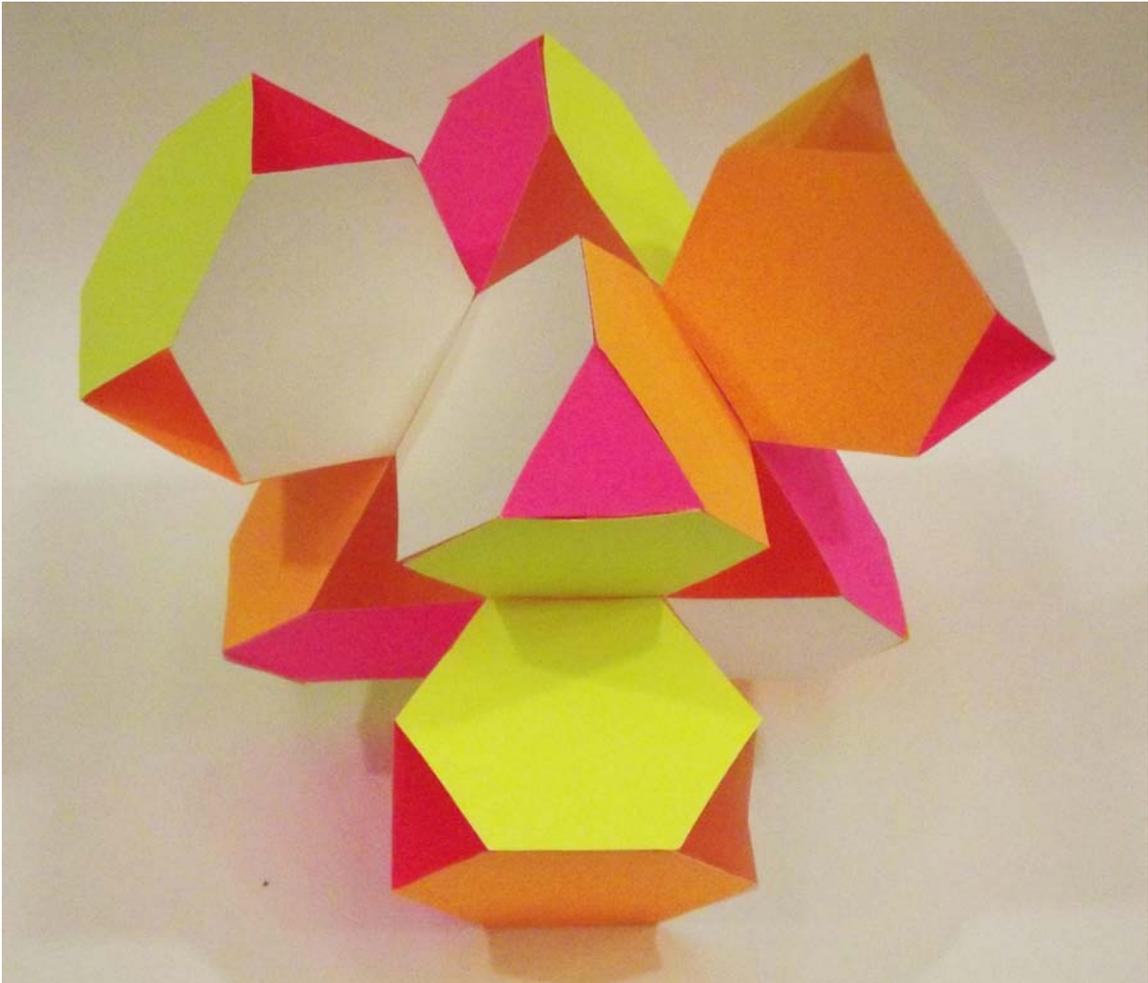


A Ten-Cell Ornament

G4GX Gift from Thomas Banchoff



Martin Gardner loved to play with ideas, especially ideas that could be visualized using real objects looked at in many different ways. For this tenth Gathering for Gardner, I have been playing around for several weeks with an ornament made from ten congruent three-dimensional objects, all simple to design and relatively simple to put together. Well, not really that simple since actually putting them together would require

going into four-dimensional space, but, making some compromises, we can get a very good image of the total object by unfolding it into the third dimension. Any unfolding expresses a break in the actual symmetry of the object, and studying the different ways that the overall symmetry is broken gives an appreciation for the symmetry as a whole. Decorating the object in different ways to reflect the various symmetries expresses the richness of its geometry, in particular the geometry connected with the celebrated Hopf mapping from the three-dimensional sphere to the two-dimensional sphere.



Figure 1: Unfolding of the ornament into three-space, top view

So, what is the object? Basically it is a central slice of a five-dimensional cube perpendicular to a long diagonal, giving a four-dimensional convex body with a boundary three-dimensional sphere made up of ten truncated tetrahedra. It is a maximally symmetric configuration of ten cells that can be decomposed in various ways to exhibit its fundamental structure. As usual, the way to approach such an object is by building it up using the Dimensional Analogy.

Slicing a square perpendicular to a diagonal produces first a vertex, then a point, then a small segment that grows to the perpendicular diagonal with two more of the vertices, then small segments receding to the fourth vertex. The sequence of vertices in slicing lines is 1-2-1 giving the total of 4.

Slicing the cube perpendicular to a long diagonal gives a familiar sequence, first a vertex, then an equilateral triangle growing to a maximal size when it hits three more vertices whereupon it experiences truncation to form hexagons with all angles equal and opposite sides parallel and congruent. The middle slice will be a regular hexagon with edges half the length of a diagonal of a square. The rest of the slicing sequence reverses the process, first hitting three more vertices and then the final vertex. The vertex sequence is 1-3-3-1. Note that the central slice does not contain any vertices of the original cube.

In “The Hypercube: Projections and Slicing” featured by Martin Gardner in one of his Scientific American columns, the final movement in this 1978 computer animation made with computer scientist Charles Strauss gives the central slices of a four-dimensional cube. The sequence begins with a vertex that then grows to a small regular tetrahedron reaching its maximal size when it hits four vertices. It then experiences truncation to form a truncated tetrahedron with four triangular faces and four hexagonal faces, each face being a slice of one of the eight cubes in the boundary of the hypercube. The slice three-eighths of the way through will have four equilateral triangles and four regular hexagons, all with 18 congruent edges and 12 vertices, none of which is a vertex of the original hypercube. The central slice halfway through the hypercube is a regular octahedron with six vertices all of which are vertices of the original hypercube. The sequence reverses the form of the slices, giving the partition of the 16 vertices of the hypercube 1-4-6-4-1.

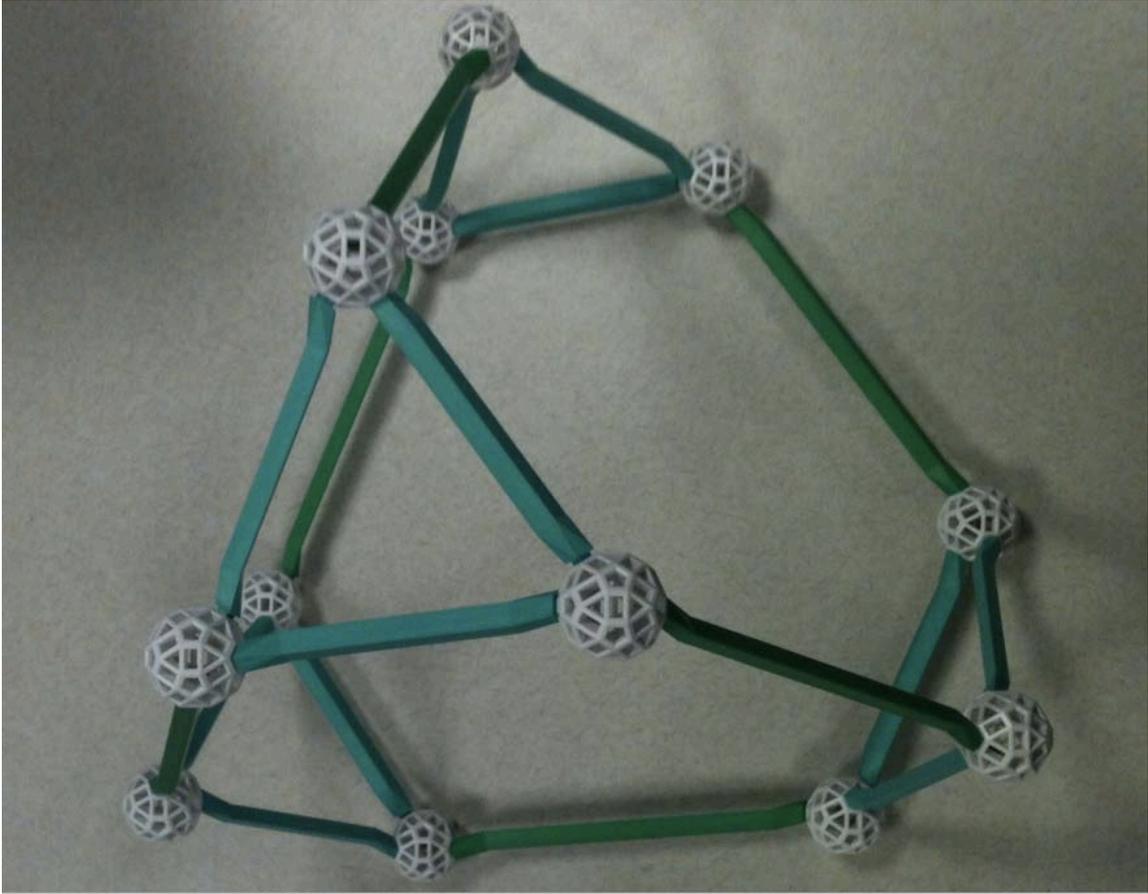


Figure 2: Constructing the truncated tetrahedron with Zometools presents its own challenges.

Now we get to the slices of the 5-cube perpendicular to one of its longest diagonals. We start with a vertex, which then grows into a regular 4-simplex with five vertices and five tetrahedral faces. The maximal such slice will contain five vertices of the original 5-cube. The next slices are truncated 5-cells with five growing tetrahedra and five truncated tetrahedra shrinking down to octahedra in the slice that contains ten of the original 32 vertices of the 5-cube. This highly symmetrical slice two-fifths of the way through the 5-

cube has five regular octahedra and five regular tetrahedra. The five tetrahedra experience truncation and the five octahedra continue to form truncated tetrahedra, producing, in the central slice, an object with ten congruent 3-cells, each one a semi-regular Archimedean polyhedron with four regular triangles and four regular hexagons. This is the ten-cell ornament.

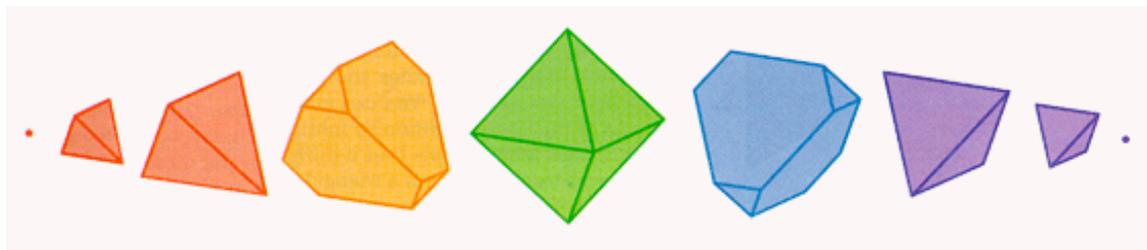


Figure 3: Slices of the Hypercube (from “Beyond the Third Dimension”)

Note that none of the vertices of this object is a vertex of the original 5-cube. As always, the slicing sequence continues in the reverse order. The partition of the 32 vertices encountered in the slicing sequence is the predictable arrangement of binomial coefficients 1-5-10-10-5-1.

So now we have an object we can play with. For a start, we can make a number of truncated tetrahedra out of paper, each one of which is obtained by taking a strip of four regular hexagons in a zig-zag pattern and making the appropriate identifications, similar to the way that a regular tetrahedron can be described as a strip of four

regular triangles with identified sides. These truncated tetrahedra are missing their four triangular faces, but that is all right since we can attach them together along these triangles to form surfaces that divide the full object into 3-dimensional pieces. We will call such a truncated tetrahedron a “basic cell”.

The first decomposition of the ten-cell ornament begins with a white basic cell in the middle and four other white basic cells, one on each of the triangles of the middle cell. This gives five basic cells in a white “tetrad” arrangement. Similarly we can form a tan tetrad arrangement of five tan basic cells. These two tetrads fit together partially by putting one white hexagon of the white middle basic cell onto a tan hexagon in the tan middle cell. Note that it takes a bit of flexing of the paper in order to accomplish this with two actual tetrads.

The resulting figure is an unfolding of the ten-cell ornament into three-dimensional space. It can be viewed as a column of four basic cells, with two white cells attached triangle to triangle and two tan cells similarly attached. A white hexagon from one of these units attaches to a tan hexagon of the other with a “twist” so that each edge of the resulting figure is adjacent to one triangle, alternately a white triangle or a tan one. The remaining six basic cells form a ring of six cells, three white and three tan.

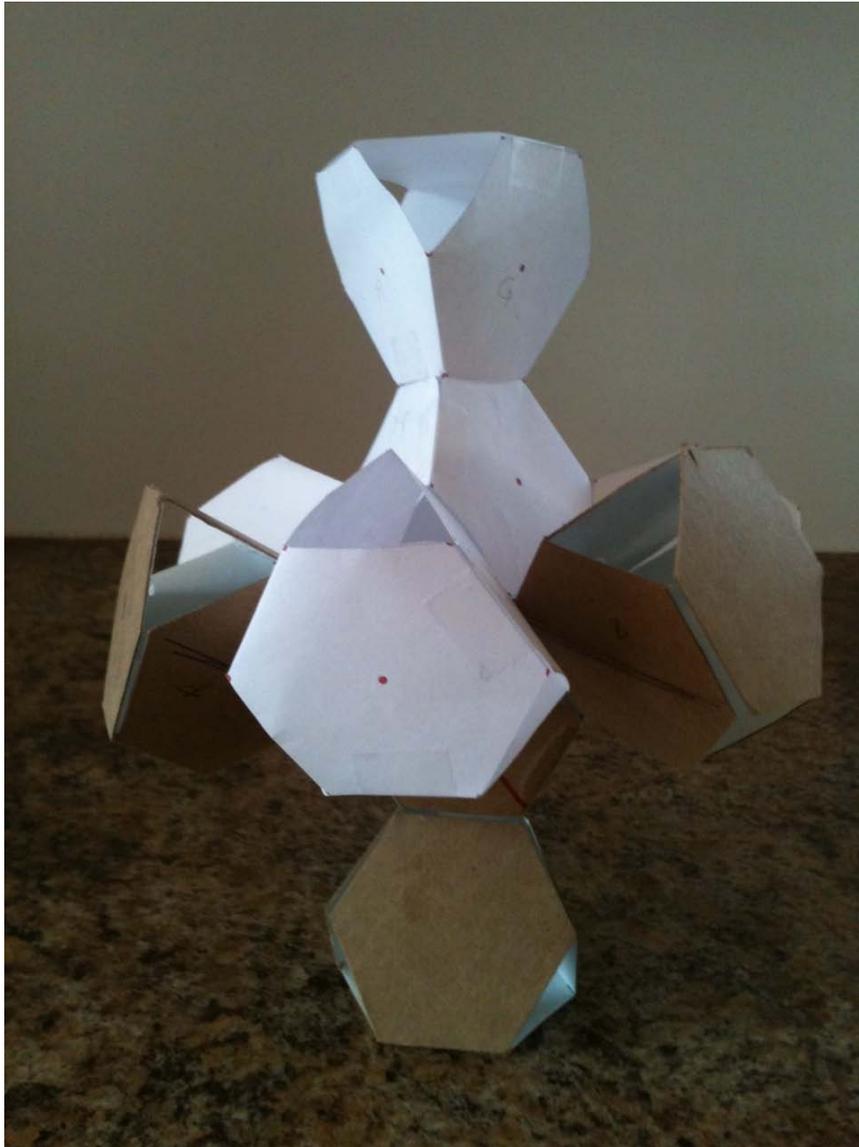


Figure 4: Unfolding the ornament as two tetrads

This last configuration is especially important since it expresses the 3-sphere boundary of a convex 4-dimensional ball as a union of two solid tori joined along a torus.

The comparable decomposition of the boundary of a 4-cube consists of a column of four 3-cubes with the top square identified with the bottom, forming one of the solid tori. This figure is surrounded by a ring of the other four 3-cubes to produce the “hypercubical cross” made famous in the 1954 painting “Corpus Hypercubicus” by Salvador Dali featured in the 1962 Martin Gardner column “The Church of the Fourth Dimension”.

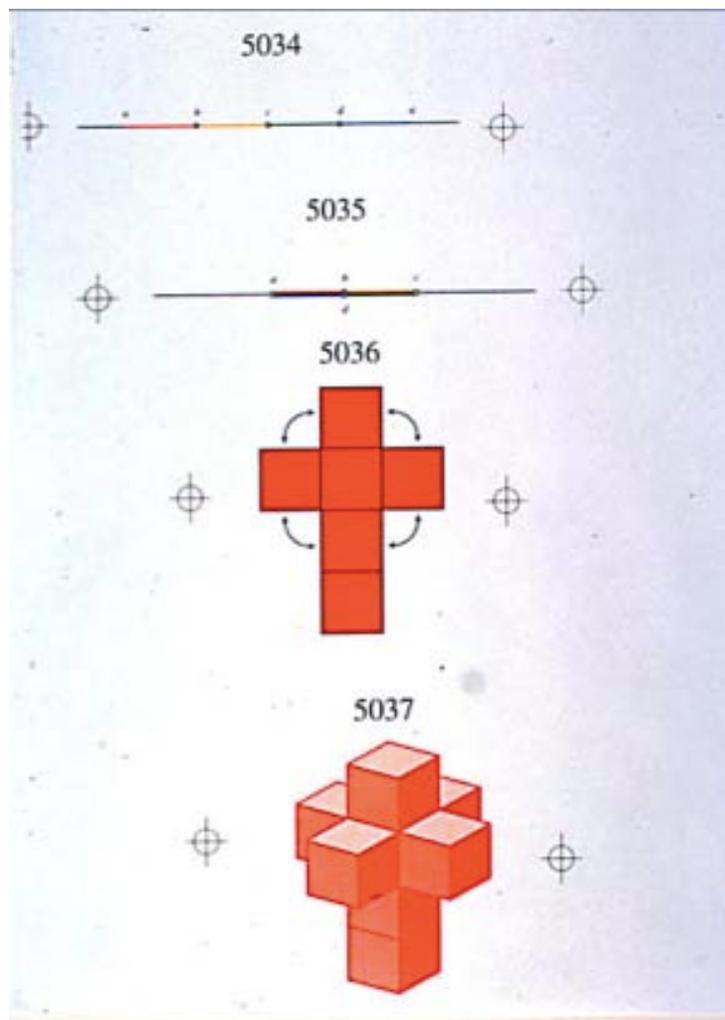


Figure 5: The unfolded hypercube, from “Surfaces Beyond the Third Dimension”

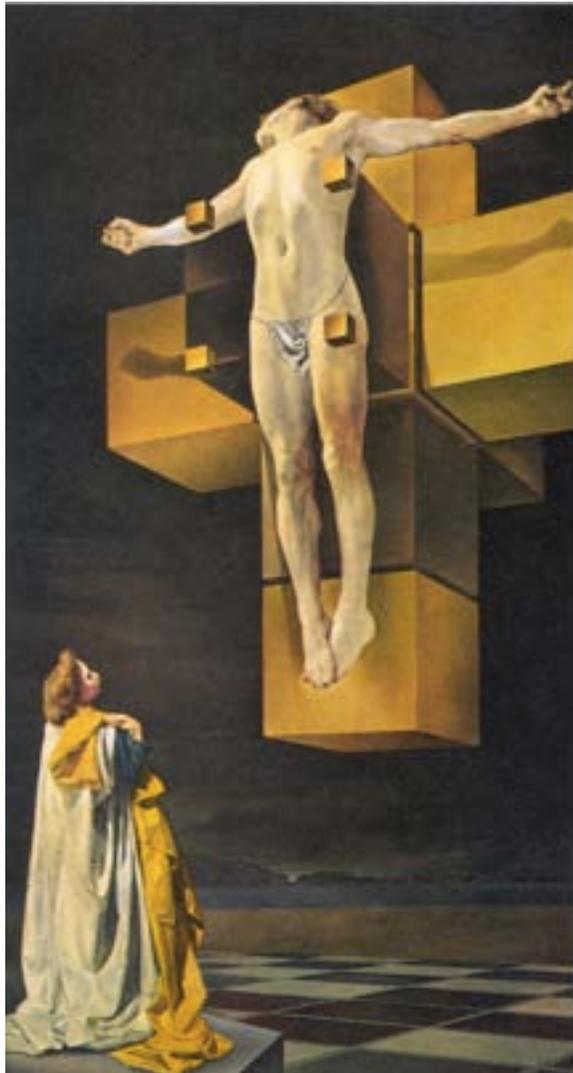


Figure 6: Corpus Hypercubicus (1954) by Salvador Dali, described by the author in the gift for G4G8

Note that if we remove one cell from the ten-cell ornament, we have a symmetric configuration of a central basic cell with eight basic cells attached. Folding this together in four-dimensional space will give an object with a boundary consisting of the boundary of the tenth basic cell. For such a basic cell

boundary, there are eight faces, four triangles and four hexagons, with one triangle opposite each hexagon. We can “decorate” the symmetric object using four different colors so that the triangle and its opposite hexagon have the same color. This makes it clear that the ten-cell ornament can be thought of as a central column of four cells surrounded by a ring of six other cells in four different ways.

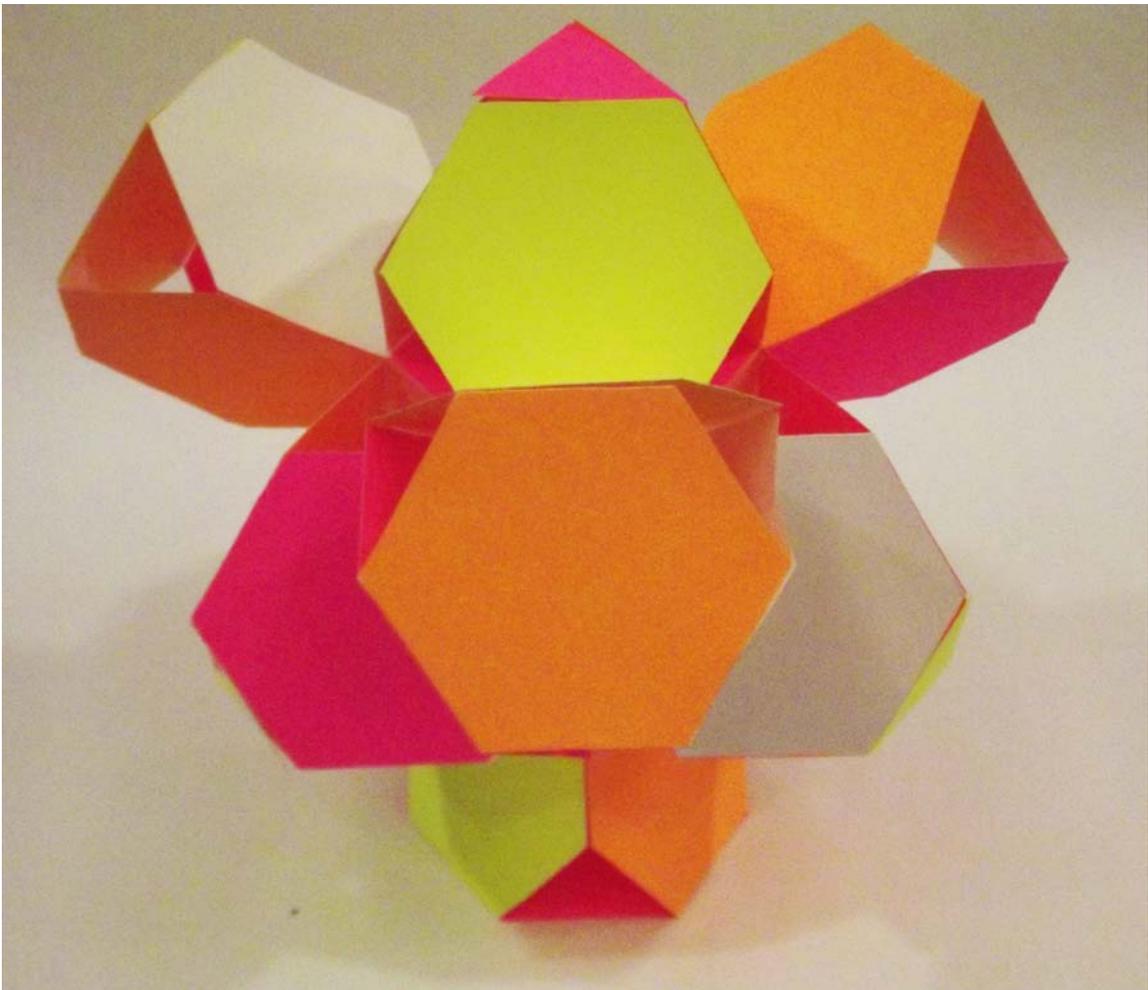


Figure 7: Color-coding the unfolded ornament to exhibit its ring structure

It is possible to draw diagonals on the hexagons of the column in such a way that they form disjoint closed polygons, each of which is a one-one curve on the torus boundary of the column. This relates the decorated structure of the ten-cell ornament to the toroidal decomposition of the regular polytopes in four dimensions, as described in the article by the author in the volume "Shaping Space".

As of the evening of March 23, I have established the property that I claimed in earlier documents, namely that the structure consisting of ten identical truncated tetrahedra can be decomposed in four different ways in to a column of four vectors and a ring of the six remaining tetrahedra to form the complementary torus.

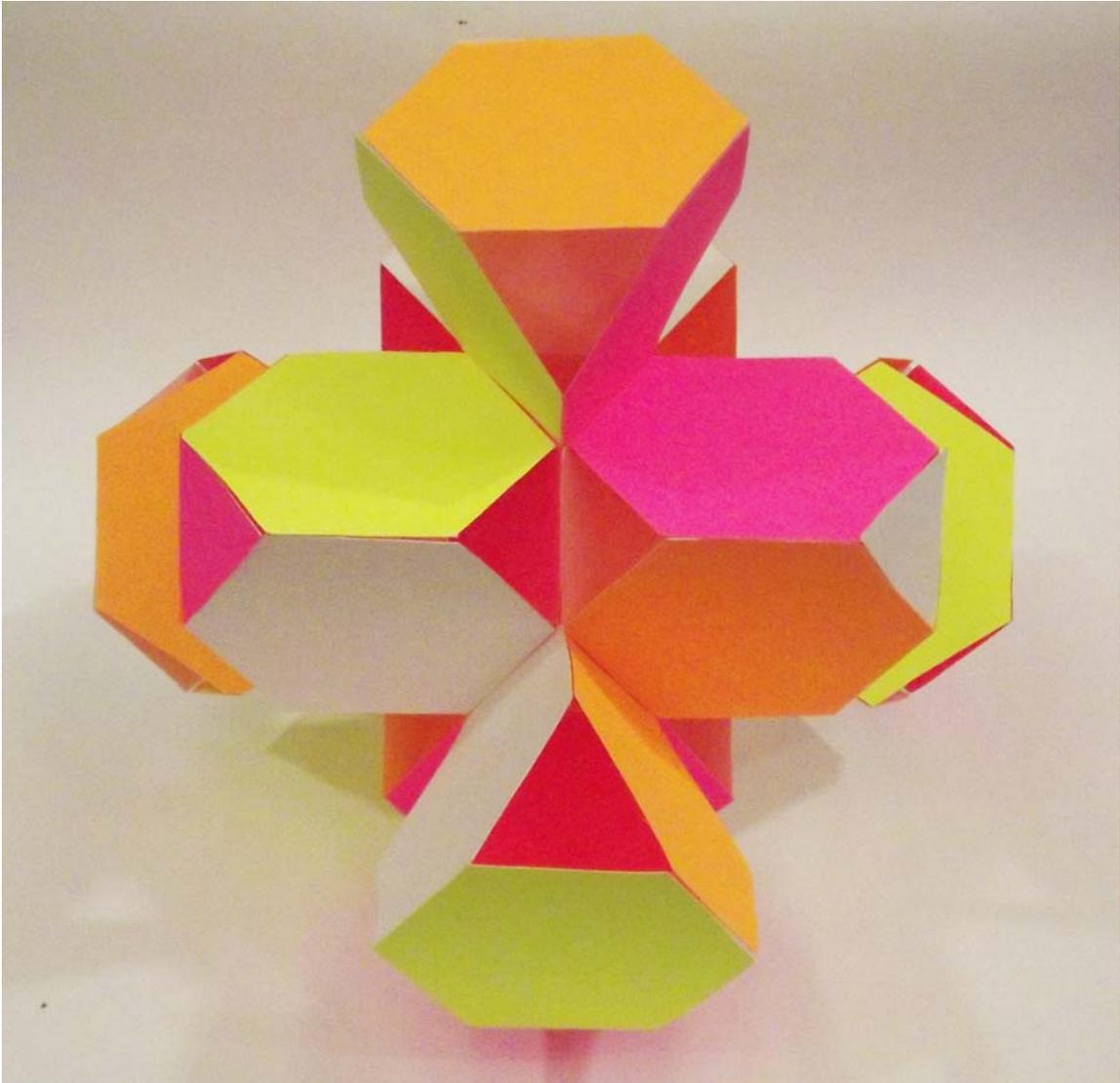


Figure 8: Ring Structure via Color-Coding

This structure, as mentioned earlier, is connected with the Hopf mapping, from the three-dimensional sphere to the two-dimensional sphere. The most direct way of seeing this is to consider the three-sphere as the collection of pairs of complex numbers (z,w) such that $|z|^2 + |w|^2 = 1 = zz^* + ww^*$ where $w^* = (u + iv)^* = u - iv$ indicates the operation of conjugation. We then map

the pair (z,w) to $h(z,w) = (|w|^2 - |z|^2, 2zw^*)$, where the first component is a real number and the second is a complex number, in \mathbb{R}^3 . It is straightforward to show that the image of the mapping is the two-sphere in three-space. Note that the preimage of the point $(1,0,0)$ is the circle where $z = 0$ and $|w| = 1$, and the preimage of $(-1,0,0)$ is the circle where $|z| = 1$ and $w = 0$. The equator on the two-sphere has as its preimage the “flat torus” in the three-sphere, given by the collection of points with $|z|^2 = |w|^2 = \frac{1}{2}$. This torus separates the three-sphere into two solid tori, each a preimage of one of the hemispheres of the two-sphere. The preimage of any single point on the two-sphere is a circle on the three-sphere, and any two such circles link, in the sense that they bound discs in four-space intersecting at a single point.

As indicated in the article in “Shaping Space”, it is sometimes difficult to find a parametrization of a three-dimensional sphere that relates specifically to the Hopf mapping. In particular if the three-sphere is given as a simplicial polyhedron, we can ask if it is possible to situate the vertices so that the preimages of point on the two-sphere appear as polygons in some natural way. So far the fourfold decomposition of the three-sphere into ten congruent truncated tetrahedra produces ten color-coded cells indicating the spiraling behavior of preimages of vertices.

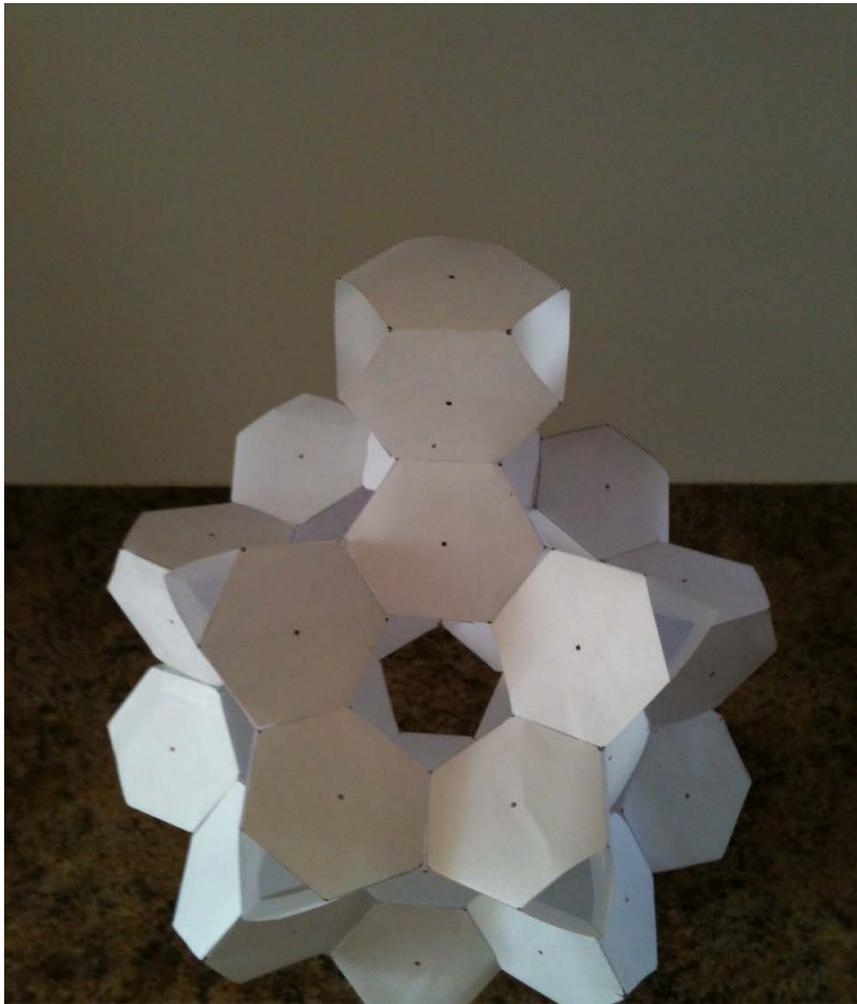


Figure 9: A compound of truncated tetrahedra and truncated dodecahedra formed by a union of tetrads

Martin Gardner loved to contemplate fascinating objects, in particular objects from higher dimensions. This is one object that especially fits that description, on the occasion of G4GX.