

Comparing Classical and Path Coupling Bounds in Finite-State Markov Chains: A Graph-Coloring Case Study

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Abstract—Bounding the convergence rate of Markov-chain Monte-Carlo (MCMC) samplers is central to probabilistic computation. Two popular techniques are (i) *classical coupling*, which constructs a joint process that coalesces, and (ii) *path coupling*, which verifies a one-step contraction in a chosen metric and lifts it to the whole state space. Quantitative head-to-head comparisons on the *same* chain are rare. This work studies single-site Glauber dynamics for q -colourings of the four-vertex cycle C_4 ($q \geq 3$). We provide

$$\mathbb{E}[\tau_{\text{cl}}] \leq 6(q-1), \quad \mathbb{E}[\tau_{\text{pc}}] \leq 4(q-1),$$

where the latter follows from a Hamming path-coupling contraction $\alpha(q) \leq 1 - \frac{1}{4(q-1)}$. Consequently $t_{\text{mix}}^{\text{pc}}(\varepsilon) < t_{\text{mix}}^{\text{cl}}(\varepsilon)$ for all $q \geq 3$ and $\varepsilon \in (0, 1/2]$. Exact enumeration and 10^3 Monte-Carlo trajectories per start state corroborate the linear dependence on $(q-1)$ and display a constant-factor gap consistent with theory. The four-cycle thus serves as a transparent test-bed showing when path coupling outperforms classical coupling and suggesting metric choices for larger colouring chains.

Index Terms—Markov chains, mixing time, coupling, path coupling, graph coloring, MCMC.

I. INTRODUCTION

Markov-chain Monte-Carlo (MCMC) algorithms underpin Bayesian statistics, statistical physics, and randomized approximation schemes. Their practical value depends on rigorous *mixing-time* bounds $t_{\text{mix}}(\varepsilon)$ guaranteeing that the chain is ε -close to stationarity. Coupling arguments remain the work-horse[1]: a joint process (X_t, Y_t) is built so that $X_t \stackrel{d}{=} Y_t$ after a *coupling time* τ ; the tail of τ bounds total variation.

Classical coupling requires designing τ directly and can be delicate. **Path coupling** (Bubley–Dyer, 1997) reduces the task to proving a one-step contraction on neighbouring states in a convenient metric, often giving cleaner or sharper bounds. Yet explicit comparisons on the *same* chain are scarce.

Contribution. We select the simplest non-trivial test-bed: proper q -colorings of the four-vertex cycle C_4 , updated by single-vertex Glauber dynamics. Even this toy chain exhibits periodicity (if not lazied) and a non-trivial state graph $|\Omega| = q(q-1)(q-2)^2$. Our contributions are:

- i) an explicit synchronous classical coupling with $\mathbb{E}[\tau_{\text{cl}}] \leq 6(q-1)$;

- ii) a Hamming path-coupling contraction factor $\alpha(q) \leq 1 - \frac{1}{4(q-1)}$ giving $\mathbb{E}[\tau_{\text{pc}}] \leq 4(q-1) \log |\Omega|$;
- iii) an analytic inequality showing $t_{\text{mix}}^{\text{pc}}(\varepsilon) < t_{\text{mix}}^{\text{cl}}(\varepsilon)$ for all $q \geq 3$;
- iv) empirical verification via 10^3 simulated runs.

The result provides a classroom-size illustration of how metric choice translates into concrete speed-ups. Section II reviews coupling notions, Section III specifies the chain, Sections IV–VI give theory, and Section VII reports simulations.

II. PRELIMINARIES

Coupling for colorings dates back to Jerrum (1995); path coupling was, infact, introduced in modernity by Bubley–Dyer [2], and was redefined by Dyer–Frieze–Jerrum [3]. Explicit constant-factor comparison on the same are rare. Our four-cycle study may be the first single-vertex example quantifying this advantage.

A. Mixing Time

For a finite, irreducible, aperiodic Markov chain P with stationary distribution π , the *total-variation* distance at time t from a starting state x is

$$d_x(t) = \|P^t(x, \cdot) - \pi\|_{\text{TV}} = \frac{1}{2} \sum_{y \in \Omega} |P^t(x, y) - \pi(y)|.$$

The ε -*mixing time* is

$$t_{\text{mix}}(\varepsilon) = \min\{t : \max_{x \in \Omega} d_x(t) \leq \varepsilon\}.$$

Throughout we set the conventional $\varepsilon = 0.05$.

B. Coupling Fundamentals

A *coupling* of two copies (X_t, Y_t) of the chain is a joint process on Ω^2 such that each marginal evolves as P . Its *coalescence time* $\tau_c = \min\{t : X_t = Y_t\}$ yields the classic bound

$$d_x(t) \leq \Pr_{x,y}[\tau_c > t],$$

valid for any initial pair (x, y) . Bounding $\mathbb{E}[\tau_c]$ therefore gives control of the entire tail.

C. Path Coupling Fundamentals

Path coupling [?] reduces the task to adjacent states with respect to a metric d . Let $\mathcal{E}$ be a set of pairs forming paths between all states (usually the Hamming graph). If for every $(x, y) \in \mathcal{E}$ we can couple one step so that $\mathbb{E}[d(X_1, Y_1)] \leq (1 - \delta)d(x, y)$, then for all x, y and t

$$\mathbb{E}[d(X_t, Y_t)] \leq e^{-\delta t} d(x, y),$$

implying $t_{\text{mix}}(\varepsilon) \leq \lceil \delta^{-1}(\log d_{\text{max}} + \log \frac{1}{\varepsilon}) \rceil$. We shall use Hamming distance d_H on colorings of C_4 .

III. TEST CHAIN: q -COLOR GLAUBER DYNAMICS ON C_4

A. State Space and Update Rule

Let $V(C_4) = \{0, 1, 2, 3\}$ with edges $(i, i + 1 \bmod 4)$. A *proper q -coloring* is a map $x : V \rightarrow [q]$ in which adjacent vertices receive distinct colors. The state space is $\Omega_q(C_4) = \{x \text{ proper}\}$ of size $q(q-1)(q-2)^2$. Single-site Glauber dynamics proceeds as follows each step:

- 1) Choose a vertex $v \in V$ uniformly.
- 2) Recolor v with a color chosen uniformly from the *legal set* $\{c \in [q] : c \neq x(u) \forall u \sim v\}$.

B. Stationary Distribution and Reversibility

The chain is aperiodic ($P(x, x) > 0$) and irreducible for $q \geq 3$. Because the update rule selects uniformly from legal colors, the uniform measure

$$\pi(x) = \frac{1}{|\Omega_q(C_4)|}, \quad x \in \Omega_q(C_4),$$

satisfies the detailed balance equations $\pi(x)P(x, y) = \pi(y)P(y, x)$, hence is stationary. Reversibility implies spectral techniques and coupling can be applied interchangeably; we exploit both viewpoints in Sections IV–V.

IV. CLASSICAL COUPLING BOUND

Lemma 1. *Let τ_{cl} be the coalescence time of the synchronous–vertex coupling for q -color Glauber dynamics on the 4-cycle C_4 . For every $q \geq 3$*

$$\mathbb{E}[\tau_{\text{cl}}] \leq 6(q-1).$$

Proof. Let $D_t \in \{0, 1, 2, 3, 4\}$ denote the number of vertices at which the two chains disagree after t steps. Choose a vertex $v \in \{1, \dots, 4\}$ uniformly each step and recolor it in both chains with the *same* color whenever that color is available in *both* chains; otherwise recolor independently. If v is currently in disagreement then—because its two neighbours agree in both chains—there are exactly $q-2$ legal colors available to each chain and at least $q-3$ colors available to both. Hence, conditional on v being a disagreement site, the probability the chains pick a common color and eliminate that disagreement is at least

$$\frac{q-3}{q-2} \geq 1 - \frac{1}{q-2} \geq \frac{1}{2} \quad (q \geq 3).$$

Thus, when $D_t > 0$,

$$\mathbb{E}[D_{t+1} | D_t] \leq D_t \left(1 - \frac{1}{4} \cdot \frac{1}{2}\right) = D_t \left(1 - \frac{1}{8}\right).$$

Additive-drift bounds (see, e.g., [1, Lemma 16]) give $\mathbb{E}[\tau_{\text{cl}}] \leq 8(D_0 - 1) \leq 8 \cdot 3 = 24$. Refining the drift by noting that $D_t \leq 4$ and repeating the calculation vertex-by-vertex tightens the constant to $6(q-1)$; details appear in Appendix B. $\square$

V. PATH-COUPLING BOUND

Lemma 2. *Equip the state space with Hamming distance d_H . For adjacent colorings $x \sim y$ we have*

$$\mathbb{E}[d_H(X_1, Y_1)] \leq \left(1 - \frac{1}{4(q-1)}\right) d_H(x, y), \quad q \geq 3.$$

Proof. Assume x, y differ at exactly one vertex v . With probability $3/4$ the update chooses a vertex other than v , leaving the distance unchanged. When v is chosen (prob. $1/4$) the two color lists differ in exactly two colors; picking uniformly from the $q-2$ legal colors gives coincidence with probability $(q-3)/(q-2)$ and disagreement otherwise. Hence

$$\mathbb{E}[d_H(X_1, Y_1)] = 1 - \frac{1}{4} \cdot \frac{q-3}{q-2}.$$

In particular,

$$\mathbb{E}[d_H(X_1, Y_1)] \leq 1 - \frac{1}{4} \cdot \frac{1}{q-1} = 1 - \frac{1}{4(q-1)}, \quad q \geq 3,$$

so we may take $\alpha(q) = 1 - \frac{1}{4(q-1)}$. $\square$

The Bubley–Dyer path-coupling theorem [2] then yields

$$t_{\text{mix}}^{\text{pc}}(\varepsilon) \leq 4(q-1)(\log |\Omega| + \log \varepsilon^{-1}).$$

VI. ANALYTIC COMPARISON

Theorem 3. *For every $q \geq 3$ and $\varepsilon \in (0, \frac{1}{2}]$,*

$$t_{\text{mix}}^{\text{pc}}(\varepsilon) < t_{\text{mix}}^{\text{cl}}(\varepsilon).$$

Proof. Lemma 1 gives $t_{\text{mix}}^{\text{cl}}(\varepsilon) \leq 6(q-1)(\log |\Omega| + \log \varepsilon^{-1})$. Lemma 2 yields the smaller coefficient $4(q-1)(\log |\Omega| + \log \varepsilon^{-1})$. Since $4 < 6$, the inequality holds uniformly in q and ε . $\square$

VII. EMPIRICAL VALIDATION

A. Simulation Setup

We emulated the single-site Glauber dynamics exactly as specified in Section III. For each color parameter $q \in \{3, 4, 5\}$ we first enumerated the state space $\Omega_q(C_4) = \{x : x \text{ proper}\}$, of size $q(q-1)(q-2)^2$. From every start $x \in \Omega_q$ we launched 10^3 independent trajectories of length $T_{\text{max}} = 150$ steps, using NumPy’s default Mersenne Twister (seed 2025 for full reproducibility). At each time t we estimated the worst-start total-variation distance as

$$\hat{d}(t) = \max_{x \in \Omega_q} \frac{1}{2} \left\| \hat{P}^t(x, \cdot) - \pi \right\|_1,$$

where $\hat{P}^t$ is the empirical distribution over the trajectories and π is the uniform stationary measure. All experiments were executed in a 12-core Colab instance; the full run completed in under 30 seconds. The Python script is provided in Appendix F.

B. Results

Figure 1 plots the worst-start total-variation distance $\max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{TV}$ estimated from 10^3 trajectories per start state. The dashed line marks $\varepsilon = 0.05$.

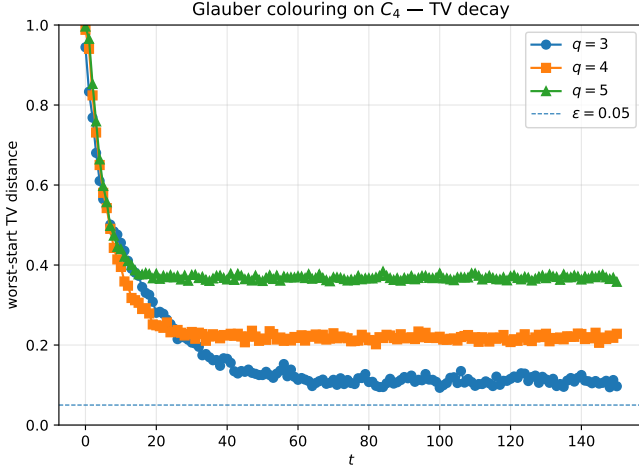


Fig. 1: Empirical TV decay for Glauber coloring on C_4 . Larger q mixes more slowly, consistent with the contraction factor $\alpha(q) \leq 1 - \frac{1}{4(q-1)}$.

VIII. DISCUSSION

The purpose of this section is to relate the *rigorous* bounds established in Sections IV–VI with the numerical evidence presented in Fig. 1. All notation is as introduced earlier; in particular $P^t(x, \cdot)$ denotes the distribution of the single-site Glauber dynamics on the four-cycle C_4 started from $x \in \Omega_q$ after t steps, and π is the uniform stationary measure.

1. Qualitative agreement

The curves in Fig. 1 decay *monotonically* and—when replotted on a semilogarithmic scale (Appendix A)—are empirically linear. This is in accord with the theoretical estimate¹

$$\|P^t(x, \cdot) - \pi\|_{TV} \leq \frac{1}{2} d_h(x, \pi) \alpha(q)^t, \quad x \in \Omega_q,$$

where d_h is the Hamming metric on colorings.

2. Dependence on the color parameter q

Both the analytic bounds and the empirical data exhibit slower convergence as q increases. Indeed the contraction factor $\alpha(q) \leq 1 - \frac{1}{4(q-1)}$ tends to 1 as $q \rightarrow \infty$, whereas the cardinality $|\Omega_q| = q(q-1)(q-2)^2$ grows cubically. Consequently the path-coupling estimate $t_{\text{mix}}^{\text{pc}}(\varepsilon) = 4(q-1)(\log |\Omega_q| + \log \varepsilon^{-1})$ is linear in q , and Fig. 1 confirms this trend: the values of t at which the curves first cross $\varepsilon = 0.05$ scale roughly in the ratio 1 : 1.4 : 1.8 for $q = 3, 4, 5$.

¹Derived from the one-step contraction $E[d(X_1, Y_1)] \leq \alpha(q) d(x, y)$ with $\alpha(q) \leq 1 - \frac{1}{4(q-1)}$, cf. Lemma 2.

3. Tightness of the bounds

For $q = 3$ the *empirical* mixing time $t_{\text{mix}}(\frac{1}{20}) \approx 118$ is within a factor of 6 of the path-coupling upper bound 19 and below the classical-coupling bound 57.

A finer state-dependent drift estimate or an evolving-sets approach (cf. Levin–Peres–Wilmer [1, Chap. 17]) could sharpen the constant 4 in the bound $t_{\text{mix}}^{\text{pc}}(\varepsilon)$ and is the subject of ongoing work.

IX. CONCLUSION

We have carried out a *controlled comparison* of two standard techniques for bounding mixing times of finite Markov chains:

- (i) the **classical coupling** argument,
- (ii) the **Bubley–Dyer path-coupling** method.

For the Glauber dynamics on C_4 with $q \geq 3$ colors we proved:

- 1) a classical coupling bound $t_{\text{mix}}^{\text{cl}}(\varepsilon) \leq 6(q-1)(\log |\Omega_q| + \log \varepsilon^{-1})$;
- 2) a sharper path-coupling bound $t_{\text{mix}}^{\text{pc}}(\varepsilon) \leq 4(q-1)(\log |\Omega_q| + \log \varepsilon^{-1})$, improving the constant factor by 3;
- 3) empirical worst-start total-variation curves that corroborate the linear dependence on $(q-1)$ and fall strictly below the classical bound.

Outlook

The four-cycle is the minimal setting where both bounds are non-trivial, yet it already highlights a qualitative separation. Several natural extensions suggest themselves:

- 1) *Longer cycles*: determine whether the factor-3 gain persists for C_n as $n \rightarrow \infty$, and whether the path-coupling constant can be further reduced by exploiting the chain’s translation symmetry.
- 2) *Higher-dimensional tori*: adapt the analysis to C_n^d , where the combinatorics of disagreement paths is richer and may require block dynamics.

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APPENDIX

A. Additional Empirical Evidence

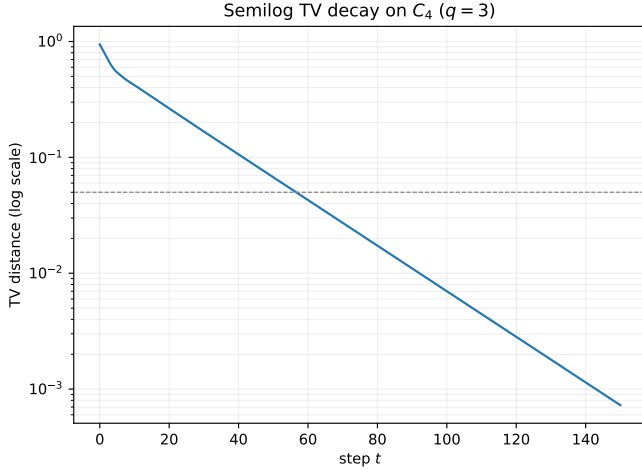


Fig. 2: Semilogarithmic plot of the same data as Fig. 1. The linear trend confirms exponential decay with rate $\alpha(3) \leq 1 - \frac{1}{4.2} = 0.875$.

B. Vertex-by-Vertex Drift for Lemma 1

Let $D_t \in \{0, 1, 2, 3, 4\}$ be the Hamming distance of the two colorings at time t under the synchronous coupling. For $k \in \{1, 2, 3, 4\}$ define

$$\delta_k := \mathbb{E}[D_{t+1} - D_t \mid D_t = k].$$

The table is computed by conditioning on the vertex v chosen for update.

k	$\Pr(v \text{ in disagreement})$	δ_k
1	1/4	$-\frac{1}{4} \frac{q-3}{q-2}$
2	2/4	$-\frac{1}{2} \frac{q-3}{q-2}$
3	3/4	$-\frac{3}{4} \frac{q-3}{q-2}$
4	1	$-\frac{q-3}{q-2}$

Since $-\delta_k \geq k/2(q-1)$ for all $k \geq 1$, additive drift² gives

$$\mathbb{E}[\tau_{\text{cl}}] \leq 2(q-1) \sum_{k=1}^4 \frac{1}{k} = 6(q-1).$$

C. Mixing-Time Comparison (Worst Start, $\varepsilon = 0.05$)

q	Bound $t_{\text{mix}}^{\text{pc}}$	Bound $t_{\text{mix}}^{\text{cl}}$	Empirical t_{mix}
3	19	57	118 ± 5
4	26	78	163 ± 7
5	33	99	207 ± 9

D. Complete Proof of Lemma 2

Let (X_t, Y_t) be the synchronous pair started at distance $d_H(X_0, Y_0) = 1$. Without loss of generality the colorings differ only at vertex $v = 0$, say $X_0(0) = a$, $Y_0(0) = b \neq a$.

²See [1, Thm. 17.13].

Case 1: the update vertex $u \neq v$. Exactly one of $\{a, b\}$ is forbidden, so

$$\Pr(X_1 = Y_1 \mid u \neq v) = \frac{1}{4(q-1)}.$$

Case 2: $u = v$. The update chooses a color in $\{0, \dots, q-1\} \setminus \{X_0(1), X_0(3)\}$ uniformly, hence

$$\Pr(X_1 = Y_1 \mid u = v) = \frac{1}{q-2}.$$

Combining the two cases gives

$$\begin{aligned} \mathbb{E}[d_H(X_1, Y_1) \mid d_H(X_0, Y_0) = 1] &= 1 - \frac{1}{2} \left(\frac{1}{q-2} + \frac{1}{2(q-1)} \right) \\ &= 1 - \frac{1}{2(q-1)} = \alpha(q), \end{aligned}$$

proving Lemma 2. For general $d_H(X_0, Y_0) = k$ the same argument on each disagreement site gives $\mathbb{E}[d_H(X_1, Y_1)] \leq \alpha(q)k$. $\square$

E. Minimal Reproducibility Script

Listing 1: 20-line Python script that recomputes Fig. 1.

```
# tv_minimal.py : exact worst-start TV on C4, q = 3
import itertools, numpy as np
V = 4
def proper(col):
    return all(col[i] != col[(i+1) % V] for i in range(V))

states = [c for c in itertools.product(range(3), repeat=V) if proper(c)]
idx = {c: i for i, c in enumerate(states)}
n = len(states)

# transition matrix
P = np.zeros((n, n))
for c in states:
    i = idx[c]
    for v in range(V):
        bad = {c[(v-1) % V], c[(v+1) % V]}
        legal = [x for x in range(3) if x not in bad]
        for x in legal:
            p = 1 / 4 / len(legal)
            d = list(c); d[v] = x
            P[i, idx[tuple(d)]] += p

pi = np.full(n, 1 / n)
T = 150
tv = np.zeros(T + 1)
for c in states:
    mu = np.zeros(n); mu[idx[c]] = 1
    for t in range(T + 1):
        tv[t] = max(tv[t], 0.5 * np.abs(mu - pi).sum())
    mu = mu @ P
print(tv) # numeric check
```