A Chip-Firing Puzzle on Graphs

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A Chip-Firing Puzzle is played on a graph consisting of a collection of nodes with edges between them; one can think of the nodes as representing people and the edges representing relationships between them. At the beginning of the puzzle, each node has associated to it an integer; I like to think of positive integers representing an amount of money that the node has in the bank and a negative integer representing an amount of money it owes to the bank.

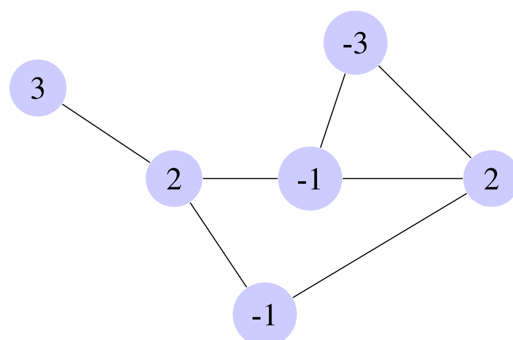


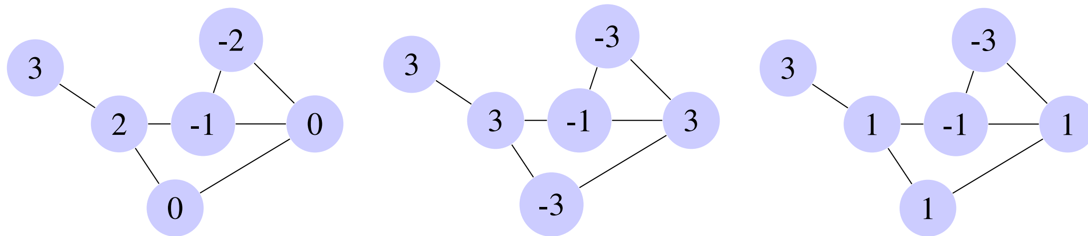
Figure 1: Example of Initial Position of Chip-Firing Puzzle

At each turn, the player can do one of the following moves:

- Lending: Choose a node and give one dollar from this node to each of its neighbors.
- Borrowing: Choose a node and have this node take one dollar from each of its neighbors.

If we start with the initial configuration from Figure 1 then some legal configurations after one move include the following:

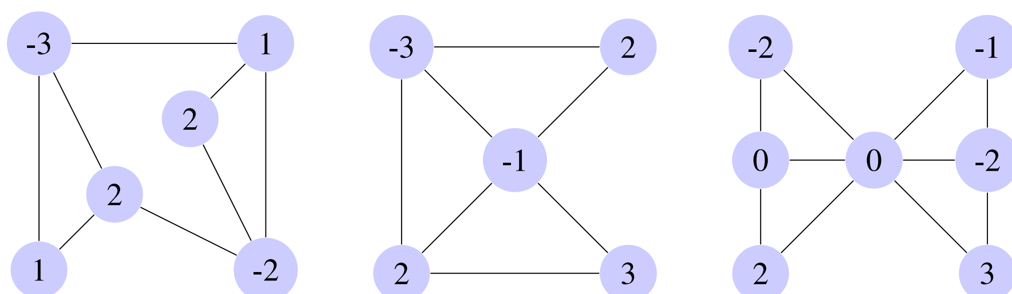
Note:I had originally planned to include as my gift a paper based on my talk at Gathering For Gardner 10 about 'The Secretary Problem From The Applicants Point of View'. However, once I realized that the MAA's gift was a copy of the issue of *College Mathematics Journal* in which my original article [2] appeared I opted to instead write this note about a kind of puzzle that I find fun to play.



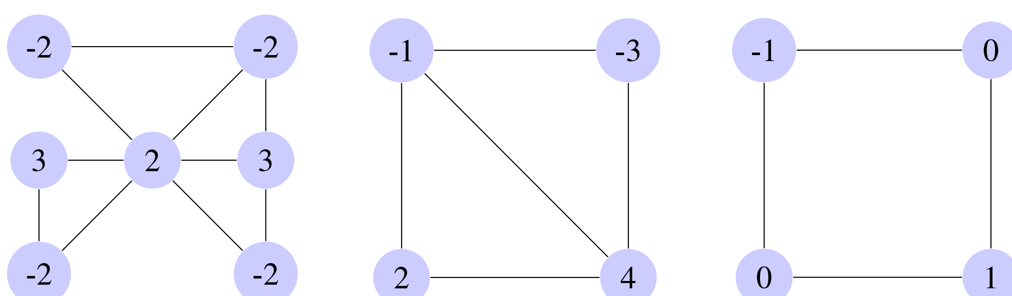
(a) Lending from right node (b) Lending from bottom node (c) Borrowing to bottom node

Note that the moves of borrowing and lending both preserve the total number of chips. In particular, if the total number of chips in the initial configuration is a non-negative number then it might be possible to do a sequence of moves that will eventually lead to none of the nodes being 'in debt.' We will refer to such a position as a 'winning position' and a sequence of moves that leads to a winning position will be called a 'winning strategy.' In our example, we begin with a total of two dollars to be shared between the nodes – can you find a legal sequence of moves that will result in all of the vertices being labelled with nonnegative numbers?

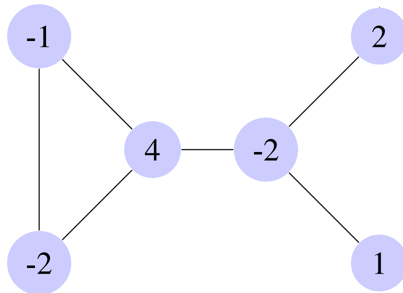
Here are several more examples to play with. For each example, try to either find a sequence of moves that will lead you to a winning position *or* explain why no such sequence exists.



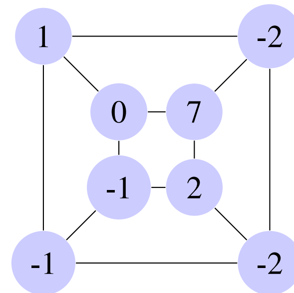
(a) Puzzle 1 (b) Puzzle 2 (c) Puzzle 3



(d) Puzzle 4 (e) Puzzle 5 (f) Puzzle 6



(a) Puzzle 7



(b) Puzzle 8

As you have likely already realized, it is not always clear when a winning strategy exists and when it does not. The following theorem due to Baker and Norine, appearing in [1], gives a partial answer to this question.

Theorem 0.1. *For a given graph, let E be the number of edges and N be the number of nodes of the graph. Furthermore, for any initial configuration on the graph let D be the total number of dollars in the system.*

1. *If $D > E - N$ then there is a winning strategy.*
2. *For any $D \leq E - N$ there will be initial configurations of D dollars with no winning strategy.*

However, like many existence theorems in mathematics, the proof is entirely nonconstructive and does not give any insight into how to find a winning solution. Doing so makes for a fun puzzle, and knowing this theorem gives one a quick way to come up with puzzles that can keep you occupied on long airplane flights or particularly boring committee meetings. Just draw a graph with N nodes and E edges and an initial configuration with a total of $E - N + 1$ dollars shared between the nodes and try to find a winning strategy. It is also interesting to try to come up with examples with $E - N$ dollars for which no winning strategy exists!

References

[1] Matthew Baker and Serguei Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. **215** (2007), no. 2, 766–788. MR 2355607 (2008m:05167)

[2] Darren Glass, *The secretary problem from the applicants point of view*, College Math J. **43** (2012), no. 1, 76–81.