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*Ten Terribly Tempting
Elementary Number Theory Tidbits*

by John J. Watkins

Martin Gardner's *Mathematical Games* columns were usually devoted to single topic such as the four color theorem, M. C. Escher, or card shuffling; but, every so often he would toss in a column containing a bunch of unrelated problems. These puzzle collections were great fun and included many problems that have now become old favorites, such as the one that asked whether a termite could bore just once through each of the 26 small wooden cubes forming the outer portion of a large $3 \times 3 \times 3$ cube and end up in the center cube.

These columns always delighted me and later, when I began teaching, became a prime source of good problems for me to use in the classroom. So, this paper is intended as a similar collection of problems and ideas — in this case from elementary number theory — and on this occasion of *G4GX* features ten 'tidbits', each of which in some way involves the number 10.

1. Pythagorean Triangles

A *Pythagorean triangle* $\{x, y, z\}$ is a triangle having integer sides that satisfy

$$x^2 + y^2 = z^2,$$

and such a triangle is called *primitive* if the three integers have no common factor greater than 1. Thus, $\{3, 4, 5\}$ and $\{8, 15, 17\}$ are examples of primitive Pythagorean triangles.

In the late fourth century B.C., Euclid provided us with a marvelous characterization of these triangles:

for any primitive Pythagorean triangle $\{x, y, z\}$, one of the numbers x or y must be even, and the other odd, so let x be the even number; then, there exist two positive integers s and t, $s > t$, one even and the other odd, with s and t having no common factor other than 1, such that

$$x = 2st, \ y = s^2 - t^2, \ z = s^2 + t^2;$$

moreover, if s and t are any two such positive integers, then these formulas produce a primitive Pythagorean triple. For example, if $s = 2, t = 1$ we get the $\{3, 4, 5\}$ triangle, while if $s = 4, t = 1$ we get the $\{8, 15, 17\}$ triangle.

Find all Pythagorean triangles having 10 as one of the three sides.

2. The Infinitude of Primes

Euclid's proof that there are infinitely many primes $2, 3, 5, 7, 11, 13, \ldots$ is widely viewed as one of the most elegant proofs in all of mathematics. It uses contradiction and assumes that there are only finitely many primes $p_1, p_2, p_3, \ldots, p_n$, but then the number $N = p_1 p_2 p_3 \cdots p_n + 1$ must have a prime factor other than $p_1, p_2, p_3, \ldots, p_n$; hence, the contradiction.

As appealing as Euclid's proof is to mathematicians, it often leaves students unconvinced. There is a new proof by Filip Saidak (*The American Mathematical Monthly*, vol. 62, no. 5, May 2006, p. 353) that I think is even simpler than Euclid's. It, like Euclid's proof, depends on two fundamental facts about primes: any integer greater than 1 has a prime factor, and two consecutive positive integers are always relatively prime (that is, they have no common factor greater than 1), but Saidak's proof does not need contradiction. Here is his proof.
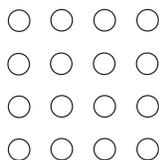
Consider the number $a = 10$. Since 10 is divisible by a prime, we know there is at least one prime. But $a + 1 = 11$ is relatively prime to $a = 10$, so there must be a second prime. Hence, the number $b = 10 \cdot 11 = 110$ is divisible by at least two primes. But, $b + 1 = 111$ is relatively prime to $b = 110$, so there must be

a third prime. Hence, the number $c = 110 \cdot 111 = 12\,210$ is divisible by at least three primes. But, $c + 1 = 12\,211$ is relatively prime to $c = 12\,210$, so there must be a fourth prime. Hence, $d = 12\,210 \cdot 12\,211$ is divisible by at least four primes. Since we can repeat this process forever, there are infinitely many primes.

Do one more step in this proof and produce a number e that is divisible by at least five primes and, although this is completely irrelevant to the proof, find out exactly how many prime factors the number e has.

3. Square Numbers

Today, we think of a number such as 16 being a *square* because it is equal to 4^2 — that is, 4 'squared'. The ancient Greeks, on the other hand, thought of 16 being a square because it is possible to arrange 16 stones in a square array:

$$
\begin{matrix}
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc
\end{matrix}
$$

Give a visual proof, in the ancient Greek style, to explain why the sum of the first ten odd integers is equal to 10^2 — that is, $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 10^2$.

4. Sums of Cubes

In Problem 1, we saw that it is possible for the sum of two squares to be a square; in fact this happens infinitely often. However, in the 1630s, Fermat conjectured that it is *not* possible for the sum of two cubes to be a cube, or for that matter the same is true for any higher power as well. This famous conjecture, now a theorem, we call *Fermat's last theorem.*

But what about three cubes? Is it possible for the sum of *three* cubes to be a cube? In other words, is there a solution to the equation
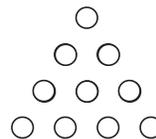
$$x^3 + y^3 + z^3 = w^3$$

in the positive integers? The answer, perhaps surprisingly, is yes.

Find a solution to this equation where one of the positive integers x, y, z, w is 10.

5. Triangles and Tetrahedrons

In addition to square numbers, the ancient Greeks thought of other numbers in geometric ways too. For example, they thought of the number 10 being a *triangular* number because ten stones can be arrange in a triangle:

$$
\begin{matrix}
& & & \bigcirc & & & \\
& & \bigcirc & & \bigcirc & & \\
& \bigcirc & & \bigcirc & & \bigcirc & \\
\bigcirc & & \bigcirc & & \bigcirc & & \bigcirc
\end{matrix}
$$

$$10 = 1 + 2 + 3 + 4$$

Thus, the *triangular numbers* are the numbers

$$1, \ 3, \ 6, \ 10, \ 15, \ \ldots \ .$$

It turns out that we can also arrange ten stones into a different geometric shape by stacking the ten stones into a *tetrahedron,* which we do by placing six stones in a triangle at the bottom, then nestling another three stones into the three spaces formed by the six bottom stones, and finally adding a tenth stone at the very top in the single space formed by the triangle of three stones in the middle.

Thus, the *tetrahedral numbers* are the numbers

$$1, \ 4, \ 10, \ 20, \ 35, \ \ldots$$

where each number is a sum of consecutive triangular numbers beginning with 1. For example, $10 = 1 + 3 + 6$, and $35 = 1 + 3 + 6 + 10 + 15$.

An extremely important formula for the triangular numbers was known by the Pythagoreans: the nth triangular number is given by

$$\frac{n(n+1)}{2}.$$

For example, the 4th triangular number is $\frac{4(4+1)}{2} = 10$. A similar formula for the tetrahedral numbers was apparently known in Egypt about 300 B.C. and also discovered in India by Aryabhata around 500 A.D.: the nth tetrahedral number is given by

$$\frac{n(n+1)(n+2)}{6}.$$

For example, the 3rd tetrahedral number is $\frac{3(3+1)(3+2)}{6} = 10$.

By the way, the triangular numbers and the tetrahedral numbers appear in Pascal's triangle as the third and fourth diagonals!

```
                    1

                1       1

            1       2       1

        1       3       3       1

    1       4       6       4       1

1       5      10      10       5       1

1   6      15      20      15       6       1

1   7   21      35      35      21       7       1

                    ⋮
```

Note that 10 appears in both of these diagonals in the above diagram; that is, as we have already observed, 10 is *both* triangular and tetrahedral.

Are there any other numbers (besides 1, of course) that are both triangular and tetrahedral?

6. $p^k + 1$

Numbers of the form $p^k + 1$ and $p^k - 1$, where p is a prime, have been extensively studied. For example, one of Fermat's most famous conjectures (which turned out to be false) was that $2^{2^k} + 1$ is prime for all values of k; and numbers of the form $2^k - 1$ are called Mersenne numbers and are particularly useful for finding enormously large primes.

It turns out that the triangular number 10 is of the form $p^k + 1$ since $10 = 3^2 + 1$. Are there any other triangular numbers that are also of the form $p^k + 1$, where p is prime?

7. Is 10^k a sum of two tetrahedrons?

Find a power of 10 that is the sum of two tetrahedral numbers.

8. Sums of Cubes

Express the sum of the first <u>ten</u> cubes in terms of the tenth triangular number.

9. Perfect Numbers

The concept of perfect numbers is extremely old, perhaps going back to Archytas, one of the last of the Pythagoreans. A number is *perfect* if it is the sum of its proper divisors; so, for example, 6 is a perfect number because $6 = 1 + 2 + 3$. The next perfect number is 28, since $28 = 1 + 2 + 4 + 7 + 14$. Then come the next two perfect numbers: 496 and 8128. These four perfect numbers were known well over two thousand years ago. But, the fifth perfect number didn't appear until the fifteenth century!

Book IX of Euclid's *Elements* contains as its very last proposition the following extraordinary theorem about perfect numbers: if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is a perfect number. Furthermoe, Fermat proved much later that $2^n - 1$ can be prime *only if* n is prime. When $2^n - 1$ is prime, it is called a *Mersenne prime* after the 17th century French friar who made a bold, though slightly flawed, conjecture claiming exactly which values of $2^n - 1$ are prime for all $n \leq 257$.

Euclid's proposition seems to make it quite easy to produce perfect numbers, one after another:

1) For $n = 2$, $2^n - 1 = 2^2 - 1 = 3$ is prime, so

$$2^{2-1}(2^2 - 1) = 6$$

is a perfect number.

2) For $n = 3$, $2^n - 1 = 2^3 - 1 = 7$ is prime, so

$$2^{3-1}(2^3 - 1) = 4 \cdot 7 = 28$$

is a perfect number.

3) For $n = 5$, $2^n - 1 = 2^5 - 1 = 31$ is prime, so

$$2^{5-1}(2^5 - 1) = 16 \cdot 31 = 496$$

is a perfect number.

4) For $n = 7$, $2^n - 1 = 2^7 - 1 = 127$ is prime, so

$$2^{7-1}(2^7 - 1) = 64 \cdot 127 = 8128$$

is a perfect number.

Why did it take so long to find the fifth perfect number? Well, maybe it is because the fifth perfect number is $33\,550\,336$, which is pretty big; and to find this perfect number you would need to know that $2^{13} - 1 = 8191$ is prime. (Note, by the way, that $2^{11} - 1 = 2047 = 23 \times 89$ is not prime.)

Find the tenth perfect number.

10. $\sqrt{10}$

It is easy to evaluate an infinite *repeating* continued fractions such as

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}},$$

assuming it converges to a real number, because

$$x - 1 = \cfrac{1}{1 + \cfrac{1}{2 + (x-1)}},$$

and so,

$$x - 1 = \cfrac{1}{1 + \frac{1}{x+1}} = \frac{x+1}{x+2},$$

which means that $x^2 + x - 2 = x + 1$, and then that $x^2 = 3$. Thus, $x = \sqrt{3}$.

Find an infinite *repeating* continued fraction that represents the number $\sqrt{10}$.

1. First, let's assume that $\{x, y, z\}$ is primitive. If 10 is one of the legs, then since 10 is even we have, using Euclid's characterization, $x = 10 = 2st$ and so $s = 5, t = 1$, *both* of which are odd. So 10 is cannot be a leg of a primitive Pythagorean triangle. Similarly, we see that 10 cannot be the hypotenuse of a primitive Pythagorean triangle because if $z = 10 = s^2 + t^2$, then $s = 3, t = 1$, which are again both odd.

Therefore, any Pythagorean triangle having 10 as a side must be a non-primitive triangle such as $\{6, 8, 10\}$, which is a multiple of the primitive triangle $\{3, 4, 5\}$ having 5 as its hypotenuse. Note that if 5 is the hypotenuse of a primitive triangle, then $z = 5 = s^2 + t^2$, so $s = 2, t = 1$, and the triangle is the $\{3, 4, 5\}$ triangle. On the other hand, if 5 is a leg of a primitive triangle, then $y = 5 = s^2 - t^2$, so $s = 3, t = 2$, which yields the $\{5, 12, 13\}$ triangle; this means that 10 is a leg of the $\{10, 24, 26\}$ triangle.

Another possibility is that 10 is a side of a non-primitive triangle that is a multiple of a primitive triangle having 2 as a side. So, either $x = 2 = 2st$, which is impossible since $s > t$; or, $z = 2 = s^2 + t^2$, which is impossible for the same reason. Finally, the only other possibility is that 10 is a side of a non-primitive triangle that is a multiple of a primitive triangle having 1 as a side, but it is easy to see that neither $y = 1 = s^2 - t^2$ nor $z = 1 = s^2 + t^2$ is possible.

Hence, the only Pythagorean triangles having 10 as a side are the $\{6, 8, 10\}$ triangle and the $\{10, 24, 26\}$ triangle.

2. The first number $a = 10$ has two prime factors, 2 and 5. The next number $b = 10 \cdot 11$ has a third prime factor, 11. Then, $c = 110 \cdot 111 = 12210$ has five prime factors since $111 = 3 \cdot 37$. Next, $d = 12210 \cdot 12211$ has six prime factors since 12211 itself is prime. Thus, $e = 149\,096\,310 \cdot 149\,096\,311$ has <u>ten</u> prime factors, since 149 096 311 factors into four primes: $7 \cdot 13 \cdot 103 \cdot 15907$.

3. Just to save space, we present only a visual proof

that sum of the first four odd integers is equal to 4^2 — that is, $1 + 3 + 5 + 7 = 4^2$.
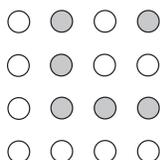
$$1 + 3 + 5 + 7 = 16$$

4. Of course, one solutions for $x^2 + y^2 = z^2$ is $x = 3, y = 4, z = 5$, which consists of three *consecutive* integers. Amazingly, the four consective integers $x = 3, y = 4, z = 5, w = 6$ happen to be a solution to the equation $x^3 + y^3 + z^3 = w^3$ because $3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$. Therefore, it follows that

$$6^3 + 8^3 + 10^3 = 12^3,$$

and $x = 6, y = 8, z = 10, w = 12$ is the solution we are looking for.

5. We are looking for solutions in the positve integers for the equation $\frac{m(m+1)}{2} = \frac{n(n+1)(n+2)}{6}$, which we can rewrite as

$$3m(m + 1) = n(n + 1)(n + 2).$$

So, for example, we can easily see that 10 is both triangular and tetrahedral because $m = 4, n = 3$ is an obvious solution to this equation (since we get $3 \cdot 4 \cdot 5$ on both sides). Similarly, $m = 1, n = 1$ is another obvious solution to this equation (since $1 \cdot 2 \cdot 3$ is on both sides); hence, 1 is also both triangular and tetrahedral.

A much less obvious solution is $m = 15, n = 8$, yielding the triangular tetrahedral number 120. The only other solutions are $m = 55, n = 20$ and $m = 119, n = 34$, which yield 1540 and 7140 as also being both triangular and tetrahedral.

6. There are three others besides 10, namely:

$$3 = 2^1 + 1, \quad 6 = 5^1 + 1, \quad \text{and} \quad 28 = 3^3 + 1.$$

Let's prove this. Since a triangular number is of the form $\frac{n(n+1)}{2}$ we are looking for solutions to $\frac{n(n+1)}{2} = p^k + 1$, which we rewrite as

$$(n - 1)(n + 2) = 2p^k.$$

There are several cases to consider.

Case 1: $p = 2$. In this case, $(n - 1)(n + 2) = 2 \cdot 2^k = 2^{k+1}$ is a power of 2. But, $n - 1$ and $n + 2$ are of opposite parity, so the only way $(n - 1)(n + 2)$ can be a power of 2 is if $n = 2$, in which case $k = 1$, and we have $3 = 2^1 + 1$.

Case 2: $p = 3$. In this case, 2 must divide either $n-1$ or $n+2$. First, let's suppose that 2 divides $n+2$; then $n + 2 = 2 \cdot 3^j$ and $n - 1 = 3^i$, where $i + j = k$. Thus, we have

$$3 = (n + 2) - (n - 1) = 2 \cdot 3^j - 3^i,$$

which we can write as

$$1 = 2 \cdot 3^{j-1} - 3^{i-1}.$$

This means that $j = 1$ (otherwise 3 divides 1, which is impossible), and so $i = 1$, $k = 2$, and $n = 4$. This gives us the triangular number $\frac{4(4+1)}{2} = 10$ represented as $3^2 + 1$.

Next, let's suppose that 2 divides $n-1$; then $n-1 = 2 \cdot 3^i$ and $n + 2 = 3^j$, where again $i + j = k$. Thus, we have

$$3 = (n + 2) - (n - 1) = 3^j - 2 \cdot 3^i,$$

which we can write as

$$1 = 3^{j-1} - 2 \cdot 3^{i-1}.$$

This means that $i = 1$, so $j = 2$, $k = 3$, and $n = 7$. This gives us the triangular number $\frac{7(7+1)}{2} = 28$ represented as $3^3 + 1$.

Case 3: $p > 3$. In this case p cannot divide both $n - 1$ and $n+2$; therefore p^k either divides $n - 1$ or it divides $n + 2$. But, p^k cannot divides $n - 1$, because then n would be greater than 0, and so $n + 2$ would be greater than 2, which is impossible. Therefore, p^k divides $n + 2$, which must be odd, and so, $n - 1 = 2$, and $n = 3$, which means that $n + 2 = 5$; thus $p = 5, k = 1$. This gives us the triangular number $\frac{3(3+1)}{2} = 6$ represented as $5^1 + 1$.

7. Using Aryabhata's formula from Problem 5, we see that 10^4 is the sum the 8th and the 38th tetrahedral numbers:

$$\frac{8 \cdot 9 \cdot 10}{6} + \frac{38 \cdot 39 \cdot 40}{6} = 120 + 9880 = 10^4.$$

8. $1^3 + 2^3 + 3^3 + \cdots + 10^3 = 3025 = 55^2$, where 55 is the tenth triangular number.

There are several ways to prove that the sum of the first n cubes is equal to the square of the nth triangular number (for example, induction one good way, albeit completely unenlightening). Here is a visual proof due to Solomon W. Golomb, one of Martin Gardner's most frequent collaborators (*Mathematical Gazette*, vol. 49, May 1965, p. 199), and which in order to save space is presented for the case $n = 5$:
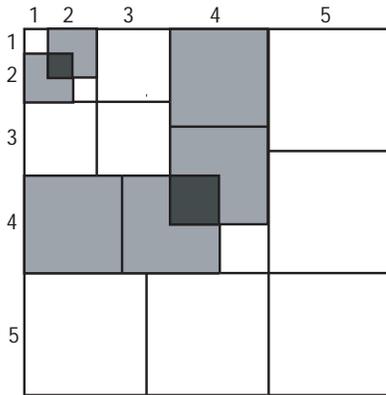
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 15^2$$

Note that each of the odd cubes is completely visible in this picture, for example, 5^3 can be seen as the sum of five 5×5 squares, and Golomb cleverly handles the even cubes in a similar way by using positive and negative area.

9. Because of Fermat's result, we need only consider values of n which are prime. The first ten values of n for which $2^n - 1$ is prime are:

$$2, \ 3, \ 5, \ 7, \ 13, \ 17, \ 19, \ 31, \ 61, \ \text{and } 89.$$

Thus, the tenth perfect number is

$$2^{89-1}(2^{89} - 1)$$

$$= 191\,561\,942\,608\,236\,107\,294\,793\,378$$
$$084\,303\,638\,130\,997\,321\,548\,169\,216 \,.$$

10.
$$\sqrt{10} = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}}},$$

because if we set x equal to this continued fraction, we have

$$x - 3 = \frac{1}{6 + (x - 3)} = \frac{1}{x + 3},$$

and so, $x^2 - 9 = 1$, and $x^2 = 10$; hence, $x = \sqrt{10}$.

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