Backwards addition

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Sum-rich circular permutations

We will treat circular permutations as a series of numbers where we go "around-the-corner" as we look at consecutive terms. For example 31254 is a permutation of the numbers 1 through 5 and the four consecutive terms starting at 5 are 5431.

Colm Mulcahy [2, June 2008] looked at the circular permutation 85417632 from g4g8 which arranges the first eight numbers in alphabetical order but also has the following curious property: If the sum of any three consecutive terms is given, the three consecutive terms can be recovered. This is because the eight different possible sums of three consecutive terms are distinct, namely we have the following:

$17 = 8+5+4$	$16 = 7+6+3$
$10 = 5+4+1$	$11 = 6+3+2$
$12 = 4+1+7$	$13 = 3+2+8$
$14 = 1+7+6$	$15 = 2+8+5$

Definition. A circular permutation p of $\{1, 2, \ldots, n\}$ is k-*sum-rich* if all the possible sums of k consecutive numbers are distinct.

So that the above permutation is an example of a 3-sum-rich permutation. Strictly speaking we do not need to restrict to the set $\{1, 2, \ldots, n\}$ but we will find it useful for our purposes and is natural.

It is easy to construct k-sum-rich permutations for some values of n. For example we have the following result.

Theorem. *If* $\gcd(k, n) = 1$ *then* $123\ldots n$ *is* k-*sum-rich.*

Proof. We need to show that the n possible sums of length k are distinct which we can do by showing they are distinct modulo n. Let L denote the sum of the first k consecutive terms. Then the sum of the next consecutive k terms will be $L+(k+1)-1 \equiv L+k \pmod n$ (i.e., we added the last term of $k+1$ and deleted the first term of 1). In general we have that modulo n, the next sum will differ by exactly k from the previous sum. We can conclude that the sum beginning at a will have value $L+(a-1)k \pmod n$.

If $L+(a-1)k \equiv L+(b-1)k \pmod n$ then it follows $(a-b)k \equiv 0 \pmod n$ which is possible if and only if $a \equiv b \pmod n$ (using $\gcd(k, n) = 1$). In particular the two sums start at the same point. □

There are of course many other possible ways to form k-sum-rich permutations. In the table below we have counted the number of circular permutations (up through rotation and reflection) which are k-sum-rich for small values of n and k. We note that if a permutation of $\{1, 2, \ldots, n\}$ is k-sum-rich then it is also $(n-k)$-sum-rich; and so our table only needs to include $2 \leqslant k \leqslant \frac{1}{2}n$.

	k=2	k=3	k=4	k=5
n=4	1			
n=5	3			
n=6	8	36		
n=7	46	76		
n=8	176	690	694	
n=9	955	2996	2529	
n=10	5446	22368	23679	67636
n=11	36122	147472	177885	184014

In particular this would indicate there are *many* k-sum-rich permutations. For example, the circular permutation 85417632 is one of 690 such permutations of $\{1, 2 \ldots, 8\}$ that are 3-sum-rich.

Super sum-rich permutations

The preceding considered permutations which have distinct sums for all possible sums of k consecutive terms. We can generalize this definition as follows.

Definition. Given a set S of numbers, a circular permutation p of $\{1, 2 \ldots, n\}$ is S-*sum-rich* if all the possible sums of ℓ consecutive numbers for $\ell \in S$ are distinct.

As an example, the permutation 124653 is 2-sum-rich and 3-sum rich because we have the following which shows that the sum of any two consecutive terms are distinct, and similarly the sum of any three

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consecutive terms are distinct.

$$\begin{aligned} 3 &= 1+2 & 7 &= 1+2+4 \\ 6 &= 2+4 & 12 &= 2+4+6 \\ 10 &= 4+6 & 15 &= 4+6+5 \\ 11 &= 6+5 & 14 &= 6+5+3 \\ 8 &= 5+3 & 9 &= 5+3+1 \\ 4 &= 3+1 & 6 &= 3+1+2 \end{aligned}$$

On the other hand this permutation is not $\{2,3\}$-sum-rich because given the sum value of 6 we cannot determine if it came from $2+4$ or from $3+1+2$.

In general the requirement to be S-sum-rich is very restrictive, particularly when the elements of S are close together. Note that the possible sums of $\{1,2,\ldots,n\}$ will lie between 1 and $1+2+\cdots+n = \frac{1}{2}n(n+1)$. On the other hand there are n^2 possible ways to sum consecutive terms in a cyclic permutation, i.e., pick a starting point (in n ways) and pick the number of terms to add (also n ways). This shows that we must have repetition in the set of all possible sums, and also rules out the possibility of being able to distinguish *each* possible consecutive sum based just on the total.

In general, the following problem is very hard and currently the best tool is brute force computation.

Problem. Given S and n determine whether there is an S-sum-rich permutation of $\{1,2,\ldots,n\}$.

Examples of S-sum-rich permutations

n	S	S-sum-rich
6	$\{2,4\}$	126543
6	$\{1,3,5\}$	135624
7	$\{1,3,5,7\}$	1357246
7	$\{0,2,4,6\}$	1357246
8	$\{1,3,5,7\}$	14563278
8	$\{2,4,6\}$	14386275
8	$\{3,4\}$	12387645*
8	$\{4,5\}$	12387645*
9	$\{2,3\}$	136857924*
9	$\{3,4\}$	125369784
9	$\{4,5\}$	152498637
10	$\{1,3,5,7,9\}$	138479256 10
10	$\{2,3\}$	12876 10 9543*
10	$\{4,5,6\}$	12589 10 6473*

Those permutations marked with a "*" are the *unique* such S-sum-rich permutation for the given n.

Backwards addition

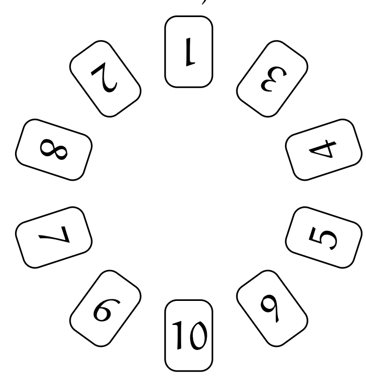
We have talked about some mathematics, but the question naturally arises: "What's the trick?"

The fact that sums are distinct allows us to recover the terms that were added together and can be the basis for a good trick. This idea has been used to good effect by Colm Mulcahy [2, February 2008] and also forms magic tricks based on de Bruijn cycles or more generally universal cycles (an excellent source of material in this direction is in the recent book of Persi Diaconis and Ron Graham [1]).

The following is based on the (unique!) $\{2,3\}$-sum-rich circular permutation for $n = 10$ which we will call "backwards addition".

Take cards numbered one through ten in order (you can use cards from a poker deck with an ace as a 1), move the card 1 down one position below 2 and move cards 9 and 10 together up to immediately follow 5. We are now ready to perform.

The first thing to do is to arrange the cards in a circular order. This can be done by the audience member, you can have them cut the deck as many times as they want and then deal in a circular pattern in either direction with the cards face down. Ultimately the cards will more or less be arranged as shown below (or in the reverse order).



Now start a dialogue similar to the following.

"We have all learned in school how to take two or three numbers and add them up to get a new number. But we cannot reverse that process because if I tell you the sum was eight you cannot know if this came from two plus six, or three plus five or one plus two plus five or any other number of possibilities. However I have mastered the art of backwards addition that makes this possible, and I will need your help to demonstrate.

"First, you need to pick some numbers to add. Pick one of the ten card below and also pick up either the card on the left or the right. If you want you can also pick up a third card that is next to one of the two cards that you picked up. At this point you should have either two or three cards in your hand. Now add up the numbers on the card and tell me the total. I will then use backwards addition to tell you which cards you are holding."

To help add to the mystique you do not need to watch them select cards and can do this blindfolded showing that you do not know if they have two or three cards.

Let us suppose they say the sum is twelve. (Hopefully they added the two or three small numbers in their head correctly or the trick doesn't work!)

"Twelve? What a great number, there are lots of ways to get twelve. You could have added four and eight or added two, three and seven. However that is not what you did ..." insert comments involving three, four, and five related to something that can be seen on the person or in the room leading up to the end "... so by the method of backwards addition you must have added the numbers three, four, and five to get twelve. Am I right?"

Get confirmation from the other person.

"And that is the art of backward addition!"

Of course the trick works by recognizing that we are given a sum of either two or three consecutive numbers from the circular permutation and these are all distinct. Once we have the total we can recover the original sum, the rest is showmanship!

Below is a chart of the 20 possible sum totals and how they correspond to a sum of two or three consecutive terms in the above circular permutation.

$3 = 1 + 2$	$14 = 5 + 9$
$4 = 1 + 3$	$15 = 7 + 8$
$6 = 1 + 2 + 3$	$16 = 6 + 10$
$7 = 3 + 4$	$17 = 2 + 8 + 7$
$8 = 1 + 3 + 4$	$18 = 4 + 5 + 9$
$9 = 4 + 5$	$19 = 9 + 10$
$10 = 2 + 8$	$21 = 6 + 7 + 8$
$11 = 1 + 2 + 8$	$23 = 6 + 7 + 10$
$12 = 3 + 4 + 5$	$24 = 5 + 9 + 10$
$13 = 6 + 7$	$25 = 6 + 9 + 10$

The hardest part about this trick is memorizing the twenty different sums, alternatively one can write them down on a piece of concealed paper or quickly find them in the circular pattern shown above.

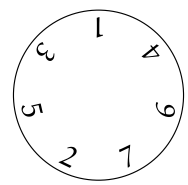
One can do this trick similarly for $n = 9$ with the (unique!) $\{2, 3\}$-sum-rich permutation 136857924. There is *no* permutation of $\{1, 2, \ldots, n\}$ that works for this trick for $n = 4$, 5, 6, 7, 8, 11 or 12!

Neighbors and strangers

The S-sum-rich circular permutations can also be used as the basis for simple puzzles and we finish with an example of one such puzzle.

Seven people have moved onto a circular cul-de-sac and are deciding how to label their houses with the numbers $1, 2, \ldots, 7$. They can do it in a number of different ways and have decided to take advantage of this opportunity to help easily identify groups of houses by the sum of the house numbers. In particular there are two types of groups they want to be able to quickly identify, "a house and its two neighbors" and "the houses that are not a neighbor of a particular house" or more simply "the strangers to a particular house".

For example if they chose to label as



then 4 and its neighbors are 1, 4, and 6 which would be identified with 11; while the strangers of 6 are 1, 2, 3 and 5 which would also be identified with 11. This is not a good labeling because if we were told 11 we would not be able to tell which of these two groups we would be referring to.

Find a labeling of the houses with $1, 2, \ldots, 7$ so that all possible neighbors and strangers can be identified by only using the sum of the house numbers. (In other words, find a $\{3, 4\}$-sum-rich permutation for $n = 7$.)

Acknowledgment

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References

[1] Persi Diaconis and Ron Graham, *Magical Mathematics, The mathematical ideas that animate great magic tricks*, Princeton University Press, Princeton, 2011, 258 pp.

[2] Colm Mulcahy, *Card Colm*, available online at `www.maa.org/columns/colm/cardcolm.html`.