (a, b), $t_c = (148 - \sqrt{979})/90 \approx 1.296789158951955$ min.
 For (a, c), $t_c = (101 - \sqrt{301})/60 \approx 1.394177473785042$ min.
 For (b, c), $t_c = (2111 - \sqrt{133})/180 \approx 11.6637079855852$ min.
 $t_e = t_m = 213.125$ min.,
 $t_v = 426.25$ min., $t_l = 639.375$ min., $t_s = 1278.75$ min.



**PUZZLES AND
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