The Penta-Hex Tetrahedral Tile: A Self-Similar 3D Tile as the Shadow of a 4D Fractal

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1 Introduction

It is well known that regular tetrahedra cannot tile three-dimensional space. However, this raises a compelling question: Is there any solid with tetrahedral symmetry that can tile three-dimensional space?

One such example is the triakis truncated tetrahedron, which corresponds to the Voronoi cell of the diamond lattice. Yet, are there any other objects that satisfy these conditions? In this article, we introduce a self-similar 3D tile with tetrahedral symmetry. It is a self-replicating tile: it can be decomposed into 27 smaller copies of itself, each scaled by a factor of 1/3. For convenience, we will refer to it as the Penta–Hex Tetrahedral Tile. Figure 1 shows a 3D-printed model of this solid.

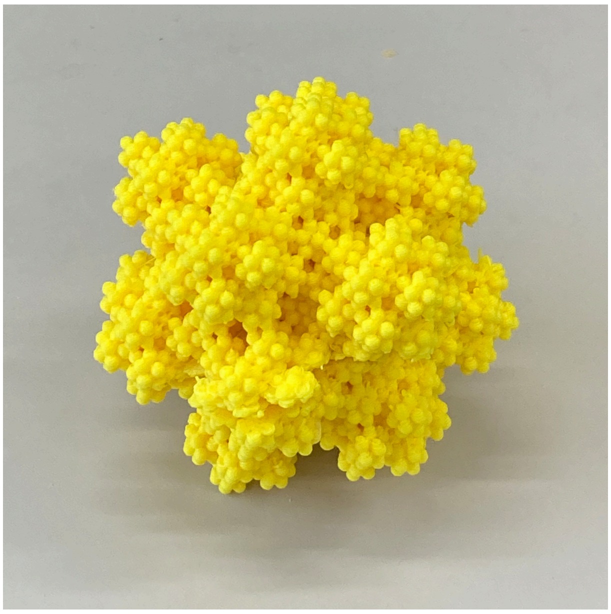


Figure 1: A 3D-printed model of the Penta–Hex Tetrahedral Tile.

The Penta–Hex Tetrahedral Tile has a fractal boundary and contains an infinite number of holes. It has a striking visual appeal and remarkable symmetry, and it arises as the 3D projection of a 4D fractal. In this article, we introduce its geometric properties and tell the story of how I discovered it.

2 A Self-similar 3D Tile

Looking at Figure 1, you might see a sphere covered with numerous protrusions, which I will call *bumps*. In fact, if you count them, you will find that the Penta–Hex Tetrahedral Tile has 16 bumps. Furthermore, each of these bumps is, in turn, covered with even smaller bumps. In the ideal mathematical object, this structure continues indefinitely.

You will notice that these bumps have the same overall shape as the Penta–Hex Tetrahedral Tile itself. Moreover, there are eleven additional components of the same shape located in the interior of the solid. In this way, the solid is self-similar: it consists of 27 smaller copies of itself (each scaled by 1/3) that fill the volume without gaps or overlaps.

Since the Penta–Hex Tetrahedral Tile consists of 27 smaller copies of itself, one can imagine a larger version composed of 27 such tiles, and then even larger and larger versions beyond that. In this way, the Penta–Hex Tetrahedral Tile tiles three-dimensional space. Such a set is called a *self-similar tile* in fractal geometry. Looking closely at its surface, one can find holes that pass through the object. Due to its recursive structure, the solid contains infinitely many such holes. Can you imagine how these holes interlock to form a larger object?

3 Symmetry

One may find the Penta–Hex Tetrahedral Tile both beautiful and strange, but perhaps its most striking feature is that it appears impossible from the standpoint of symmetry. In three dimensions, the possible finite rotational symmetry types of a solid are: cyclic, dihedral, tetrahedral, octahedral (cubic), and icosahedral (dodecahedral). Notably, none of these can make all 16 bumps equivalent. This means that, although the bumps are identical in shape, they do not all have the same local surroundings. Upon closer inspection, you will notice that some bumps are surrounded by six others (call these *Type A*), while others are surrounded by only five (call these *Type B*). There are four Type A bumps, arranged at the vertices of a regular tetrahedron. On each of the four faces of this tetrahedron, there are three Type B bumps. A Type A bump is surrounded by six Type B bumps, while a Type B bump is surrounded by two Type A and three Type B bumps. Consequently, the five neighbors of a Type B bump are not equivalent, and there

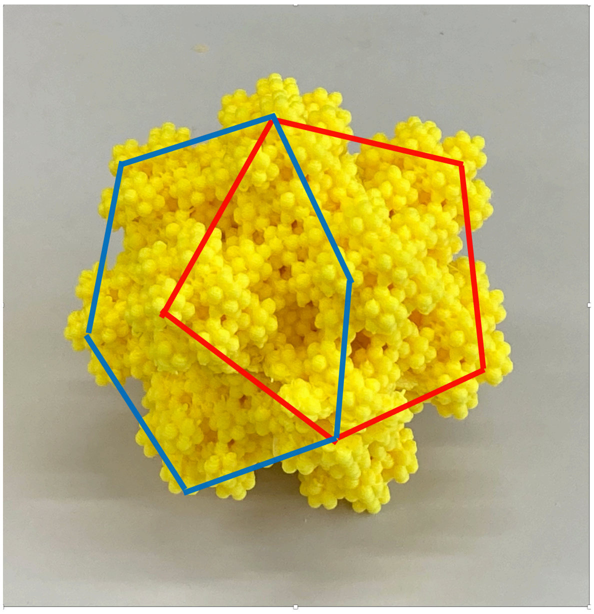


Figure 2: Near fivefold and near sixfold patterns of the Penta–Hex Tetrahedral Tile.

is no rotational symmetry centered at a Type B bump.

Thus, the Penta–Hex Tetrahedral Tile as a whole possesses tetrahedral symmetry. Keeping this symmetry in mind allows for a deeper understanding of its structure. Yet there is more: Figure 3 shows the view from a Type A bump. Although it has true 3-fold symmetry, it looks almost 6-fold. Why does it look that way?

4 The Logic Behind Structure

To clarify its structure, we now give a formal definition of this self-similar object.

The Penta–Hex Tetrahedral Tile is a self-similar set that is formed by the union of 27 copies of itself, each scaled by a factor of 1/3. Crucially, these smaller copies are only translated (not rotated); thus, each map can be written as

$$f_{k,d}(x) = \frac{x+d}{k} \qquad (d \in \mathbb{R}^3).$$

In particular, here $f_{3,d}(x) = \frac{x+d}{3}$. For a set $X \subset \mathbb{R}^3$, we write $f_{k,d}(X) = \{f_{k,d}(x) \mid x \in X\}$.

For a set $D \subset \mathbb{R}^3$ consisting of 27 points, the object is the fixed point of the mapping:

$$F(X) = \bigcup_{d \in D} f_{3,d}(X).$$

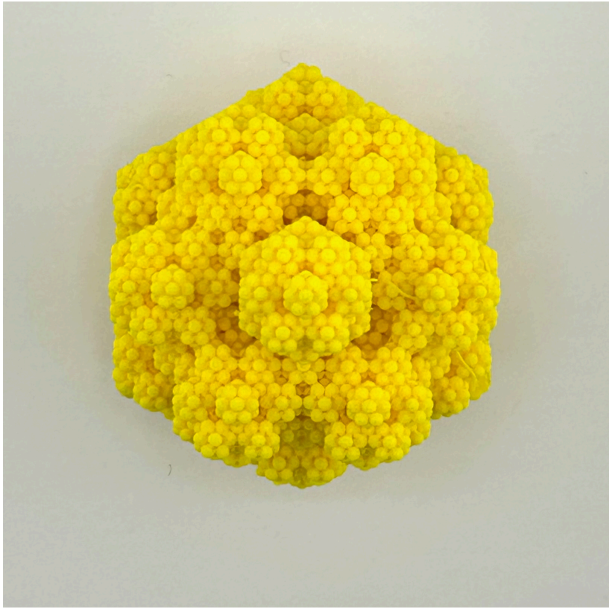


Figure 3: View from a Type A bump: visually close to 6-fold symmetry.

In fact, it is the unique fixed point of this map. This set D is referred to as the *digit set*. The collection of functions $\{f_{k,d} \mid d \in D\}$ is known as an *Iterated Function System (IFS)*, and its unique fixed point is called the *attractor* of the IFS. If the attractor has positive volume and tiles space by translations, it is called a *self-similar tile*. For further details on these concepts and the conditions for tiling, consult [1], [2], and [3].

In the following, we denote the attractor of an IFS $\{f_{k,d} \mid d \in D\}$ by $\mathcal{F}(k, D)$. In the Penta–Hex Tetrahedral Tile, the 27 small copies are positioned according to the digit set D, which is composed of five distinct subsets $D = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$. The coordinates of these subsets are defined as follows:

- D_1:

$$\begin{aligned} D_1 = \{&(3,1,-1),\ (3,-1,1),\ (-3,1,1),\ (-3,-1,-1), \\ &(-1,3,1),\ (1,3,-1),\ (1,-3,1),\ (-1,-3,-1), \\ &(1,-1,3),\ (-1,1,3),\ (1,1,-3),\ (-1,-1,-3)\}. \end{aligned}$$

  These 12 points form the vertices of a *truncated tetrahedron*.

- D_2:

$$D_2 = \{(2,2,2), (2,-2,-2), (-2,2,-2), (-2,-2,2)\}.$$

  These are the vertices of a *regular tetrahedron*.

- D_3:

$$D_3 = \{(2,0,0),\ (-2,0,0),\ (0,2,0),\ (0,-2,0),\ (0,0,2),\ (0,0,-2)\}$$

  These are the vertices of a *regular octahedron*.

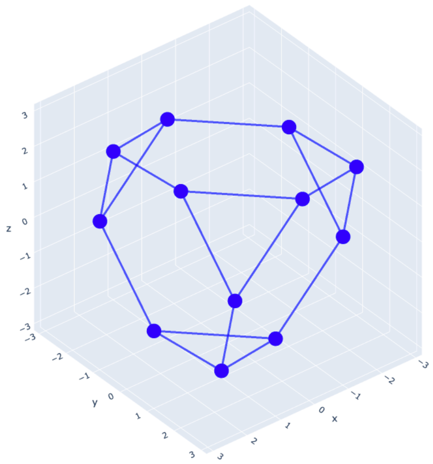
- D_4:

$$D_4 = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}.$$

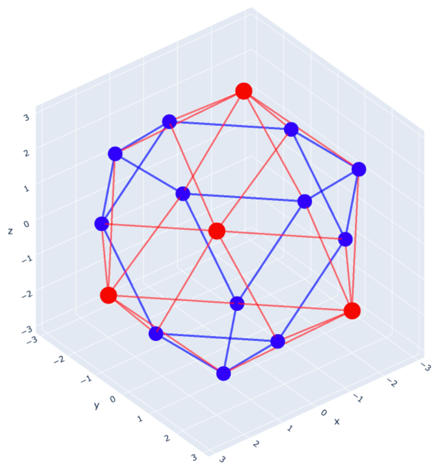
  This set forms a *regular tetrahedron*, exactly half the size of D_2.

- D_5:

$$D_5 = \{(0,0,0)\}.$$



D_1



$D_1 \cup D_2$

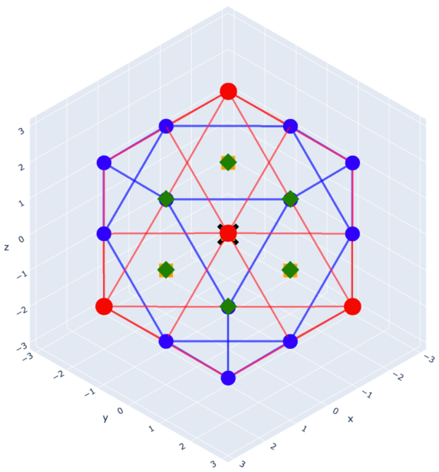
From this definition, it is evident that D possesses *tetrahedral symmetry*. Consequently, the resulting self-similar set $\mathcal{F}(3, D)$ also exhibits the same tetrahedral symmetry. Among these, the 16 points in $D_1 \cup D_2$ correspond to the 16 surface bumps discussed earlier. Specifically, the 4 digits in D_2 give the Type A bumps, and the 12 digits in D_1 give the Type B bumps.

According to the theory of self-similar tiles, if a digit set D is chosen such that it contains exactly one element from each coset of $\Lambda/(k\Lambda)$ for some lattice Λ, then $\mathcal{F}(k, D)$ has a positive volume and forms a self-similar tile. In our case, let $k = 3$ and consider the lattice Λ generated by the basis:

$$\vec{a} = (2,0,0), \quad \vec{b} = (0,2,0), \quad \vec{c} = (1,1,1).$$

This lattice contains all points in D. When we express each point $d \in D$ in the form $l\vec{a} + m\vec{b} + n\vec{c}$, the resulting coefficients (l, m, n) represent each of the 27 cosets of $\mathbb{Z}^3/3\mathbb{Z}^3$ exactly once. Therefore, we can conclude that the self-similar set $\mathcal{F}(3, D)$ is indeed a self-similar tile.

When these 27 points are viewed from the direction of a vertex in D_2, several points overlap, leaving only 19 distinct positions visible. These 19 points are arranged in a beautiful, exact six-fold symmetry. Although the entire solid is composed of 27 small copies of itself, only 19 of these copies are visible from this specific vantage point. Because these copies overlap along the line of sight, the resulting image is not a perfect six-fold symmetry in a literal sense; however, their underlying allocation follows an exact six-fold symmetric pattern. This explains why the object appears to possess nearly six-fold symmetry when viewed from a Type A bump.



5 How did I find this?

I believe it is perhaps more important to share the journey of how I discovered this object. My research is centered on "imaginary cubes"—objects whose projections along three mutually orthogonal directions are squares. In particular, I am fascinated by fractal imaginary cubes, such as the Sierpiński tetrahedron.

I am intrigued by the shadows of fractal imaginary cubes, as they occasionally form two-dimensional self-similar tiles other than squares. I have previously characterized the specific projection directions that generate such self-similar tiles as shadows for cases where the digit set possesses a layered structure [4].

A layered digit set is defined as:

$$D_{k,l}^{(3)} = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 0 \le x_1, x_2, x_3 < k, \sum_{i=1}^{3} x_i \equiv l \pmod{k}\}$$

for $k \ge 2$ and $0 \le l < k$. (Note that our definition differs slightly from that in [4].) The cardinality of this set is always k^2. For $k = 2$, both $\mathcal{F}(2, D_{2,0}^{(3)})$

and $\mathcal{F}(2, D^{(3)}_{2,1})$ result in the Sierpiński tetrahedron. Moving to $k = 3$, we obtain two distinct fractal imaginary cubes: $\mathcal{F}(3, D^{(3)}_{3,0})$ and $\mathcal{F}(3, D^{(3)}_{3,1})$ (since $\mathcal{F}(3, D^{(3)}_{3,2})$ is congruent to $\mathcal{F}(3, D^{(3)}_{3,1})$). Due to their elegant structures, I have named them the H-fractal and the T-fractal, respectively[1]. As detailed in [4], the T-fractal projects to a self-similar tile along a vector (a, b, c) if a, b, c are coprime integers whose sum is not divisible by 3.

These results can be generalized to four-dimensional space. For $k \geq 2$ and $0 \leq l < k$, we consider the fractals generated by the following digit set:

$$D^{(4)}_{k,l} = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid 0 \leq x_i < k, \sum_{i=1}^{4} x_i \equiv l \pmod{k}\}$$

which constitutes a fractal imaginary hypercube. In the case of $k = 2$, $\mathcal{F}(2, D^{(4)}_{2,0})$ and $\mathcal{F}(2, D^{(4)}_{2,1})$ are congruent and represent 16-cell fractals. For $k = 3$, since $\mathcal{F}(3, D^{(4)}_{3,1})$ and $\mathcal{F}(3, D^{(4)}_{3,2})$ are congruent, we obtain two distinct fractal imaginary hypercubes: $\mathcal{F}(3, D^{(4)}_{3,0})$ and $\mathcal{F}(3, D^{(4)}_{3,1})$. We refer to these as the 4D H-fractal and 4D T-fractal, respectively. It can be shown that both the 4D H-fractal and 4D T-fractal project onto a three-dimensional self-similar tile along a vector (a, b, c, d) if the components are coprime integers whose sum is not divisible by 3. Notably, the projected image possesses a positive volume when projected along the vector $(1, 1, 1, 1)$.

Given that the four-dimensional digit set is defined by such simple and elegant symmetries, and that the projection along $(1, 1, 1, 1)$ is highly symmetric, one might naturally expect the resulting 3D self-similar tile to possess high symmetry as well.

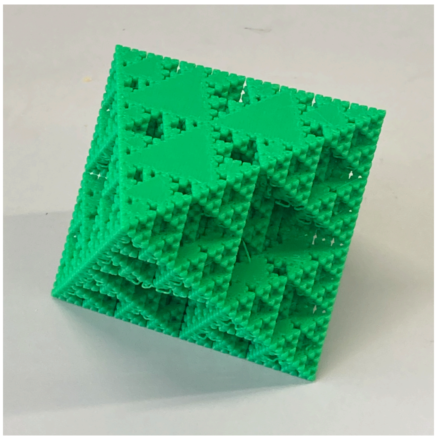
In fact, the Penta–Hex Tetrahedral Tile is precisely the image of the 4D T-fractal projected along $(1, 1, 1, 1)$. The 3D projection of the 4D H-fractal along $(1, 1, 1, 1)$ is equally striking, exhibiting cubic symmetry. Figure 4 shows 3D-printed models of this projected tile. We also include the projection of the 16-cell fractal along $(1, 1, 1, 0)$: its projection along $(1, 1, 1, 1)$ has zero volume, but the projection along $(1, 1, 1, 0)$ has positive volume. Finally, Figure 4(c) shows two copies of the latter object interlocked with one another.

References

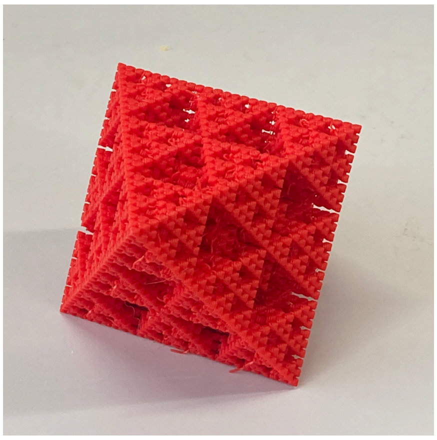
[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, 1988.

[2] C. Bandt, "Self-similar sets 5. Integer matrices and fractal tilings of $\mathbb{R}^n$," *Proc. Amer. Math. Soc.*, vol. 112, no. 2, pp. 549–562, 1991.

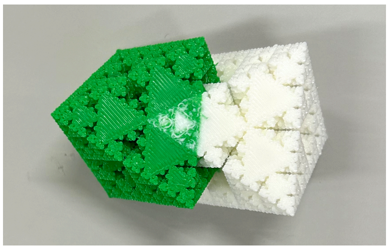
[1] These names are derived from the first letters of their base polyhedra: the Hexagonal bipyramid imaginary cube and the Triangular antiprismoid imaginary cube.



(a) Projected image of H-fractal



(b) Projected image of 16-cell fractal



(c) Two interlocked copies of (b).

Figure 4: Other projected tiles discussed in this section.

[3] J. C. Lagarias and Y. Wang, "Self-affine tiles in $\mathbb{R}^n$," *Adv. Math.*, vol. 121, no. 1, pp. 21–49, 1996. DOI: 10.1006/aima.1996.0045.

[4] H. Tsuiki, "Projected images of the Sierpinski tetrahedron and other layered fractal imaginary cubes," *J. Fractal Geom.*, vol. 12, no. 3/4, pp. 303–339, 2025. DOI: 10.4171/JFG/163.