An Intergalactic Franchise War

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Burger Queen and Dairy King, two failed fast food chains on Earth, receded to outer space. By a twist of fate, they both set eyes on the same asteroid belt, and locked horns in an intergalactic franchise war.

The asteroid belt consisted of eleven big asteroids, each with significant population, and countless small asteroids, each with no or negligible population. Various pairs of asteroids were linked by space shuttle service. The President of the governing Federation declared that no asteroid may be served by both chains. Each chain in turn chose one asteroid. The first one for each chain might be chosen arbitrarily, but subsequent choices must have space shuttle link to an asteroid already chosen by the chain.

Dairy King fired the first salve by bribing the Minister of Administrative Affairs, who granted Dairy King the privilege of choosing first. It would appear that Burger Queen was doomed to a minority share, namely at most five of the big asteroids. In desperation, Burger Queen bribed the Minister of Transport, who let them submit a revised space shuttle network. The task for Burger Queen was to come up with a network which would work to their advantage.

Accordingly, the top brains from within the organization joined force to form a joint committee, an *ad hoc* body called the **Burger Joint**. It started on the right foot by adopting the standard procedure in problem-solving, namely, downsizing. If there were only one big asteroid, Dairy King would get it, and there was nothing Burger Queen could do about it. If there were two, each chain would get one, and there was nothing to worry about either. So the first interesting scenario was if there were three big asteroids, and Burger Queen would like to secure two of them, despite choosing second.

After much deliberation, the Burger Joint came up with the network shown in Figure 1. Three small asteroids were included. All other small asteroids were left out of the loop.

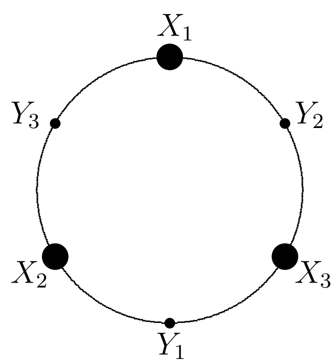

Figure 1

The analysis that Figure 1 would work was written out as follows.

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Case 1. Dairy King takes a Y, say Y_1.
We can take X_2 and will win the race to X_1.

Case 2. Dairy King takes an X, say X_1.
We will take Y_1 and win both of the races to X_2 and X_3.

In summary, we can get two big asteroids no matter what Dairy King does.

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In the next scenario, there were five big asteroids, and Burger Queen would like to secure three of them, despite choosing second. It would appear that the network in Figure 2 should work.

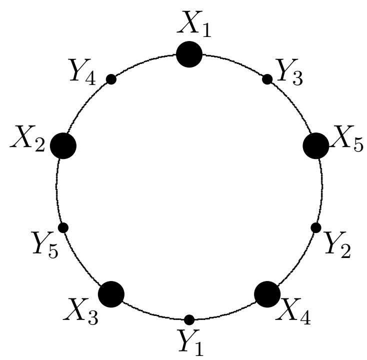

Figure 2

The analysis that Figure 2 would work was written out as follows.

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Case 1. Dairy King takes a Y, say Y_1.
We can take X_3 and will win both of the races to X_2 and X_1.

Case 2. Dairy King takes an X, say X_1.
We can take Y_2 and win all of the races to X_3, X_4 and X_5.

In summary, we can get three big asteroids no matter what Dairy King does.

* *

Actually, the Buger Joint had a flawed strategy earlier, where the proposed response in Case 2 was to take Y_1 as in the previous scenario. However, Burger Queen would only get X_3 and X_4. It was then realized that the significance of Y_1 in the first scenario was not because it was opposite to X_1, but because it was the third one from X_1 in either clockwise or counter-clockwise order. The corrected strategy would allow Burger Queen to get four out of seven, five out of nine and in the actually problem on hand, six out of eleven. Interestingly, the flawed strategy would work in every other scenario, including the last one. So the Burger Joint prepared to submit to the Minister of Transport a network generalized from Figures 1 and 2.

With six big asteroids in the pocket, Burger Queen became curious and wondered if seven were possible. The Burger Joint went back to the drawing board, and soon came up with an idea. Burger Queen would get two of the three big asteroid in Figure 1. All that was needed was to attach eight other big asteroids to them in a distribution which was as even as possible, say two to X_1 and three to each of X_2 and X_3. The best Dairy King could do was to secure either X_2 or X_3 and get four big asteroids, leaving seven for Burger Queen. So Figure 3 became the latest plan.

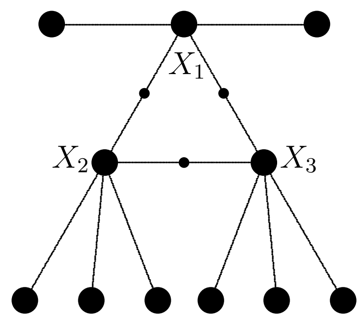

Figure 3

With seven of the big asteroids in the pocket, Burger Queen became greedy, and told the Burger Joint to push for eight. This was a tall order. Again, the Burger Joint resorted to downsizing and tried to get three big asteroids out of four. After a prolonged brown study, a break-through came when a three-dimensional model was introduced. Instead of using triangle $X_1X_2X_3$ as in the critical portion of Figure 3, they now use a tetrahedon $X_1X_2X_3X_4$. Figure 4 was obtained by cutting it open along the edges X_1X_2, X_1X_3 and X_1X_4, and flattening the four faces.

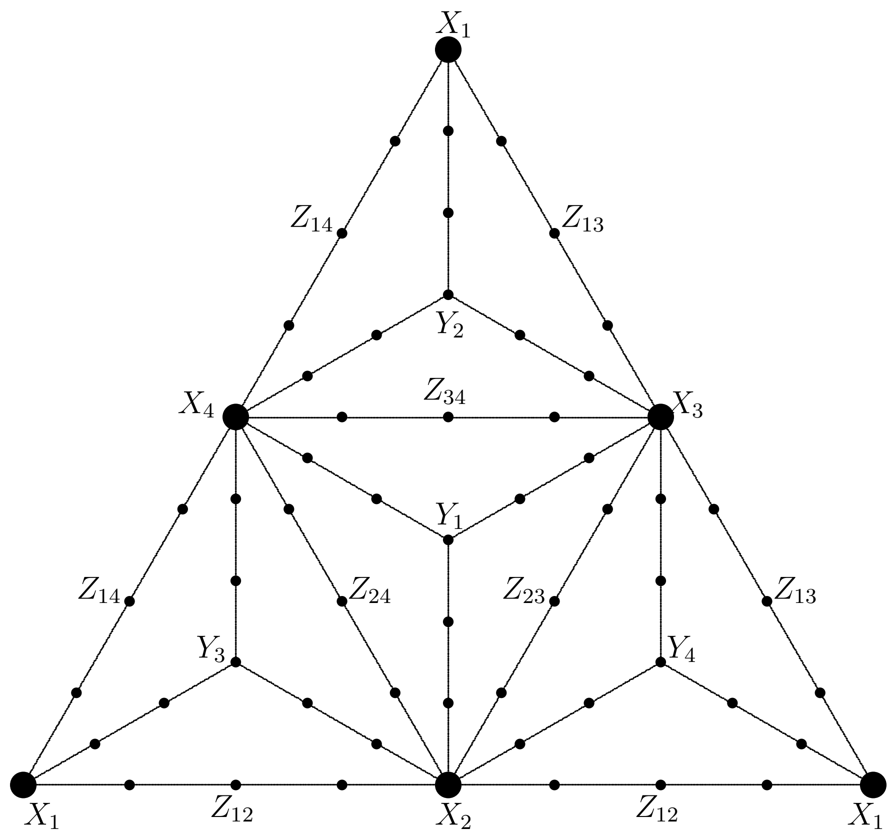

Figure 4

The analysis that Figure 4 would work was written out as follows.

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Case 1. Dairy King takes an X, say X_1.
We will take Y_1 and win all of the races to X_2, X_3 and X_4.

Case 2. Dairy King takes a Y, say Y_1.
We will take Z_{34} and win both of the races to X_3 and X_4. Moreover, we will get to one of them before Dairy King can get to either of them. Hence we will also win the race to X_1.

Case 3. Dairy King takes a Z, say Z_{34}.
We can take X_4. By the time Dairy King gets to X_3, we will have taken one step towards each of X_1 and X_2. Hence we will win both of these races.

Case 4. Dairy King takes an unlabelled asteriod adjacent to a Y, say the one between Y_1 and X_2.
We can take the unlabelled vertex between X_2 and Z_{24}. We will win both of the races to X_2 and X_4. Moreover, we will get to X_2 before Dairy King can get to X_3. Hence we will also win the race to X_1.

Case 5. Dairy King takes an unlabelled asteroid adjacent to an X.
There are two subcases.

Subcase 5(a). The asteroid is between this X and a Y, say the one between X_2 and Y_1.
We will take X_2. We will win the race to X_3 and get there before Dairy King can get to X_4. Hence we will also win the race to X_1.

Subcase 5(b). The asteroid is between this X and a Z, say the one between X_2 and Z_{24}.
We will take X_2. By the time Dairy King gets to X_4, we will have taken at least one step towards each of X_1 and X_3. Hence we will win both of these races.

In summary, we can get three big asteroids no matter what Dairy King does.

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It was then easy to modify Figure 4 to yield the desired result. As in Figure 3, a big asteroid was attached to X_1 and two big asteroids were attached to each of X_2, X_3 and X_4. The best Diary King could do was to secure one of X_2, X_3 and X_4, and end up with only three big asteroids.

* *

NOW WE NEED SOMETHING WHICH WILL GIVE BURGER QUEEN NINE BIG ASTEROIDS. A PLAN FOR FIVE OUT OF SIX WILL DO.

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Obviously, it would not be possible for Burger Queen to get all eleven big asteroids, as Dairy King could always take one in the first move. The ultimate goal then was to get ten out of eleven. This was seemingly impossible, but the Burger Joint, encouraged by the recent success, decided to give it a try anyway. So they sought an alternative approach to the downsized problem of getting two big asteroids out of three, and at long last, Figure 5 emerged.

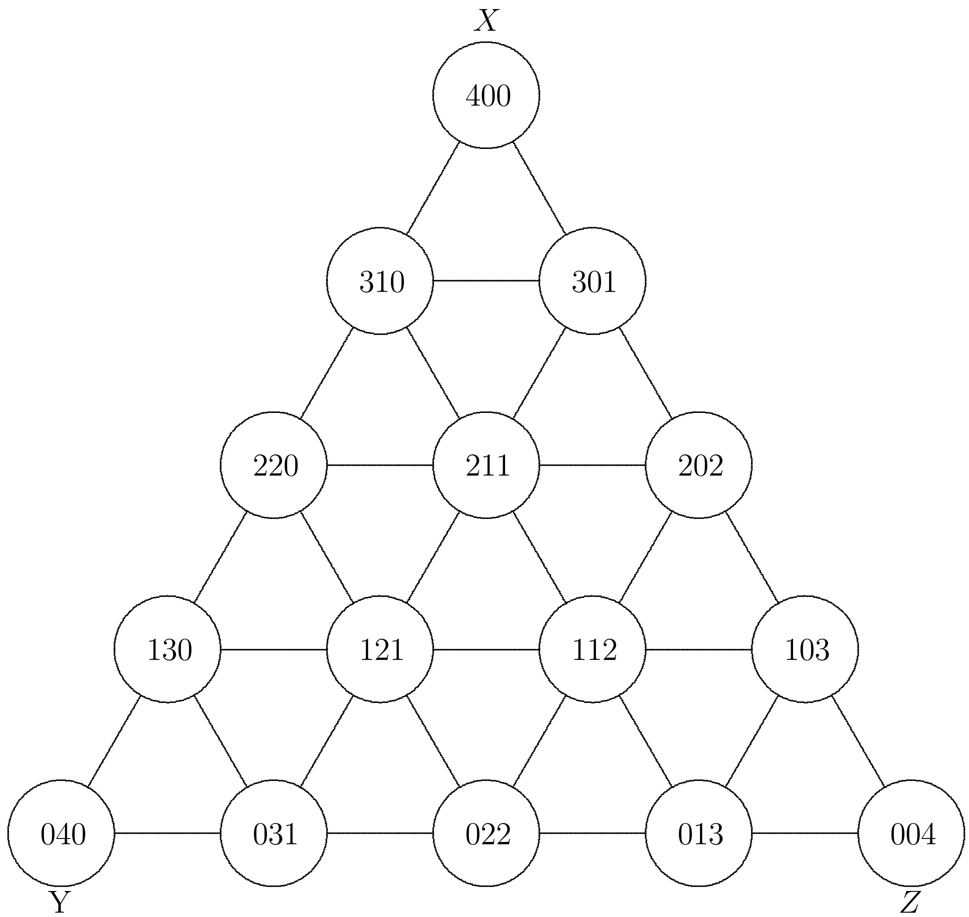

Figure 5

The analysis that Figure 5 would work, and how it might be generalized to yield a solution to the actual problem, was written out as follows.

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This network adds twelve small asteroids to the three big ones at the corners, X, Y and Z. Each asteroid is given three non-negative coordinates, representing how many levels it is from YZ, ZX and XY, respectively. Note that the sum of all three coordinates is always 4, and all sets of three coordinates with sum 4 are used. Two asteroids are linked by space shuttle service if and only if one pair of their coordinates is the same while each of the other two pairs differs by 1. By symmetry, we may assume that Dairy King chooses first one of the six asteroids at the top, that is, (x, y, z) with $x \geq 2$. We shall respond by choosing the asteroid $(x-2, y+1, z+1)$. We will concede X to Dairy King, but are one stop ahead in the races for Y and Z.

With eleven big asteroids, we just use a ten-dimensional simplex which has eleven vertices representing the big astroids. Each asteroid, big or small, will have eleven non-negative coordinates with sum 100, and all such sets of coordinates are used. The coordinates of the big asteroids consist of one 100 and ten 0s. Two asteroids are linked by space shuttle service if and only if each pair of their coordinates is the same except for two, where the difference is 1 in each of these two cases. Consider the first asteroid chosen by Dairy King. By the generalized Pigeonhole Principle, one of its coordiantes is at least 10. We can take the asteroid which is 10 less in this coordinate but 1 more in every other coordinate. We can then secure ten big asteroids out of eleven.

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So Burger Queen submitted a draft proposal to the Minsiter of Transport while the Burger Joint prepared the actual network, which was necessarily very complicated. However, the work was put to a stop because Burger Queen got the following reply from the Minsiter of Trsnaport. “Our asteroid belt does not have $\binom{110}{10} - 11 = 46897636623970$ small asteroids!”