How to Draw a Hypercube

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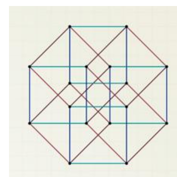
Every two years I draw a logo for the next Gathering 4 Gardner event. The G4G16 logo is a hypercube, because a hypercube has 16 corners. Here's the story behind the logo.

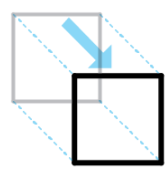
I've been drawing hypercubes all my life. I especially like this version of the hypercube, which is beautifully symmetrical, but tends to look flat, rather than dimensional.



To understand why this is a picture of a 4D cube, start from a 1D cube and build up . Stretching a line (a 1D cube) creates a square (a 2D cube). A square has 4 sides.

2D cube
4 vertices, 4 sides

Stretching a square makes a 3D cube. A cube has 6 faces, all of which are squares. Of course this isn't really a cube — it's a projection of a cube into a plane.

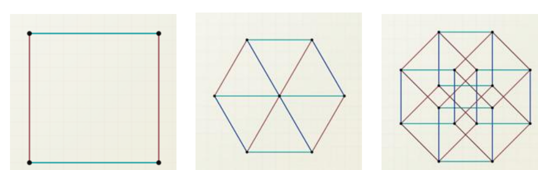


3D cube
8 vertices, 6 sides

And stretching a cube makes a 4D cube — a hypercube. A hypercube has 8 hyperfaces, all of which are cubes. Again, this isn't really a hypercube — it's a 2D picture of a hypercube.

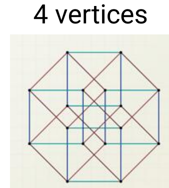
4D cube
16 vertices, 8 sides

Notice that the number of vertices keeps doubling. That's because to get from an n-cube to an (n+1)-cube, we double the n-cube and join corresponding vertices with new edges.



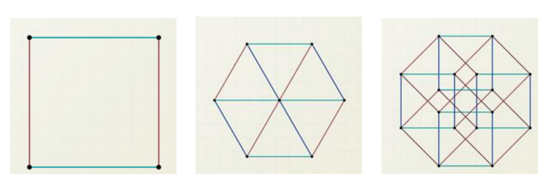
4 vertices 8 vertices 16 vertices

One day I was musing on the fact that the perimeter of this figure is an octagon. 8 is twice 4, and this is a picture of a 4-dimensional cube. Does this pattern keep going?



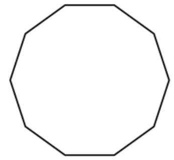
The perimeter of a **4**-cube
Has **8** sides

So far, yes. The perimeter of a picture of a 2D cube has 4 sides. The perimeter of a picture of a 3D cube has 6 sides. And The perimeter of a picture of a 4D cube has 8 sides.

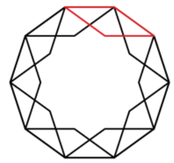


4 sides 6 sides 8 sides

So what if I drew a regular decagon? Could I fill in the inside of the figure to make a projection, of a five-dimensional cube? I had to know. I got out a ruler and protractor and started drawing.



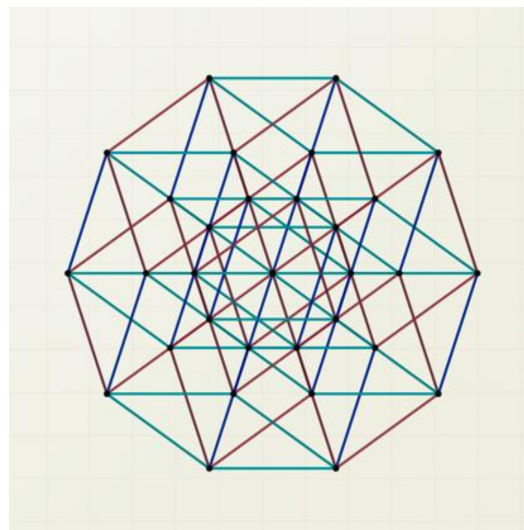
I started by completing all the rhombuses determined by 2 adjacent edges of the decagon. That makes 10 rhombuses. But we're just getting started.



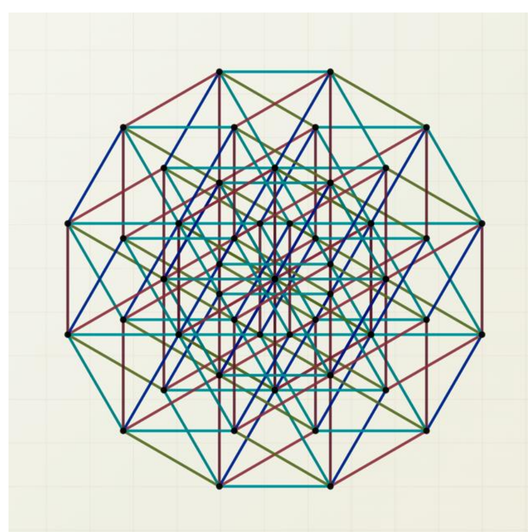
Then I completed all the squashed cubes determined by 3 adjacent edges of the decagon. That makes 10 squashed cubes. I kept completing every rhombus and cube I could see…

…and when I finished, I ended up with this figure, which has a lot of coincident lines, making the cubes and hypercubes hard to see. Look closely, and you'll see 40 cubes altogether.

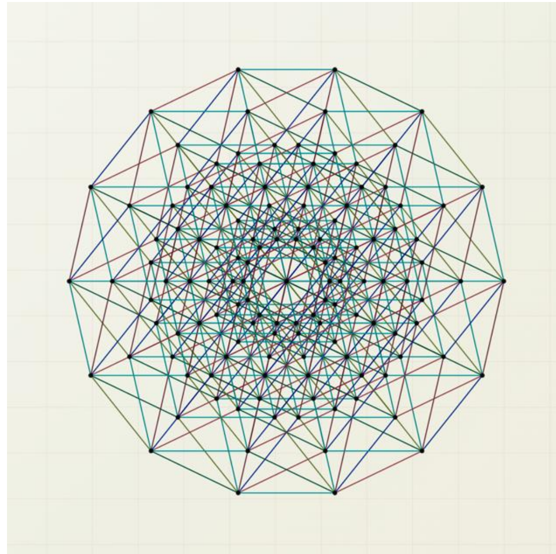
A 5D cube has 32 vertices. But this picture shows only 31 black dots. That's because 32 vertices doesn't divide evenly by 10-fold symmetry, so the remaining 2 vertices are both at the center.



But I didn't stop there. The 6D cube yields a much prettier picture, with squares & hexagons everywhere. Again, some vertices coincide, because 12 doesn't divide evenly into 64.

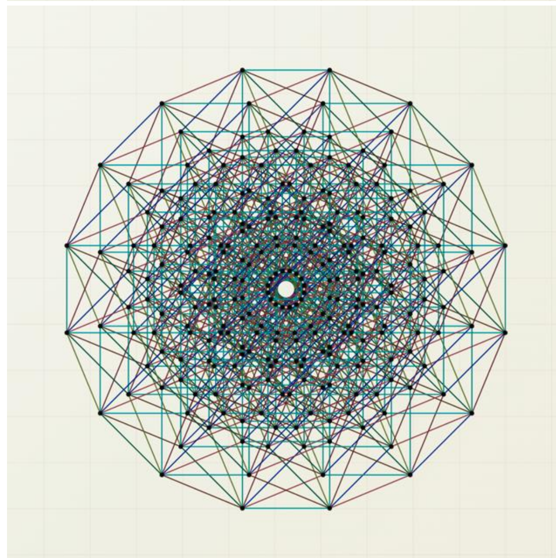


Seven, like five, is a prime number, so a 7D cube has a lot of coincident lines. But you can still see many rhombuses (2D faces) and cubes (3D faces here).

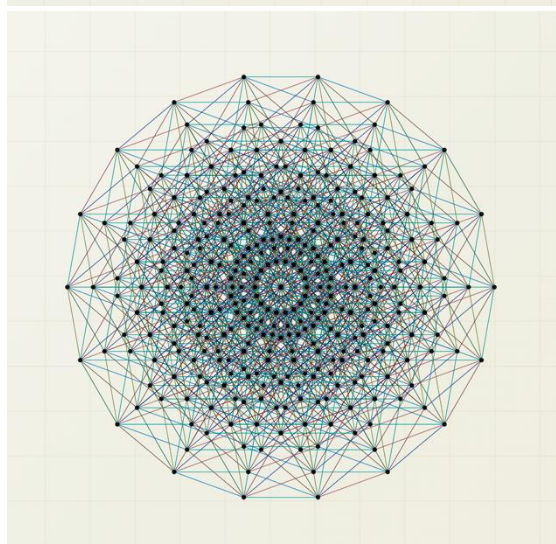


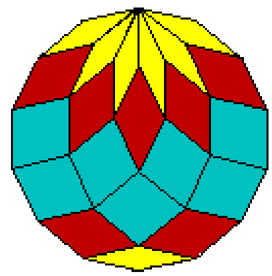
Something magic happens at 8. Eight is a power of 2, so the 256 vertices of an 8D cube fit neatly around a circle in 8-fold symmetry, without requiring that there be a vertex in the middle. This is the first projection since the 4D hypercube to have a hole in the center. And this won't happen again until we get to the 16-cube.

Also, 8 is a multiple of 4, so if you look closely, you'll find plenty of perfect squares.



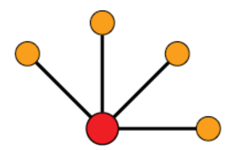
The last hypercube I drew by hand was a 9-cube. 9 is a multiple of 3, so if you look closely, you'll find equilateral triangles and even complete equilateral hexagons



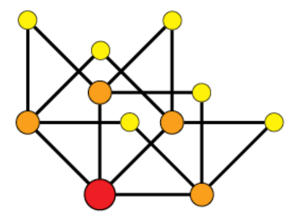


Puzzle and zonohedra enthusiasts may recognize that these figures are similar to popular rhombic puzzles like Pocket Rhombs™ by Kate Jones and Fractiles-7™ by Marc Pelletier and Beverly Johnson (both available from gamepuzzles.com). These rhombuses combine easily in many different ways because they are projections of higher-dimensional cubes.

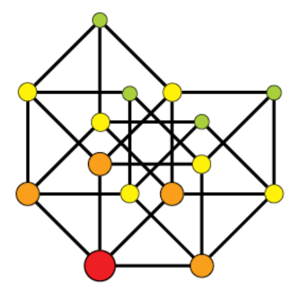
Here's another way to think about hypercubes. The corners of an n-cube are all possible sums of unit vectors in each of n different directions. To draw a picture of an 4-cube projected flat into a plane, draw 4 unit vectors equally spaced at 45° angles. That's **1** vertex at the origin & **4** vertices at the ends of four basis vectors.



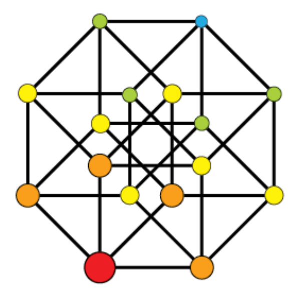
Then draw all possible sums of two vectors. The yellow dots are the opposite tips of rhombuses emanating from the **red** origin. That's **6** new vertices, or 4 choose 2.



There are **4** ways to sum three vectors. The **green** dots are the opposite tips of cubes emanating from the **red** origin.



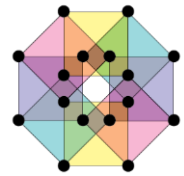
and just **1** way to sum all four vectors. The **blue** dot is the sum of all four vectors emanating from the **red** origin — it's the corner furthest away from the red corner in the hypercube.



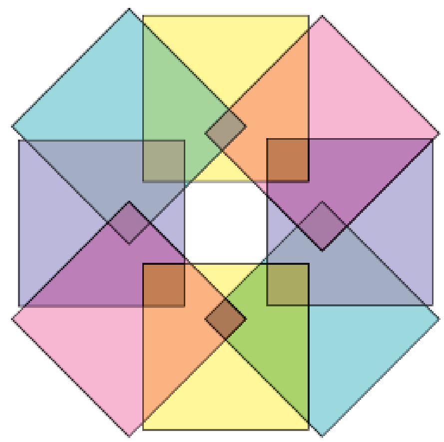
Notice a pattern? **1 4 6 4 1** — that's a row of Pascal's triangle. Every row of Pascal's triangle sums to a power of 2. That's because we're counting the number of ways to choose k vectors from a set of n basis vectors. This is a different way to count the corners of an n-cube.

1	sum = 1
1 1	sum = 2
1 2 1	sum = 4
1 3 3 1	sum = 8
1 4 6 4 1	sum = 16
1 5 10 10 5 1	sum = 32

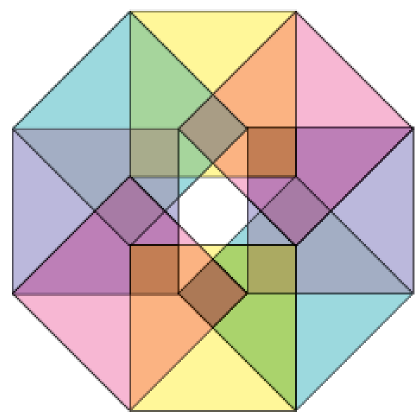
I chose the hypercube for the G4G16 logo because it has 16 vertices. When it came time for me to draw the G4G16 logo I had to decide how to construct and color this hypercube.



I chose a method that is simple to execute in a drawing program. I drew 8 squares — 4 upright and 4 turned 45° — using 3 primary colors plus a 4th color, and adding transparency so the colors would blend.



When the 8 squares snap together, they form a hypercube with a white space in the middle.



Finally, in a moment of extreme good fortune, I noticed that I could form the letter G and the number 4 by darkening some of the edges of the hypercube. That sealed the deal, and the G4G16 logo was born.

For more thoughts on matters of geometry and symmetry, check out my website, scottkim.com.