Borders
a variant of Dots & Boxes

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introduction In this article we propose a variant of the well-known game Dots & Boxes. We investigate two simple positions of the game, and conjecture a criterion for determining which player has the win (in terms of the dimensions of the rectangular board used).

description The game is played on an $a \times b$ grid (the squares being 1×1). A move consists of building a fence along one side of a 1×1 square. If the fence built completes an enclosure of one or more squares, all squares within the enclosure are claimed by that player, except for any squares already claimed by the other player.

differences from Dots & Boxes The game Borders is different from Dots & Boxes in two ways:

- The building of a single fence segment may result in the player claiming multiple squares.
- Claiming a square does not permit the player to build another fence segment.

acknowledgement In an effort to see if this game had already been described, the author found an online reference to a variant of Dots & Boxes played in Poland where multiple squares can be claimed at once, but was unable to find a more complete description of that game. It is possible that it is the same game proposed here, but it need not be, as the Polish variant may retain the rule, of Dots & Boxes, that claiming squares permits the player to build another fence segment. That is, the Polish variant may exhibit the first difference from Dots & Boxes described above, but not the second.

"k-canal" We say that one square is "accessible" from another if they are adjacent, and no fence segment has been built between them. By "k-canal", we mean a sequence of k squares bordered each by exactly two fence segments, each square accessible from the previous one, and the first and last squares of the k-sequence not accessible from any square outside the k-sequence having more than one fence segment bordering it. (Note the difference between the "canal" in Borders and what is termed a "chain" in Dots & Boxes.)

two simple positions

- A $2k$-canal has a value of 0 to the player who plays on it first. The optimal move is to split the canal in half, creating a disjunctive sum of two identical positions, which has value 0.

  If the first player had made any different move on the $2k$-canal, the second player could have fenced off an even number of squares in the middle of the canal, leaving a disjunctive sum of two identical positions (at the two ends of the canal). The first player would then have lost a number of points equal to the number of squares fenced off in the middle of the canal.

- A $(2k + 1)$-canal has a value of -1 to the player who plays on it first. The optimal move is to split the canal in two parts, one of length k and the other of length $k + 1$.

  The second player's best move is then to complete the fourth fence segment bordering the middle square in the chain, gaining him one point, and leaving a disjunctive sum of two identical positions. No other move by the second player would do better than this (for the second player).

  If the first player had made any different move on the $(2k + 1)$-canal, the second player could have fenced off an odd number of squares (more than one) in the middle of the canal, leaving a disjunctive sum of two identical positions (at the two ends of the canal). The first player would then have lost a number of points (more than one) equal to the number of squares fenced off in the middle of the canal.

conjecture Suppose the game of Borders is played on an $a \times b$ board. The game is a win for the first player if $a + b$ is odd, and for the second player if $a + b$ is even.