Representations of the **10** Geometric (**10**,3) Triangular Configurations

*For the G4G**10** Conference Honoring Martin Gardner*

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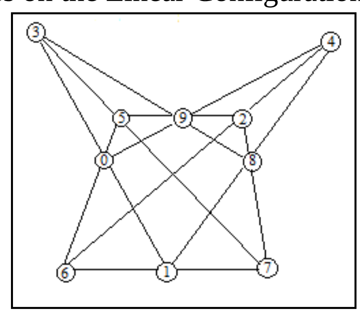
In their 2006 publication [FGR], Jeremiah Farrell, Martin Gardner, and Thomas Rodgers analyze "*(n, r)* Configurations."' Namely, any such Configuration forms a collection of *n* distinct "items," along with *n* distinct "objects" created from them, and must satisfy two conditions:

- Any two *items* are in at most one *object*, and any two *objects* have at most one *item* in common.
- Each *object* has *r* items, and each *item* is in exactly *r* objects.

The conditions restrict the number of (mathematically nonequivalent) *(n, r)* Configurations for any given values of *n* and *r*. For example, it is well known (cf. [p. 97, FGR]) that there are exactly ten nonequivalent *(10,3)* Configurations. In addition, the *items* and *objects* can be, for example, letters and words (which would then form "Word Configuration"). Or they could be vertices and lines (which then form a "Linear Configuration"). Or they could be vertices and *r*-gons—for example, when *r=3*, vertices and triangles (which then form a "Triangular Configuration"). Farrell, Gardner, and Rodgers described (in [FGR]) each of the ten *(10,3)* Linear Configurations in geometric arrangements (representations) and proved they were distinct.

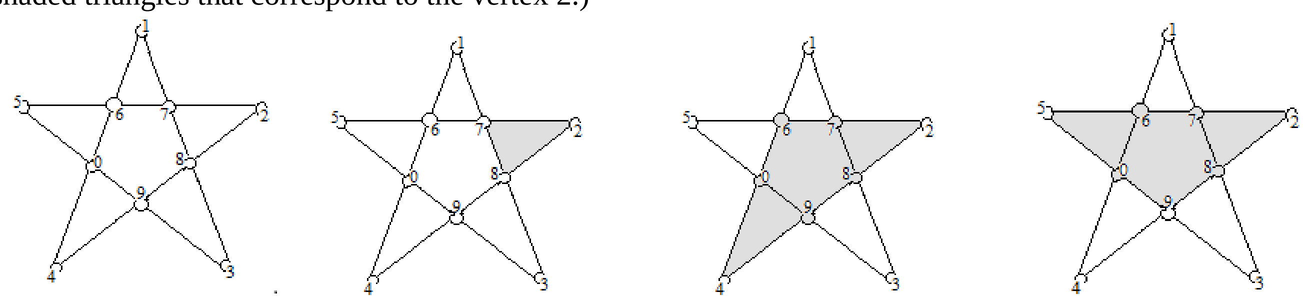
Using letters and words, there are many different two-person word games that could be created and correspond to any such Configuration. For example, one such game [from FGR] uses the Linear Configuration as a board, along with a collection of tiles, each labeled with one of the *n* distinct letters. The players alternately select from the available collection of tiles and place the tiles on the board's vertices. The first player to form one of the *n* words in the Configuration's list wins, where a word can be formed using the three letters along a line, taken in any order. Similarly, there are many one-person puzzles that correspond to any Configuration; for example, it is always possible to arrange the letter tiles on the Linear Configuration game board so that all *n* words can be spelled at once.



We use a *(10,3)* Linear Configuration example from [FGR] to illustrate, where the ten letters are from the word ELUCIDATOR, and the game board, called ``Desargues's Mitre'' in [FGR] is as shown here. The puzzle to arrange all ten letter tiles simultaneously is solved by placing the E on vertex 5, L on 6, U on 9, C on 3, I on 7, D on 2, A on 4, T on 0, O on 1, and R on 8. Then, for example, the word "DUE" is spelled along the line formed by vertices 2-9-5 (which is read from left to right on the game board as 5-9-2).

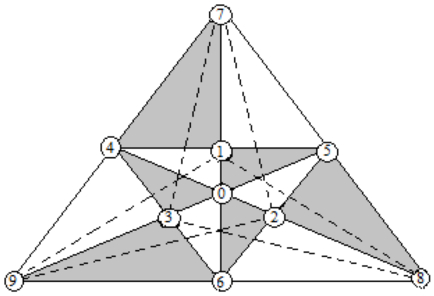
This paper describes a pictorial representation for each of the ten *(n,3)* Triangular Configurations. Each depiction is certainly not unique. The fact that they form ten distinct, nonequivalent configurations may be proven by showing each one corresponds to a different Linear Configuration presented in [FGR].

For obvious reasons, we label the first of these ``**Star**,'' shown below on the left. To illustrate how the three triangles correspond with any given vertex, the three right-hand pictures show the three triangles that contain the vertex numbered 2. The ten triangles in the configuration are listed below. (The first three are the shaded triangles that correspond to the vertex 2.)

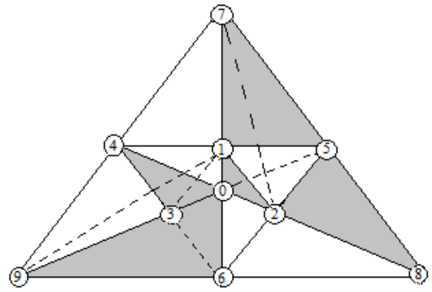


The ten triangles are 2-8-7, 2-4-6, 2-9-5, 6-1-7, 3-0-1, 1-8-4, 2-5-7, 3-9-8, 0-9-4, and 6-0-5. As required, you can see that each numbered vertex appears in three triangles in the list, each vertex appears with another vertex in the same triangle at most once, and any two triangles have at most one vertex number in common. In addition, you can compare this list of triplets of vertices with the Desargues's Mitre picture above: the ten lines in that Linear Configuration are formed exactly from those same ten triplets of vertices! Hence, the ``Star'' Triangular Configuration is mathematically equivalent to the ``Desargues's Mitre'' Linear Configuration!

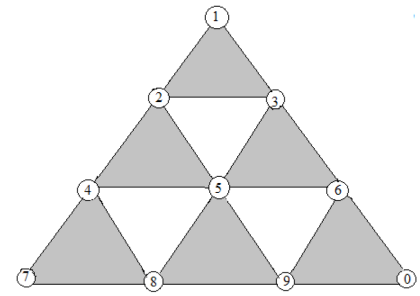
Similarly, each of the following Triangular Configurations correspond (as mathematically equivalent) to a different one of the ten depicted Linear Configurations displayed in [FGR]. Along with "Star," they provide a complete geometric (pictorial) representation list of the ten *(10,3)* Triangular Configurations. The ten triangles formed are listed below each configuration.



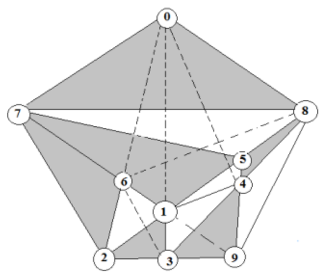
Fano's: The six shaded triangles, plus 4-5-6, 7-8-3, 2-7-9, and 1-8-9.



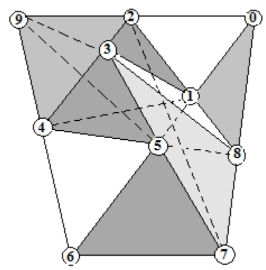
Fano's II: 0-1-2, 0-3-4, 0-6-9, 1-5-7, 2-5-8 (all shaded), plus 7-8-9, 4-6-8, 4-2-7, 3-5-6, and 3-1-9.



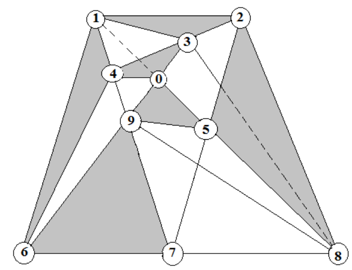
Tenpins: The six shaded triangles, plus 1-7-0, 1-4-6, 2-7-9, and 3-8-0.



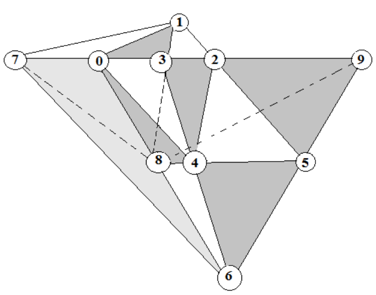
Pentagon: 1-2-3, 1-5-7, 2-6-7, 3-4-9, 4-5-8, 0-7-8 (all shaded), plus 0-1-4, 0-3-6, 9-2-5, and 9-8-6.



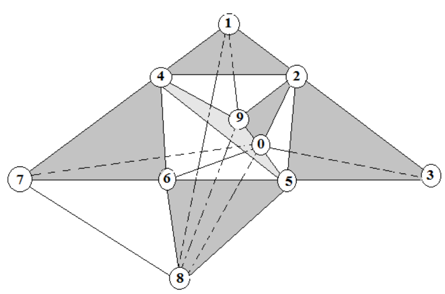
Trapezoid: 1-2-3, 2-4-9, 0-1-8, 3-4-5, 3-7-8, 5-6-7 (all shaded), plus 0-6-9, 0-2-7, 5-8-9, and 4-1-6.



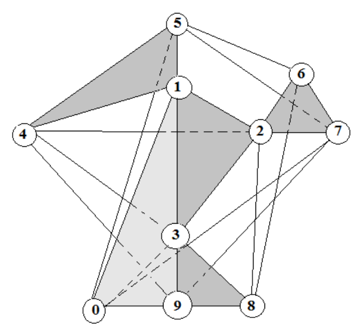
Trapezoid II: The six shaded triangles, plus 2-4-7, 1-7-5, 9-3-8, and 0-6-8.



Coffee Filter: The six shaded triangles, plus 1-2-7, 1-5-8, 7-8-9, and 3-6-9.



Batman's Kite: The six shaded triangles, plus 1-3-7, 7-8-0, 1-8-9, and 0-3-6.



Basinet: The five shaded triangles, plus 0-6-8, 0-5-7, 3-5-6, 2-4-8, and 9-4-7.

A number of interesting questions about *(10,3)* Triangular Configurations remain. For example, is it possible to obtain geometric representations for each of the ten without the need for "crossing lines" (sides of a triangle that "cross through" the interior of some other triangle)? Such lines in our depictions above are shown as dotted. (**Star** and **Tenpins** are the two configurations presented here without crossing lines.) This and other questions about *(n,k)* configurations will be enjoyable topics of further investigations.

[FGR] Farrell, J., Gardner, M., and Rodgers, T. (2005). Configuration Games. In N. Cipra et al (Eds.), *Tribute to a MatheMAGIcian*. Wellesley, MA: A K Peters