Uniform Tilings on the Sphere, the Plane, and the Hyperbolic Plane

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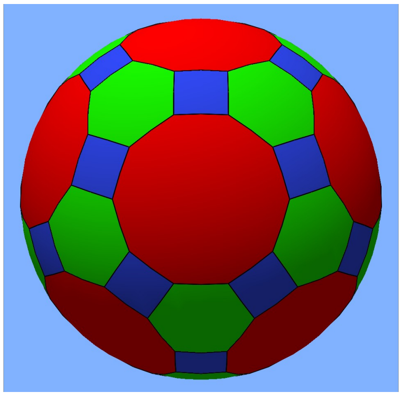
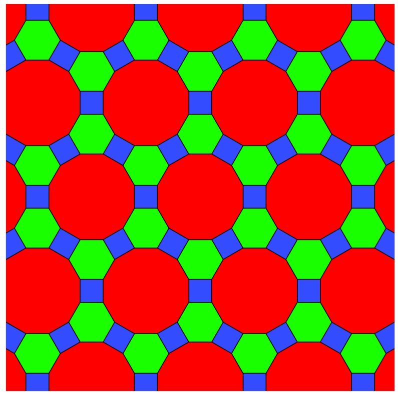
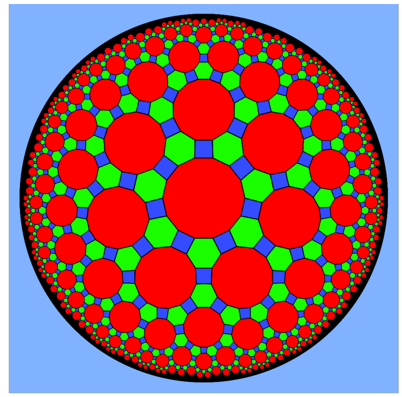






The presentation of uniform tilings at Gathering for Gardner on February 18, 2026 used interactive visualization software at https://web.cs.ucdavis.edu/~max/tiling.html. The visualization is described in detail in the MDPI Computers paper "Constructing and Visualizing Uniform Tilings", available with Open Access as https://www.mdpi.com/2073-431X/12/10/208. A keyboard and mouse input guide is at https://web.cs.ucdavis.edu/~max/QuickReference.pdf and a version of this users guide is available at https://web.cs.ucdavis.edu/~max/UserInstructions.pdf.

A tiling of a surface is a collection of polygons with disjoint interiors, whose union is the whole surface, with the intersection of any two polygons being either empty, a common vertex, or a common edge. In an Archimedean or a uniform tiling, the polygons are all regular, meaning all edges have the same length, and all vertex angles are the same. Different regular polygons in the tiling may have different numbers of edges. In an Archimedean tiling the pattern of polygons around each vertex must be the same. This pattern can be specified by a sequence of numbers, giving the numbers of sides of each polygon, counting counterclockwise around the vertex. The left image above shows a tiling of the sphere by regular spherical polygons, with pattern (10, 6, 4). Because each of the three images above has mirror symmetry, the pattern appears in clockwise order around half of the vertices, and counterclockwise order around the other half.

A uniform tiling must also satisfies the transitivity condition that there is a global symmetry of the whole tiling (a rigid mapping of the whole surface to itself taking polygons to polygons), that takes any vertex to any other vertex. That symmetry may be a reflection, that takes a counterclockwise pattern of polygons around the first vertex to a clockwise one around the second vertex.

The interactive interface allows you to specify the pattern around an initial vertex, and the computer program attempts to extend it to a uniform tiling, by adding polygons one by one, spiraling in a counterclockwise order around the outside of the pattern, and checking that the transitivity condition is satisfied. There may be more than one way to choose what kind of polygon to add next, so it keeps track of the possibilities at each location, and if eventually the next polygon addition is not possible, it backtracks to the previous addition, and tries another possibility. It stops if this backtracking process fails by deleting one of the initial polygons you specified, if the whole sphere is covered, if numerical precision errors accumulate, or if the total number 4000 of allowed polygons has been reached.

The system allows polygons of only up to 29 sides. Since there are no polygons with 1 or 2 sides, it is possible to have you enter the the vertex counts as a sequence of digits without spaces or delimiters, by considering a '1' or a '2' as the first digit of a 2 digit number. Then you presses the Enter key to terminate the list. When you do so, if the sum of the internal angles of the specified polygons is less than 360º, there will be a gap between the first and last polygons, and an animation will show continuous rotation of the the planes of the polygons around their common edges, to close the gap, and give the start of a 3D polyhedron. You can keep pressing the Enter key to add more polygons. For polyhedra or spherical tilings, you can press the ‘p’ key to toggle between the polyhedron representation, and the spherical tiling representation as in the left image.

If you press the Enter key when the last gap is perfectly closed up because the internal angles add up to exactly 360º you will have specified a planar tiling, as in the middle image above.

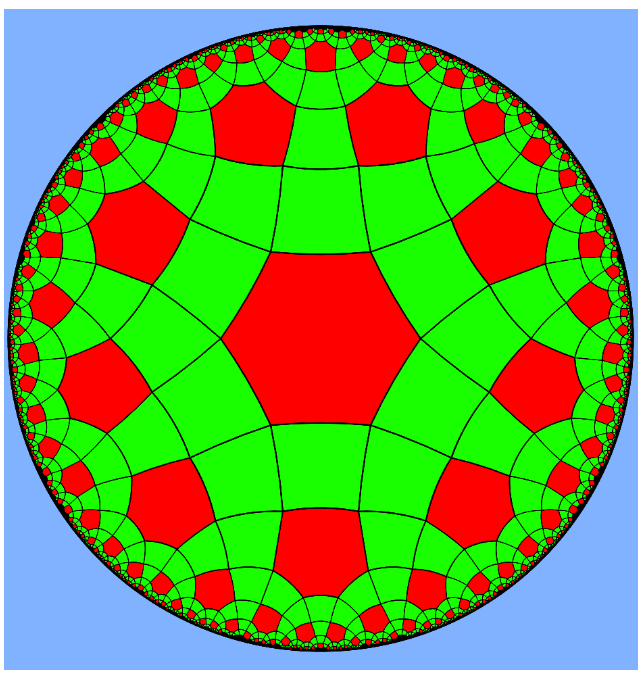
You can automatically add 200 polygons one by one by typing ‘f’, and you can add polygons until the limit of 4000 polygons is reached by typing ‘d’. At any point, if this appears to be adding more polygons than you want, you can stop the automatic addition by typing ‘s’ for “stop”. You can remove extra unwanted polygons one by one by pressing ‘r’ or the backspace key. You can start over with a new pattern of polygons around the initial vertex by typing ‘e’ for “erase” or reloading the web page, and then typing different digits.

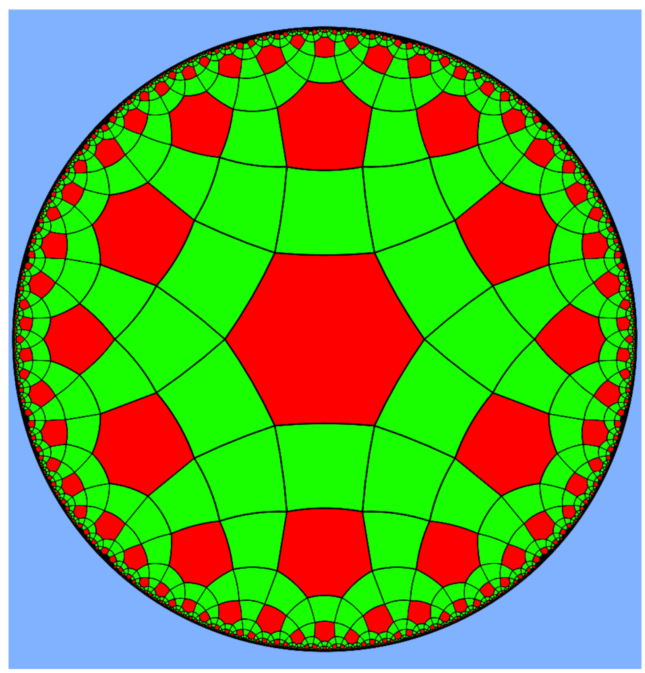
If adding an entry makes the sum if the internal angles exceed exceed 360º, a black Poincaré disc is faded in, and the existing polygons are animated to their hyperbolic shapes, opening up a gap between the first and the last just wide enough to finally add the new hyperbolic polygon. If further polygons are specified, the existing ones are again animated to create gaps for the new ones, by lengthening their edges to decrease their internal vertex angles. This process stops when you press the Enter, ‘f’, or ‘d’ key.

There are multiple ways to change the view of the tiling. If you depress the scroll wheel and roll it away from you, or else press the ‘k’ key, you will magnify the image, or in the spherical case, move closer to the sphere. If you depress the scroll wheel and roll it towards from you, or else press the ‘j’ key, you can make the image smaller, which is useful to see more of the tiling in the planer case. If you press the right mouse button and move the mouse, you can translate the image across the screen window. For the spherical case, if you press the left button and move the mouse, you can rotate the 3D sphere or polyhedron. Moving the mouse along a line through the center of the screen will rotate around an axis parallel to the screen, and moving the mouse in a circle near the edges of the screen will rotate it around an axis perpendicular to the screen. The hyperbolic case is similar, except that the left mouse button motion does not move the Poincaré disc (use the right mouse button for that). Instead it translates the hyperbolic tiling within that disc. Since only a finite number of polygons of an infinite tiling are drawn, when many polygons move close to one edge of the disk, they are faded out, and new ones are faded in where the existing ones are far from the opposite edge. The polygons appear to change size, but this translation is in fact a rigid motion in the geometry of hyperbolic plane. The apparent size change comes from squeezing the infinite hyperbolic plane into the finite Poincaré disc. Since it is sometimes difficult to get a smooth translation with the mouse, you can start a 200 frame translation to the right by pushing the right arrow or ‘h’ key. Similarly, the left arrow or ‘g’ key translates a hyperbolic tiling to the left, the up arrow or ‘v’ key translates up, and the down arrow or ‘b key translates down. Because of this fading out and in of polygons for hyperbolic tilings during rotation/translation, the program loses track of where it should add new polygons, and once you have translated or rotated a hyperbolic tiling, it is no longer possible to add more polygons. However for planar and spherical tilings, it is still possible to add new polygons after rotation or translation.

The transitivity condition for a uniform tiling is stricter then the Archimedean condition that the pattern around any vertex is the same, because it requires that the local mapping of polygons that matches the patterns at any two vertices can be extended to a global symmetry of the whole tiling. The program checks transitivity by trying to construct a global symmetry taking the initial vertex, around which the initial polygons you specified were drawn, to every other vertex. It backtracks the polygon additions if it ever fails. These multiple transitivity tests explain why the polygon additions take longer when the pattern gets large, and there are many vertices to check. (The web page currently runs more quickly on Mozilla Firefox and on Google Chrome than on Microsoft Edge.) You can toggle the transitivity test on and off by pressing the 't' key after you specify the initial pattern, before adding any more polygons. In the planar case, this makes no difference, but in the spherical case, it does make a difference in one case. If you type "3444f" and then start over with "3444tf" , you will get two different tilings. The second one is not uniform. See if you can discover why. In the hyperbolic case, you can find many tilings with the same pattern around each vertex, which are not uniform.

In the hyperbolic case, there may be more than one uniform tiling with the same vertex pattern, as demonstrated by the two tilings with vertex pattern (6, 4, 4, 4) shown below. If you type "6444f" you will get the one on the left. To get the one on the right from it, type 'n' and then 'f'. The 'n' tells the program to assume that the last polygon addition failed, even though it did not, and the 'f' tells it to keep backtracking and adding up to 200 new polygons. This will cause the program to backtrack until it constructs a different uniform tiling, if one exists, and you will get the tiling on the right. But the program does not compare patterns for symmetry, so if you type 'n' and 'f' again, you will get a tiling like the one on the left, but rotated by 60°. If you type 'n' and 'f' one further time, the program will backtrack to your initial polygons and give up.





The program can also create *k*-uniform Archimedean tilings, where there are a finite number *k* of types of vertices, so that there is a global symmetry taking any vertex to any other of the same type. To create one, start with an initial pattern around a vertex, as established when you first type the Enter key. Then type 'q'. This will create a green square marking the vertex V around which your initial loop of polygons circles, and allow you to selectively enforce the transitivity testing as new polygons are added. Whenever there is a global symmetry taking V to another vertex, that vertex will be marked in pink. If you click on a pink vertex, you can turn it green, and then the global symmetry taking V to it will be enforced as more polygons are added. (You can click any green vertex except the initial vertex V, turn it back pink, and stop this enforcement.) If you do not like any added polygon, you can type ‘n’ and the program will try another one with a different number of sides. If you choose at least two new green points, you have a good chance of creating a *k*-uniform tiling like the one below, which is 3-uniform. If you click too many pink squares, you may simply get a 1-uniform tiling, which is a uniform tiling where all vertices have the same type. If you click only one, you will probably get a tiling with potentially infinitely many vertex types.

If you type ‘q’ a second time, the global symmetries you have chosen will be enforced as new polygons are added, but the other vertices will not be tested for transitivity, so the collection of pink vertices will no longer increase, and computer time for transitivity testing will decrease. At this point, you can type ‘f’ or ‘d’ to add many more polygons, without typing Enter for each one. If you do not want to see the pink squares at vertices, type ‘i’. Typing ‘i’ again will also turn off the green squares, and typing it a third time will turn both the green and pink squares back on. In the image below, the right-most green square marks the initial vertex V, and the two ones to its left were clicked while pink to create this 3-uniform tiling.