Some New Combinatorial Games
(From the Past Ten Years)

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The past ten years have seen exciting theoretical developments in combinatorial game theory. Alongside these advances there has appeared a crop of new and fascinating examples—games that exhibit a rich, varied, and often amusing and bewildering structure. Some of these games are new variations on well-known themes; others test and expand the boundaries and capabilities of the theory.

This note surveys several of the more interesting recent inventions. The first two games in our survey, MEM and MNEM, are impartial games with extremely simple rules, and a structure that bifurcates in a surprising way into regions of order and chaos. The next, TOPPLING DOMINOES, is a partizan game in the classical sense, but one with an unusually clear structure. Finally, ENTREPRENEURIAL CHESS is an unusual CHESS variant, played on an infinitely large board, with repetition allowed. It therefore violates most of the "classical" restrictions on combinatorial games, but it nonetheless has a robust and coherent theory.

None of these games were invented by me. MNEM was introduced by Conway; TOPPLING DOMINOES by Richard Nowakowski; and ENTREPRENEURIAL CHESS by Berlekamp and Pearson. Some familiarity with combinatorial game theory is assumed; the discussion of MEM and MNEM uses only the impartial theory, while the others use the partizan theory and notation as described in *Winning Ways* [1].

Taken together, these games illustrate the boundless possibilities of combinatorial games. They inhabit a miraculous universe, in which elegant and mysterious mathematics springs from a few simple rules.

MEM and MNEM

MEM and MNEM are deceptively simple impartial games. They were introduced by Conway a few years ago and, despite their straightforward rules and "obvious" structure, it appears difficult to prove anything about them!

MEM is played with a heap of tokens. On her turn, a player must remove k tokens from the heap, provided that k is *at least as large* as the number of tokens removed on the prior turn. If played with multiple heaps, then each heap has its own "memory."

We can represent a heap of size n, with memory k, by a pair of integers n_k. Then the legal moves are given by

$$n_k = \big\{(n-i)_i : k \le i \le n\big\}.$$

Obviously n_k is a $\mathscr{P}$-position if $k > n$ (since then there are no legal moves), and an $\mathscr{N}$-position otherwise (since there is a move to 0_n).

Mnem is just the same, except that a player has the additional option of *adding* $< k$ new tokens to the heap (but at least one), instead of removing $\ge k$. When a player exercises this option, the value in "memory" decreases (which can't happen in Mem). Denoting a Mnem position by n_k^*, we have

$$n_k^* = \big\{(n-i)_i^* : k \le i \le n\big\} \cup \big\{(n+i)_i^* : 1 \le i < k\big\}.$$

A game of Mnem needn't ever end. In fact it's easy to construct sequences of moves that traverse an infinite number of distinct positions. For example:

$$4_1^* \to 0_4^* \to 3_3^* \to 5_2^* \to 6_1^* \to 0_6^* \to 5_5^* \to 9_4^* \to 12_3^* \to 14_2^* \to 15_1^* \to 0_{15}^* \to \cdots$$

Here we remove 4 tokens, then add 3, 2 and 1 tokens, leaving 6; then remove 6 tokens, and add 5, 4, 3, 2 and 1 tokens, leaving 15; then remove 15 tokens, and so on ...

But remarkably, both players have to cooperate for this to happen! For example, every 0_k^* is a $\mathscr{P}$-position: the only legal moves are to positions of the form i_i^* with $i < k$, which second player can immediately revert to 0_i^*. If second player sticks to this strategy, then eventually the position 0_1^* will be reached, which is terminal.

But from n_k^* with $k > n$, the only legal moves are to positions $(n+i)_i^*$ with $i < k$, which second player can revert to 0_{n+i}^*. So every such n_k^* is a $\mathscr{P}$-position, just as in Mem. And, likewise, every n_k^* with $k \le n$ is an $\mathscr{N}$-position, since it has a move to 0_n^*.

Beyond this, it's surprisingly hard to say anything at all about the $\mathscr{G}$-values of either variant. We don't even know how to play 2-pile Mem! Yet the experimental evidence suggests an extraordinary amount of structure.

Figure 1 shows an *intensity plot* of $\mathscr{G}(n_k)$, for all $1 \le n \le 96$ and $1 \le k \le 32$. It's a 32×96 grid of boxes, colored in grayscale; darker shades indicate lower $\mathscr{G}$-values. The black triangle in the lower-left is the space of $\mathscr{P}$-positions that we noted above.

What's striking are the thin triangular bands that extend in quadratic fashion upwards from the triangle of $\mathscr{P}$-positions. Just above the $\mathscr{P}$-region lies a region of positions with $\mathscr{G}$-value exactly 1; then a thinner region of positions with $\mathscr{G}$-value 2; and so on. Amazingly, all of these regions satisfy the following conjecture.

Conjecture 1. *If $k^2 \ge n$, then $\mathscr{G}(n_k) = \lfloor n/k \rfloor$.*

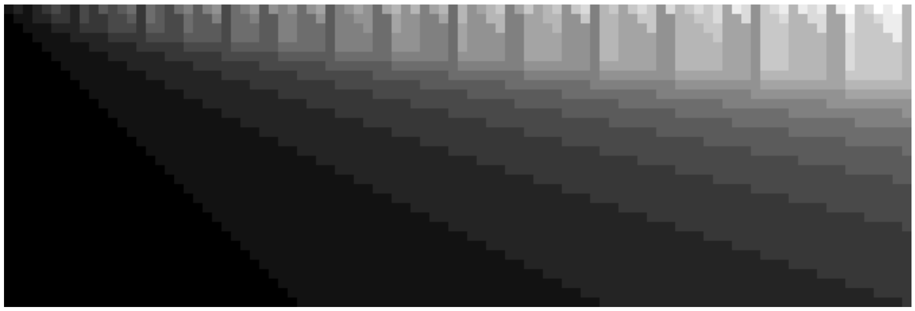


Figure 1: Intensity plot of the $\mathscr{G}$-values of Mem. The value of n_k is plotted at row k, column n, for $1 \leq n \leq 96$ and $1 \leq k \leq 32$. Black = 0, White = 14.

Within the region $k^2 < n$, the $\mathscr{G}$-values appear to have a complex structure, consisting of many interlocking, fractal-like triangles. The interested reader will enjoy computing a larger table of values and observing their striking regularity.

The geometric structure of Mnem seems very similar to Mem, and Conjecture 1 appears true for Mnem as well. However, the fine structure of the $k^2 < n$ region differs.

Here's an indication of how little is understood about these games: we can't even prove the following conjecture, which has been verified computationally for $n, k \leq 10,000$.

Conjecture 2. *Every* Mnem *position has finite $\mathscr{G}$-value.*

Toppling Dominoes

We turn now to partizan games. Toppling Dominoes, invented by Richard Nowakowski [3], is played with rows of black and white dominoes such as shown in Figure 2.

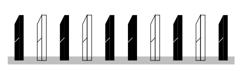
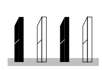


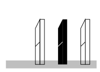
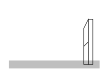
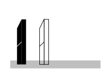
Figure 2: A typical Toppling Dominoes position.

On her turn, Left may select any black domino and "topple" it East or West (her choice). The selected domino is removed along with *all* dominoes from that row in the chosen direction.

For example, from the position



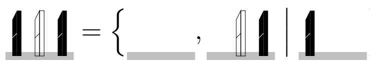
Left can move to

 or or  or 

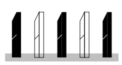
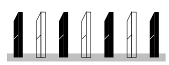
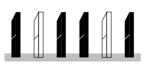
Although TOPPLING DOMINOES is a straightforward, "classical" combinatorial game, it exhibits a remarkable amount of structure. It's easy to see that every *monochromatic* row of dominoes is equal to an integer; for example

 $= 5$

since Left will prefer to topple the black dominoes one at a time. More complicated numbers can be constructed by mixing dominoes of both colors, say

 $= \{0, * \mid 1\} = \frac{1}{2}.$

Here we've omitted options that are duplicates under the obvious east-west symmetry, and used the fact that Left's move to $*$ is reversible. One can similarly show that

 $= \frac{1}{4}$ and  $= \frac{1}{8}$ and  $= \frac{3}{4}$.

(Try it!) In fact *every* dyadic rational number x (whose demoniator is a power of 2) can be expressed as a TOPPLING DOMINOES position, using a simple procedure. Write

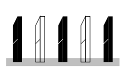
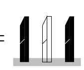
$$x = \frac{m}{2^n}$$

in lowest terms (i.e., with m odd), and let

$$y = \frac{m-1}{2^n} \quad \text{and} \quad z = \frac{m+1}{2^n}.$$

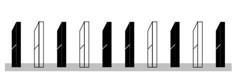
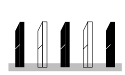
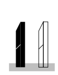
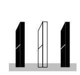
(In the notation of *Winning Ways*, $x = \{y \mid z\}$ in simplest form.) For example, if $x = \frac{3}{8}$, then $y = x - \frac{1}{8} = \frac{1}{4}$ and $z = x + \frac{1}{8} = \frac{1}{2}$.

Then y and z have smaller denominators than x, so we can recursively construct their domino sequences, say Y and Z. In the example $x = \frac{3}{8}$, we have

$Y =$  and $Z =$ 

as noted above.

Now for the trick! Write the sequences for Y and Z side-by-side, insert a black-white pair of dominoes in between them, and amalgamate the whole enlarged sequence into a single position. This gives a new position X, which miraculously has value x! For example,

 =  &  & 

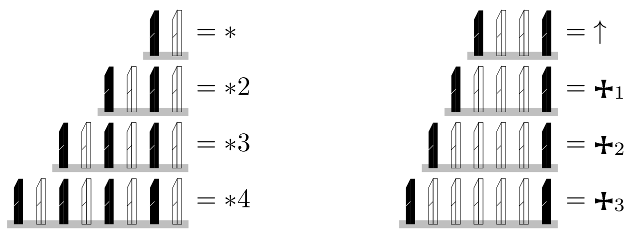
$\frac{3}{8}$ $\qquad$ $\frac{1}{4}$ $\qquad$ $\frac{1}{2}$

showing that the example in Figure 2 has value $\frac{3}{8}$.

What makes this construction truly remarkable is that it's the *only* way to represent numbers in TOPPLING DOMINOES. So every number can be represented *uniquely* as a single row of dominoes! This remarkable theorem is due to

Alex Fink, and it has an equally remarkable proof, the details of which can be found in [3].

Various other familiar values also arise naturally in Toppling Dominoes; here's a sampling:



Entrepreneurial Chess

Here's an old Chess problem, first posed by Simon Norton, and later publicized by Guy:

> With initial position WKa1, WRb2, and BKc3 . . . what is the smallest board (if any) that White can win on if Black is given a win if he walks off the North or East edges of the board? [4]

Berlekamp and Pearson invented Entrepreneurial Chess partly in response to this problem [2]. It is played on a quarter-infinite board, such as shown in Figure 3. Right (White) has a king and rook; Left (Black) has just a king; and the pieces move just as in ordinary Chess. However, instead of moving her king, Left may instead choose to *cash out*. If Left cashes out, then the entire position is replaced by an integer n, equal to the sum of the row and column numbers for her king. For example, if Left chose to cash out in Figure 3(a), the position would be replaced by the integer 7.

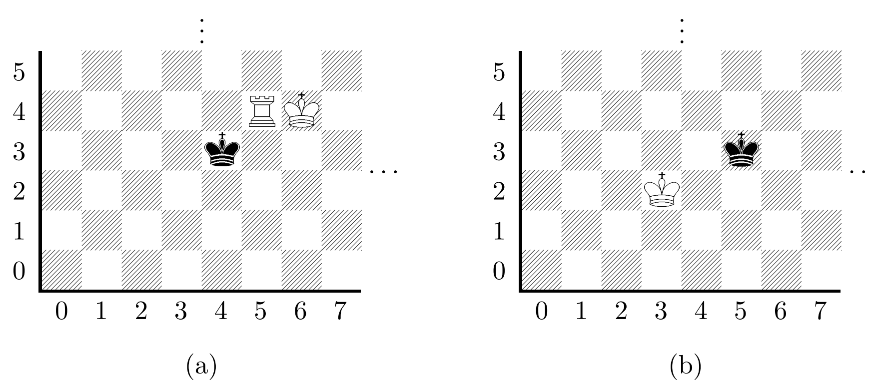


Figure 3: Entrepreneurial Chess. (a) A typical position; (b) A pathological position in which the rook has been captured.

Entrepreneurial Chess is loopy. From Figure 3(a), Left can do no better than to cash out, while Right is constrained to shuttle his king between the squares bordering his rook. Writing G for this position, we therefore have

$$G = \{7 \mid G\}$$

which according to the theory of loopy games [1, Chapter 11], has value $7 + \mathbf{over}$.

Moreover, Entrepreneurial Chess exhibits some explicitly transfinite values. Consider the position in Figure 3(b), in which Right's rook has been captured, and Left is free to run indefinitely far in the Northeast direction. Writing H for the value of this position, it's clear that

$$H > n$$

for every integer n, and its value therefore exceeds every *finite* game. However it's confused with certain *transfinite* games, such as the infinite ordinal ω:

$$\omega = \{0, 1, 2, 3, \ldots \mid\}$$

On the sum $H - \omega$, Left will prefer never to cash out and play will continue forever, so the outcome is a draw.

It can be shown that if J is any *finite* game, and $J > n$ for every integer n, then $J > \omega$ as well. Since $H > n$ for every n, but H is confused with ω, it follows that H is explicitly transfinite. In fact one can show that, in terms of the *Winning Ways* theory, H has the remarkable value

$$H = \mathbf{on} \ \& \ \{0, 1, 2, 3, \ldots \mid H\}.$$

Berlekamp and Pearson have undertaken a detailed temperature analysis of Entrepreneurial Chess positions. They also solved Norton's original problem: the answer is 8×11.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays*. A. K. Peters, Ltd., Natick, MA, second edition, 2001.

[2] E. R. Berlekamp and M. Pearson. Entrepreneurial chess. Unpublished manuscript, 2003.

[3] A. Fink, R. J. Nowakowski, A. N. Siegel, and D. Wolfe. Toppling conjectures. To appear in *Games of No Chance 4*.

[4] R. K. Guy. Unsolved problems in combinatorial games. In R. K. Guy, editor, *Combinatorial Games*, number 43 in Proceedings of Symposia in Applied Mathematics. American Mathematical Society, Providence, RI, 1991.