A STRATEGY FOR **Borders**
a variant of Dots & Boxes

Jason Colwell
Associate Professor of Mathematics
Department of Mathematics and Computer Science
Northwest Nazarene University
Nampa, ID

---

introduction In this article we describe a winning strategy for the game of Borders, a game proposed by the author in his article for the G4GX gift exchange.

First we recall the description of the game, and two differences between it and Dots & Boxes.

description The game is played on an $a \times b$ grid (the cells being 1×1). The sides of the cells we call edges. (Each edge borders one or two cells.) Their initial state is “empty”. A move consists of building a fence segment along an empty edge, after which the edge is no longer empty. If the fence built completes an enclosure of one or more cells, all cells within the enclosure are claimed by that player, except for any cells already claimed by the other player.

differences from Dots & Boxes The game Borders is different from Dots & Boxes in two ways:

- The building of a single fence segment may result in the player claiming multiple cells.
- Claiming a cell does not permit the player to build another fence segment.

The following conjecture was made by the author in the original article:
conjecture Suppose the game of Borders is played on an $a \times b$ board. The game is a win for the first player if $a + b$ is odd, and for the second player if $a + b$ is even.

In this article, we prove the conjecture by describing a winning strategy.

notation The grid is positioned in $\mathbb{R}^2$ with corners at $(0,0)$, $(a,0)$, $(0,b)$, and (a,b). For $1 \le i \le a$ and $0 \le j \le b$ we denote by

$$\overline{\boldsymbol{(i,j)}\mathbf{W}}$$

the fence segment from (i,j) "west" to $(i-1,j)$ (or the move consisting of building that fence). For $0 \le i \le a$ and $1 \le j \le b$ we denote by

$$\overline{\boldsymbol{(i,j)}\mathbf{S}}$$

the fence segment from (i,j) "south" to $(i,j-1)$ (or the move consisting of building that fence). We denote, for $1 \le i \le a,\ 1 \le j \le b$, by

$$\boxed{\boldsymbol{i,j}}$$

the cell with corners $(i-1,j-1)$, $(i,j-1)$, $(i-1,j)$, and (i,j).

"accessible" We say that one cell is "accessible" from another if they are adjacent and their common border is an empty edge.

"k-canal" By "k-canal", we mean a sequence of k cells bordered each by exactly two fence segments, each cell accessible from the previous one, and the first and last cells of the k-sequence not accessible from any cell outside the k-sequence having more than one fence segment bordering it. (Note the difference between the "canal" in Borders and what is termed a "chain" in Dots & Boxes.) Playing on a k-canal replaces it with a k-dead-end (if the new fence is built at one end) or with two canals whose lengths sum to k (if any other fence is built).

"k-dead-end" By "k-dead-end", we mean a sequence of k cells, each one accessible from the previous one, each cell except the last bordered by exactly two fence segments, the last bordered by three fence segments, and the first not accessible from any cell having more that one fence segment bordering it. To put it more intuitively, a k-dead-end is a k-canal closed off at one end. The k cells of a k-chain can be claimed in one move, and has a value of k to the player who plays on it.

"k-corral" By "k-corral", we mean a collection of k cells that can be claimed in one move. Note that a k-dead-end is a k-corral where any move will result in claiming some of the cells. Contrapositively, on any k-corral that is not a k-dead-end, a move can be made that will not result in the claiming of cells.

the strategy The strategy consists of attempting to perform the following Actions, listed in order of decreasing priority.

1. Do not create any dead-ends.
2. If there is a single corral, claim it.
3. If there are two corrals of equal size, claim one of them.
4. If there are two dead-ends of unequal length, claim enough squares from the closed end of the longer one to reduce its length to that of the shorter one.
5. if there is an edge where building a fence segment would create two corrals, one of which is not a dead-end, the two corrals containing numbers of empty edges of different parity, build a fence in the non-dead-end corral, so as not to claim any cells. This will cause the two potential corrals to contain numbers of empty edges of equal parity.
6. Play on a $2k$-canal, building a fence so as to split it into two k-dead-ends.

We shall call the "protagonist" the first player if $a + b$ is odd, or the second player if $a + b$ is even. The other player shall be called the "antagonist". The above strategy, when followed by the protagonist, will prevent him from losing. The game may end in a draw if the board has an even number of cells. In that case, an additional bit of strategy is necessary to ensure a win for the protagonist.

assumption If the protagonist follows the strategy just described, both players lose nothing by avoiding creating corrals as long as possible. We shall assume that they do this. The board's cells will then be eventually partitioned into canals (as well as, possibly, cells accessible from their ends that have no additional empty edges). If there are canals only of even length, the game will be a draw. On the other hand, as playing on an odd-length canal results in a loss of at least 1, both players will avoid this as long as possible, and play will continue until all even-length canals are split equally between the players, and only odd-length canals remain. The antagonist will have to play first on each of these, losing 1 point each time. Therefore, the game will be at best a draw for the antagonist, and a loss if there are any odd-length canals. If the protagonist can, while still following the strategy, ensure that there is at least one odd-length canal created, he can win the game. This additional bit of strategy will complete the proof that the protagonist has a win. The antagonist's possible moves are checked exhaustively. The first few moves by both players are described in the following discussion, and the subsequent moves left to the reader.

proof of strategy First note that as no more than two dead-ends can be created in a single move, Actions 1 and one of Actions 2, 3, and 4, can be performed, whenever one's opponent's move creates a dead-end. Any of these possibilities is of non-negative value to the player moving. Two corrals containing numbers of unbuilt edges of unequal parity will never simultaneously appear, due to Action 5 performed on a previous turn.

If the opponent's move does not create a dead end, then Action 1 may be performed except in the case where every place available to build a fence, belongs to a canal (and it will be to only one canal). The combination of Actions 1 and 5 will be of no loss to the player moving.

In the exceptional case, the player moving will try to perform Action 6, which has a value of 0. Only if Action 6 is impossible can there be loss to the moving player.

Note that at this point, if both players have played rationally, all squares claimed have been from even-length canals, split evenly between the two players. Each such canal now contains an even number of empty edges. Thus the number of empty edges contained in territory already claimed on the board is even. All unbuilt fence segments in unclaimed territory belong to odd-length canals, each of which has an even number of empty edges. This means that the number of fence segments is equal in parity to the total number of edges on the board, which is $2ab + a + b$. But this has the same parity as $a + b$, which means it would have to be the antagonist's, not the protagonist's, turn to move. That is, the protagonist will never have to deal with this situation where both Actions 1 and 6 are impossible.

It follows that the protagonist has at least a draw. $\square$

additional bit of strategy The strategy described so far is enough to prevent a loss, but a draw may still result – only if all the canals into which the board is partitioned are of even length. Otherwise, as described above, the antagonist – and him only – will be forced at least once to play on a canal of odd length, resulting in a strict advantage to the protagonist.

We now describe a sub-strategy, constituting "Action 7" in the overall strategy until the odd-length canal has been constructed. Effectiveness of this sub-strategy is seen by exhaustive search. Most of the essentially different possibilities are treated below. Additional possibilities are handled similarly, or are of obvious cost to the antagonist.

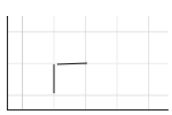
In the case where this sub-strategy is to be used, the board has an even number of cells; equivalently, at least one of a and b is even, say a. If the protagonist is the second player, there will be one fence already built when the protagonist begins using this strategy. In this case, the protagonist should choose the top row or the bottom row as necessary to avoid contact with the fence segment already built.

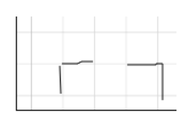
(The only way this last thing could be impossible is if $b = 2$ and the first move is some $\overline{(i,1)\mathrm{W}}$, but in this case it is relatively easy for the protagonist to make his first move $\overline{(1,1)\mathrm{W}}$ or $\overline{(a,1)\mathrm{W}}$ and ensure that a corner cell is bordered by two fences, which causes it to become a 1-canal, as desired.)

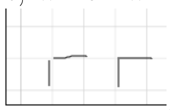
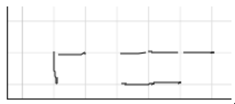
Say, without loss of generality, that the row chosen by the protagonist on which to begin play is the bottom row. The protagonist wishes to ensure that an odd canal is created in the bottom rom. The protagonist's first move is $\overline{(1,1)\mathrm{S}}$. If possible, his next move is to turn $\boxed{1,1}$ into a 1-canal by building $\overline{(1,1)W}$. But if the antagonist plays on $\boxed{1,1}$, making this impossible, then the protagonist's second move is $\overline{(2,1)\mathrm{W}}$.

The 10 essentially different states of the board after the next move by the antagonist, are depicted below. The response of the protagonist is shown in each case. Notes on the progress of the game after that, are given, with verification left to the reader.

Case 1

$\longrightarrow$  $\longrightarrow$ 

The protagonist should try to ensure that at least the fences  are built, which will force an 1-canal. Otherwise, a position containing one of

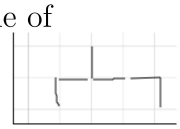
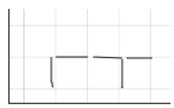
, , ,

can be achieved. All three involve shifting the play by an even number of cells to the right (2,4, and 4, respectively), where this same sub-strategy may be applied again. (If this occurs repeatedly, eventually the play will reach the right side of the board and result in an odd-length canal.)

Case 2

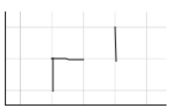
$\longrightarrow$  $\longrightarrow$ 

One of

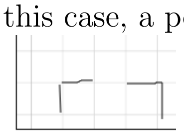
, , ,

can be achieved. In the first one, the objective has been accomplished. In the second and third, the play has been shifted two cells to the right, where this same sub-strategy may be applied again.

Case 3

$\longrightarrow$  $\longrightarrow$ 

In this case, a position containing

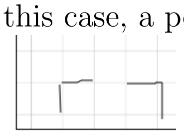
 or 

can be achieved, the first ensuring an 1-canal and the second shifting the play two cells to the right.

Case 4

$\longrightarrow$  $\longrightarrow$ 

In this case, a position containing

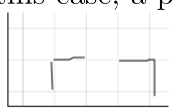
 or 

can be achieved, the first ensuring a 1-canal and the second shifting the play two cells to the right.

Case 5

$\longrightarrow$  $\longrightarrow$ 

In this case, a position containing

 or  or 

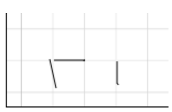
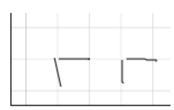
can be achieved, the first ensuring an 1-canal and the other two shifting the play two cells to the right.

Case 6

$\longrightarrow$  $\longrightarrow$ 

Here, a 1-canal is already assured.

Case 7

$\longrightarrow$  $\longrightarrow$ 

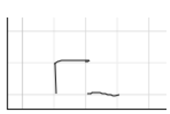
Here, the play is shifted two cells to the right.

Case 8

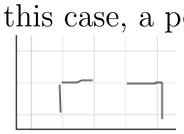
$\longrightarrow$  $\longrightarrow$ 

Here, a 1-canal is already assured.

Case 9

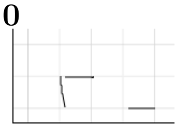
$\longrightarrow$  $\longrightarrow$ 

In this case, a position containing

 or  or 

can be achieved, the first ensuring a 1-canal and the other two shifting the play two cells to the right.

Case 10

$\longrightarrow$  $\longrightarrow$ 

Here, the play is shifted two cells to the right.

---

plans for future research The author plans to write a web program for the game of Borders, implement the strategy described in this article for the server's play against a human, and see how difficult it is, on an even-size board, for the human antagonist to force a draw (by making sure all canals created are even) if the protagonist server does not use the "additional bit of strategy". It would be nice to replace the exhaustive checking of cases by a more elegant formulation.

©2014 Jason Colwell. All rights reserved.