Gwen's 65 Puzzle

In honor of Gwen Robert's 65th Birthday
by Karl Schaffer, Aug. 25, 2007

65 is a very special number:

$65 = 8^2 + 1^2 = 7^2 + 4^2 = 1^3 + 4^3$

Not only is 65 the smallest non-trivial positive integer that is the sum of two positive squares and also the sum of two positive cubes, it is the sum of two squares in two different ways! And so it is only fitting to honor Gwen Roberts on her 65th birthday with some puzzles based on the number 65!

Just to be complete, note that the smallest number that is the sum of two squares, and also two cubes, is $2 = 1^2 + 1^2 = 1^3 + 1^3$, which we will exclude from consideration, since Gwen has not been 2 for many years, and since it's pretty darn trivial anyway! We are also excluding non-positive solutions like $1 = 1^2 + 0^2 = 1^3 + 0^3$, and $26 = 1^2 + 5^2 = (-1)^3 + 3^3$.

These facts about 65 are easily confirmed by examining the smallest sums of two cubes:
$2 = 1^3 + 1^3$, $9 = 1^3 + 2^3$, $16 = 2^3 + 2^3$, $28 = 1^3 + 3^3$, $35 = 2^3 + 3^3$, and $54 = 3^3 + 3^3$, none of which are the sum of two non-zero squares (It has long been known that the only integers that are the sum of two squares are those which contain no odd powers of primes congruent to 3, mod 4, in their factorization – we ignore numbers like 9 here since it is the sum of 3^2 and 0^2, which is useless for making a puzzle!) Since 16 is also a perfect square, it makes a very simple puzzle along the lines of those included here: find a minimal dissection, along the boundaries of edge 1 polycubes, of the 4 by 4 by 1 "square" box such that the pieces in the dissection also can be reassembled into two 2 by 2 by 2 cubes. Or here is a simple mathematics problem: prove that the minimal dissection in this case is four 2 by 2 squares. We should note that another landmark birthday problem is provided by the smallest number that is the sum of two squares in two different ways: $50 = 5^2 + 5^2 = 7^2 + 1^2$. (I made that into a puzzle for Scott Kim's 50th birthday a few years ago, using a dissection of one set of squares that reassembles to make the other two squares – but the puzzle was way too easy!)

We will call the 4 by 4 by 1 box the "4-square," and similarly for other n-squares. We will use the term "area" to indicate the surface area of the largest face of the *n* by *m* box, which thus has "area" *nm*.

Problem 1: What is the minimal number of pieces into which we can dissect the 4-cube along unit boundaries, so that the pieces can be reassembled together with a 1-cube to make a 4-square and a 7-square?

Problem 2: What is the minimal number of pieces into which we can dissect the 4-cube along unit boundaries, so that the pieces can be reassembled together with a 1-cube to make a 4-square and a 7-square, or else an 8-square and a 1-square?

Problem 3: Can such dissections make an interesting physical puzzle that one can play with on one's 65th birthday?

Fact 1. 6 is the number of pieces in the minimal dissection, along unit boundaries, of the 4-cube and the 1-cube that reassembles to form the 4-square and the 7-square.
Proof: Each piece of the 4-cube dissection must have one of its dimensions equal to 1, otherwise that piece could not be used in constructing the "flat" 4- and 7-squares. Therefore, those pieces must have width and

height both 4 or less, and cannot have "area" greater than 16. Suppose the 1-cube is included in the 4-square in the simultaneous construction of the 4- and 7-squares. The 4-square thus would require at least two pieces for its construction. But the 49 square units in the 7-square's "area" would require at least Ceiling[49/16] = 4 pieces, and so 6 pieces would be required overall for this dissection. Suppose, on the other hand, that the 1-cube is instead included in the 7-square in the minimal dissection. The other 48 square units in the 7-square might be divided among only 3 pieces only if each piece has area 16, in other words is a 4 by 4 square. But it is impossible to construct a 7-square with three 4-squares and one 1-square (the width and depth of the resulting square would both be at least 8). So at least 4 pieces are required for the other 48 square units. We can thus possibly create a 6-piece dissection by using 5 pieces, including the unit cube, in the 7-square, and keeping the 4-square as one piece. Such a solution is shown in Fig. 1. By removing a corner square of the 4-square, and attaching the unit cube to one of the 3 by 4 rectangles shown in the 6-piece solution, we can get a different 6-piece solution, also shown in Figure 1.

Conjecture 1. All possible 6-piece dissection solutions to problem 1 are shown in Figure 1 and Figure 2.

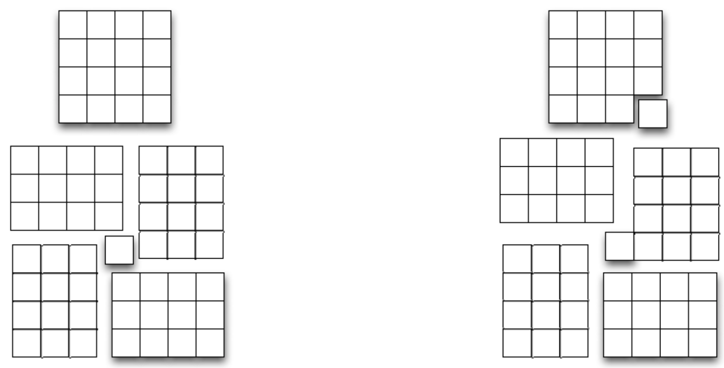

Figure 1

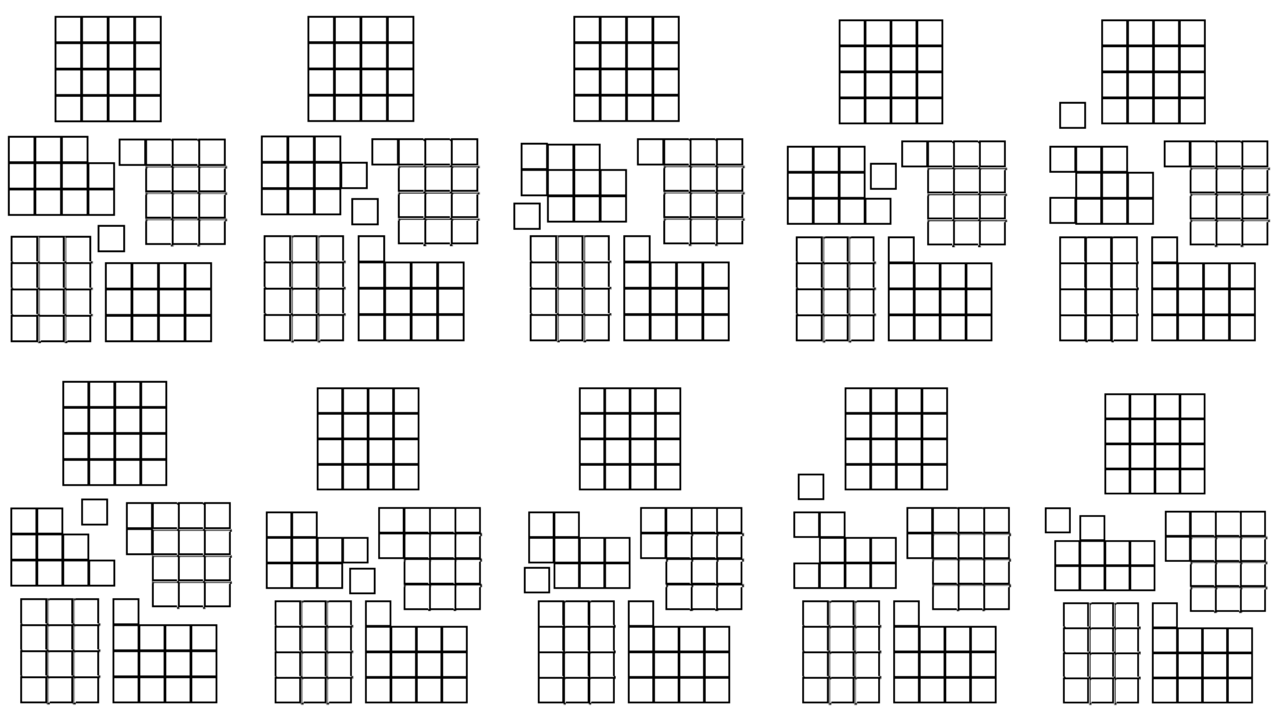

Figure 2

Conjecture 2. 7 is the smallest number of pieces that will dissect the 4-cube and the 1-cube along unit boundaries, and reassemble to form the 4- and 7-squares, or the 1- and 8-squares.
Outline of proof, based on assumption of Conjecture 1: We examine each of the possible 6-piece dissections in Conjecture 1 above. None will assemble the 1- and 8-squares. However, a 7-piece dissection is possible and is shown in Figure 3.

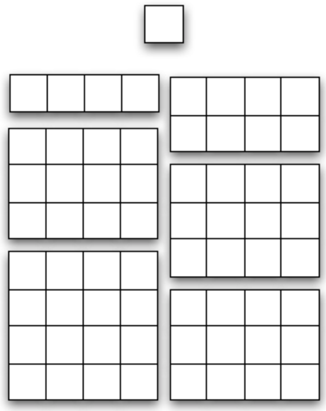

Figure 3

In order to make a puzzle that might be more fun to play with, the blocks have been colored and glued together in 32 pairs, with one remaining white cube, as shown in Figure 4 below. The manner in which they are glued is also shown in Figure 11, which also shows the solution to the 4- and 7-squares. If that manner of gluing the cubes together is too easy, then glue the pairs themselves together into 4-cube tetracubes! Some puzzles are shown on the page following Figure 4.

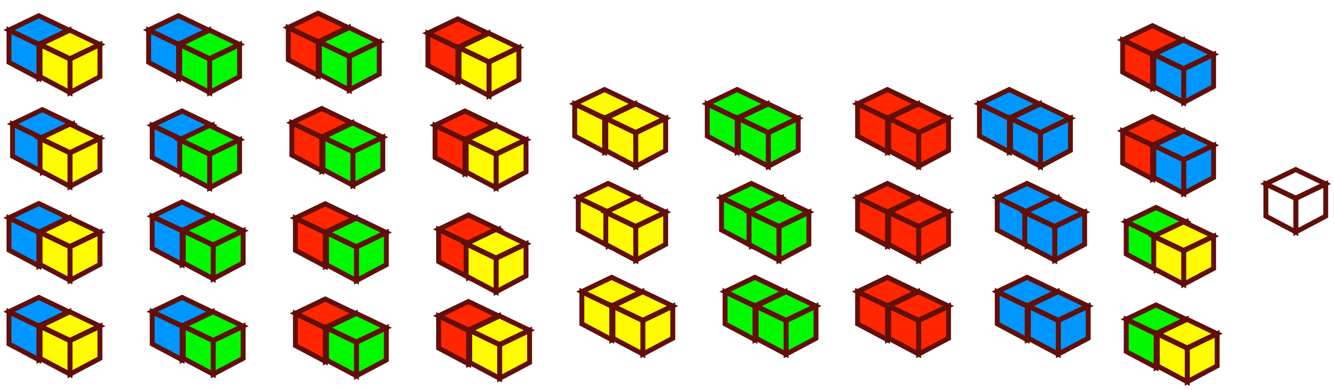

Figure 4

Puzzle 1: Use the colored blocks to form the 4 by 4 by 4 cube, with solid colored 2 by 2 by 2 cubes at each of the eight corners, shown in Figure 5. The corner not shown is green.
Puzzle 2: Use the blocks to form the 4-square and 7-square, with the pattern shown in Figure 6.
Puzzle 3: Use the blocks to form the 8-square and the 1-square with the pattern shown in Figure 7.
Puzzle 4: Use the blocks to form one set of 7 rectangles which can then be used to make the cubes or squares with the patterns shown in each of Figures 5, 6, and 7. Hint: use the solution shown in Figure 3.
Puzzle 5: Simultaneously make the five squares in Figure 8.
Puzzle 6: Simultaneously make the threee squares in Figure 9.
Puzzle 7: Make the rectangle in Figure 10.

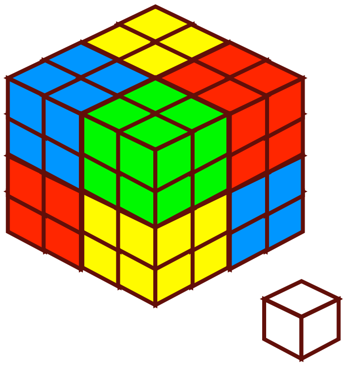

Figure 5

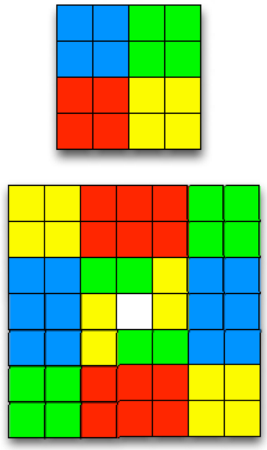

Figure 6

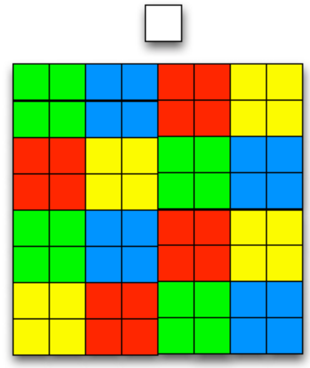

Figure 7

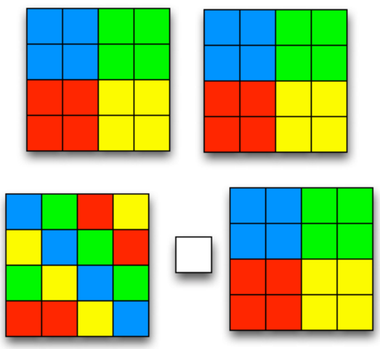

Figure 8

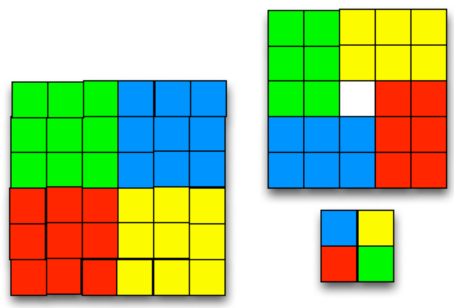

Figure 9

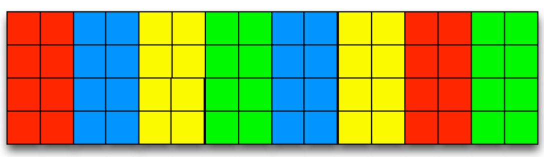

Figure 10

Figure 11 shows the solution to the 4- and 7-square problem (and also therefore shows how the cubes are glued together in pairs).

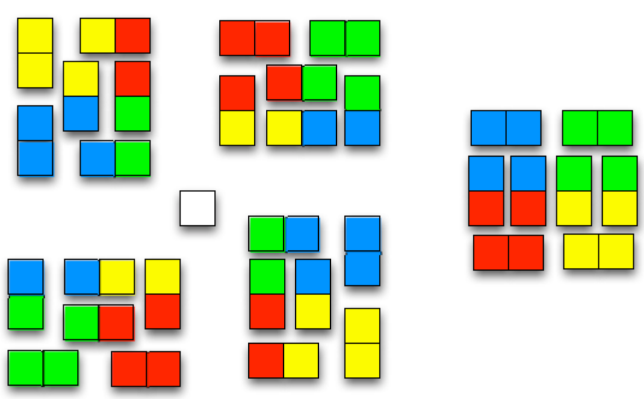

Figure 11. Solution to the 4- and 7-square problem