SOME NEW EXAMPLES OF ANOMALOUS CANCELLATION

INNA ZAKHAREVICH
ISAAC ZAKHAREVICH-LAMB

1. Introduction

Anomalous cancellation is a fancy name for a silly trick: when a fraction can be simplified by "canceling digits." In base 10 there are four two-digit examples:

$$\frac{1\mathbf{6}}{\mathbf{6}4} = \frac{1}{4} \qquad \frac{2\mathbf{6}}{\mathbf{6}5} = \frac{2}{5} \qquad \frac{1\mathbf{9}}{\mathbf{9}5} = \frac{1}{5} \qquad \frac{4\mathbf{9}}{\mathbf{9}8} = \frac{4}{8}.$$

In the first two cases we "cancel the 6's, and in the second two we "cancel the 9's"; in general we will highlight the digits we are canceling by making them bold. Anomalous cancellation with two digits was investigated in 1979 by Boas [Boa72], and others such as [FG10] have taken an interest in it, with the most recent paper we could find being [SGDC23], from 2023. Boas considered two-digit anomalous cancellation in different bases, and Saha et. al. consider larger-digit classifications where the same pattern of cancellation (last digit of numerator and first digit of denominator are canceled) are considered. Boas proved that, in prime bases, no two-digit examples of anomalous cancellation exist. The papers [FG10, SGDC23] study "pumping" of anomalous cancellation, and find examples where the "canceled digits" can be repeated to "pump" known examples of anomalous cancellation up to higher digit counts. In particular, Saha et al show that if we restrict cancellation to the last digit of the numerator and the first digit of the denominator, then all examples of anomalous cancellation appear as "pumped" examples from smaller examples, and give an upper bound on some examples. But other types of anomalous cancellation exists, with [FG10] giving many examples, in varied bases, of anomalous cancellation of intermediate digits of numbers.

In this paper we give some examples of constructions of two-digit anomalous cancellation, as well as a couple of cute examples of anomalous cancellation in base 2.

2. Constructing two-digit examples of anomalous cancellation

Boas [Boa72] showed that anomalous cancellation does not exist in prime bases, and gave some examples of ways to construct examples of cancellation in composite bases. In particular, [Boa72, Theorem 2] showed that in a composite base there is an example of anomalous cancellation corresponding to every divisor of the base. Here we give a generalization of [Boa72, Theorem 2] that does not assume that $n-1$ is prime.

Theorem. *Suppose that $n = abc$ with $1 \leq a < b < n$. Suppose in addition that*

$$a \leq \gcd(b-a, a^2c-1). \tag{$*$}$$

Let $x = ka$ and $z = kb$ for

$$k = \frac{a^2c-1}{\gcd(b-a, a^2c-1)}$$

$$y = \frac{a(n-1)}{\gcd(b-a, a^2c-1)}.$$

Then there is an example of two-digit anomalous cancellation

$$\frac{x\mathbf{y}_n}{\mathbf{y}z_n} = \frac{x}{z} = \frac{a}{b}$$

Proof. Anomalous cancellation holds if and only if $(akn+y)b = (yn+bk)a$. After factoring and using that $n = abc$, this implies anomalous cancellation happens if and only if

$$y(a^2c-1) = ak(n-1).$$

Note that

$$\gcd(b-a, a^2c-1) = \gcd(n-1, a^2c-1).$$

Our formuilas for y and k satisfy this equation, so *if* x,y, and z are well-defined digits it gives an example of anomalous cancellation. We need to prove that x, y, and z are all less than n. Consider the inequality $y < n$; simplifying gives the equivalent inequality

$$(n-1) < bc\gcd(b-a, a^2c-1),$$

which holds if and only if

$$n \leq bc\gcd(b-a, a^2c-1).$$

This is true by assumption (*), so $y < n$. Using a similar approach shows that $x, z < n$, as well. $\square$

This is clearly not a complete classification of two-digit examples, since $\frac{16_{10}}{64_{10}} = \frac{1}{4}$ is not of this form. However, it is *symmetric* to a fraction in this form: a direct consequence of Boas's formulas is that whenever $\frac{xy_n}{yz_n}$ is an example of anomalous cancellation, so is $\frac{(y-z)y}{y(y-x)}$. The fraction *symmetric* to $\frac{16}{64}$ is $\frac{26}{65}$, which *is* a fraction in the given form.

Question 1. Do there exist examples of two-digit anomalous cancellation which are not in the form of Theorem 2 nor symmetric to it?

2.1. **Application: $a = 1$.** If $a = 1$ condition (*) always holds, and Boas's observation that there is an example of anomalous cancellation for every divisor of n follows. Thus every divisor d of n has an example of anomalous cancellation reducing to the fraction $\frac{1}{d}$. Note, however, that it is not necessarily possible to make it be *equal* to $\frac{1}{d}$, as the case $n = 10$ and $d = 2$ shows. (In that case, $k = 4$ and the example is $\frac{49}{98}$.)

In the special case when $n = m^2$ the decomposition $a = 1$, $b = m$ and $c = m$ this gives the case $x = 1$, $y = m + 1$ and $c = m$.

We assume in later applications that $a > 1$.

2.2. **Application: $n = pq$, p,q prime, $p < q$.** In this case there is one possibility: $a = p$, $b = q$, $c = 1$. Then condition (*) becomes

$$p \le \gcd(q - p, p^2 - 1).$$

This can hold only if $q > 2p$. It holds, for example, when $n = 10$ or 21. If $p = 2$ it holds only when $q \equiv 2 \pmod 3$.

2.3. **Application: $n = p^m$, p prime, $m \ge 3$.** Suppose that $a = p^\alpha$, $b = p^{\alpha+\beta}$ and $c = p^{m-2\alpha-\beta}$. Then condition (*) becomes

$$\begin{aligned} p^\alpha &\le \gcd(p^\alpha(p^\beta - 1), p^{m-\beta} - 1) \\ &= \gcd(p^\alpha - 1, p^{m-\beta} - 1) \\ &= p^{\gcd(\beta, m-\beta)} - 1. \end{aligned}$$

Thus condition (*) holds if

$$\alpha < \gcd(\beta, m).$$

In particular, for (*) to hold we must have $\beta > 1$, so the theorem has no applications in the case $n = p^3$. For $n = p^4$, however, an example exists with $x = p$, $z = p^3$ and $y = p(p^2 + 1)$.

3. Base 2 examples

We give some examples of cancellation in base 2, as we felt that base 2 was underutilized for exploration of anomalous cancellation. Boas's proof that two-digit anomalous cancellation is not possible in prime bases seems to have made fewer people consider it, and we wanted to draw attention to some cute examples.

Firstly, any example in base 4 (or any power of 2) obviously leads to an example in base 2, simply by expanding the digits. Thus, for example,

$$\frac{1\mathbf{3}_4}{\mathbf{3}2_4} = \frac{1\mathbf{11}_2}{\mathbf{11}10_2} = \frac{1_2}{10_2}.$$

But this is not terribly interesting, so we focus on examples which are fundamentally base-2 examples, and would not work in other bases.

Suppose that we make the rule that only consecutive digits are allowed to be canceled. Then there is an example given by

$$\frac{\mathbf{10}1_2}{1\mathbf{10}01_2} = \frac{1_2}{101_2}.$$

We could also make a different rule that only 0's are allowed to be canceled. Then there is the example:

$$\frac{1\mathbf{00}1_2}{1\mathbf{0}1\mathbf{0}1_2} = \frac{11_2}{111_2}.$$

In fact, this example has the nice property that we cancel *all* zeroes on top and bottom to produce the example. All of the two-digit examples that we have are also of this form.

Question 2. What other examples are there of "cancelling all a's"?

References

[Boa72] R. P. Boas. Anomalous cancellation. *The Two-Year College Mathematics Journal*, 3(2):21–24, 1972.

[FG10] Michael N. Fried and Mayer Goldberg. A pumping lemma for invalid reductions of fractions. *College Math. J.*, 41(5):357–364, 2010.

[SGDC23] Satvik Saha, Sohom Gupta, Sayan Dutta, and Sourin Chatterjee. Characterising solutions of anomalous cancellation, 2023.