<u>**G4G11 Exchange Gift**</u>

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We briefly present two topics as a gift to G4G11 participants in honour of John Horton Conway and the late Martin Gardner. Both topics will be elaborated on in upcoming publications.

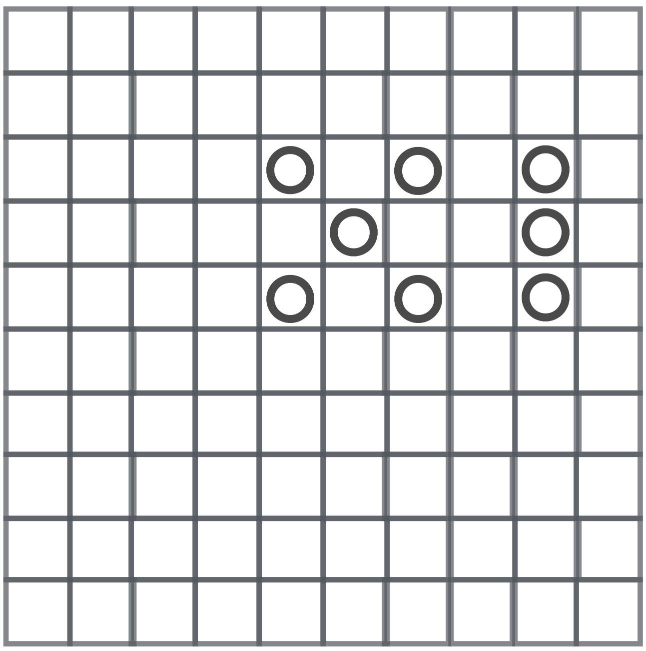
<u>1. Can you solve these “Retrolife” puzzles?</u>

Assume an infinite grid or board and some black and white counters. Each cell on the board can contain either a white counter, a black counter or the cell can be empty. An initial pattern is placed on the board using white counters. We use the term "surround" to mean "placing a counter in a neighbouring cell".

The problem:

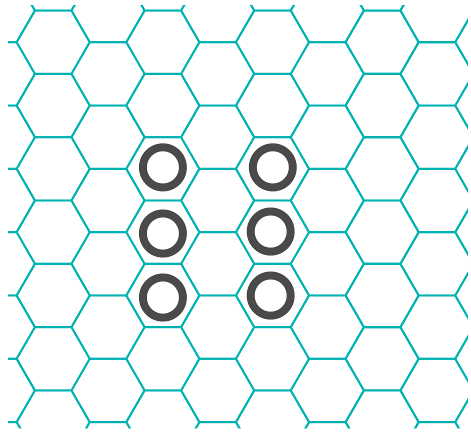
“Surround each white counter on the board with *t* black counters so that each black counter is <u>not</u> surrounded with *n* black counters and each empty square on the board is <u>not</u> surrounded by *e* black counters, where *t*, *n* and *e,* denote sets of whole numbers between 1 and 8. A formidable challenge is to find the *minimum* number of black counters that can be used to solve the puzzle”.

The following “classic” Retrolife problem uses the parameters: $t = \{3\}$, $n \neq \{2,3\}$ and $e \neq \{3\}$. The equality and inequality signs denote permitting or forbidding the surrounding of cells by the specified number of counters, respectively, and the comma is to be understood as the logical OR operation. Hence, the following puzzle is to surround each white counter on the board with 3 black counters in such a way that no black counter is surrounded by either 2 or 3 black counters, and no empty square on the board is surrounded by 3 black counters.



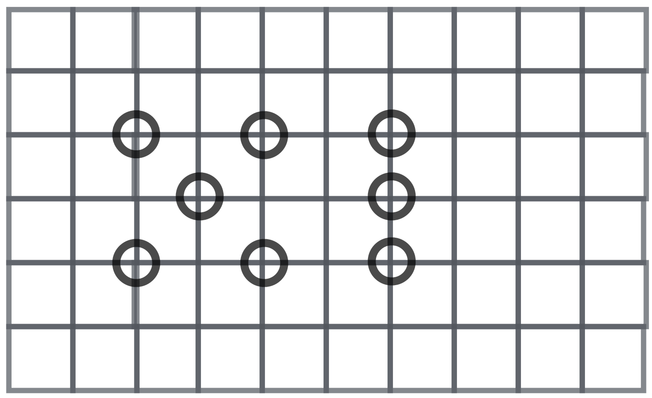
$t = \{3\}, n \neq \{2,3\}, e \neq \{3\}$

Here are two more “Retrolife” puzzles, using different topological boards.



Solve this hexagonal grid problem given:

$t = \{3\}, n \neq \{2,3\}, e \neq \{3\}$



Solve this puzzle, where the counters are placed on the gridlines of the board and with:

$t = \{2\}, n \neq \{2\}, e \neq \{2\}$

2. The Generalised Apex Magic Trick, Pascal’s Triangle and Fractals

The five-card magic trick was introduced by Martin Gardner in his book: “Mathematical Carnival” and further studied and generalised by “Card Colm Mulcahy” and others. Five cards (face valued 1-9) are chosen by a spectator and placed in a row. The magician predicts a certain card and writes down his prediction. Each pair of adjacent cards in the row are now summed and a card with the face value of the sum is chosen from the pack and placed in a row above and in between the two cards. If the sum is larger than 9, the two digits of the sum are added to get a one-digit number and this is the face value of the corresponding card in the row above. As a result of this, a four card row of the (MOD (9)) sums of the row below is created. This process is repeated thrice to create a three-card row, a two-card row and finally one card - the “apex” of the triangle of cards that has just been formed. This card of course, turns out to be the “predicted” card.

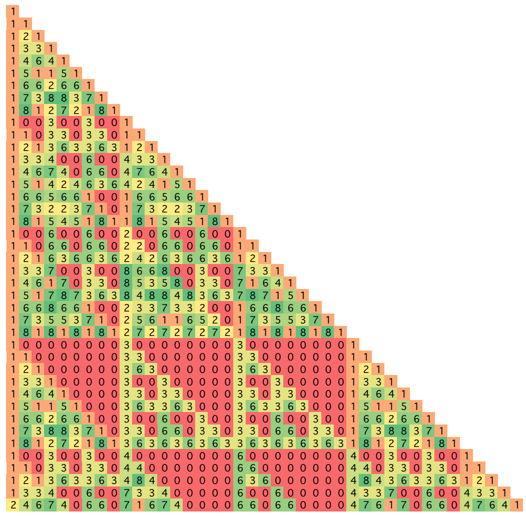
The math behind this trick, explained in many references, is based on the numbers in the fifth row of Pascal’s triangle MOD(9). This row is: 1,4,6,4,1. Consequently, the magician “predicts” the “apex” card by multiplying the face value of the spectator’s five cards by 1,4,6,4,1 respectively and adding up the sum MOD(9). Suppose these cards are: Ace, 3,8,9,8, the apex card will be (**1**x1+**4**x3+**6**x8+**4**x9+**1**x8) MOD(9) = 6.

This trick can be played with any initial number of cards using the corresponding row in Pascal’s triangle. It is, however, preferable for the magician to choose a number that

corresponds to a row in Pascals triangle that contains a lot of zeros MOD (9), so that mentally adding the numbers in the row is quick and easy. The tenth row for example is 1,0,0,3,0,0,3,0,0,1 and can be used for a ten card trick. There is an infinite number of similar rows, where the magician only needs to remember four numbers in the bottom row of the triangle of cards. The next row is the twenty-eighth row:
1,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,1

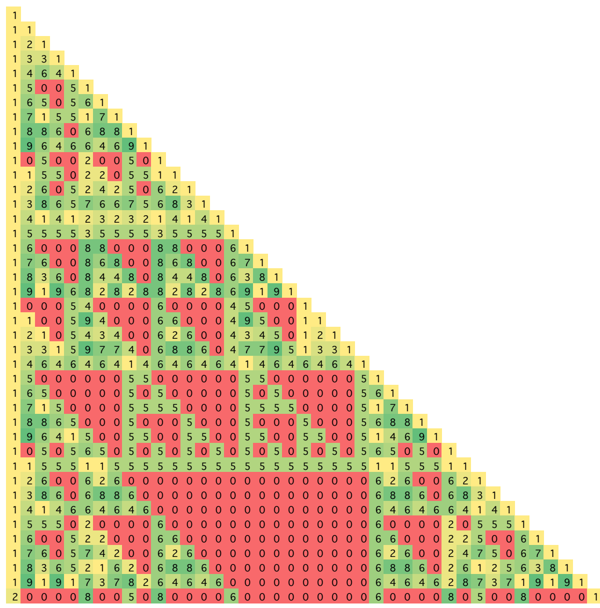
and after that 82, 244 and 2188 - in short - 9^n+1. These rows can easily be spotted by observing the fractal generated by the MOD (9) Pascal triangle. The fractal also demonstrates the general rules for the repetition of any row combination.

The five-card trick can also be performed using different moduli. In some moduli, such as MOD (3), there are rows that consist entirely of zeros except the first and last numbers in the row that are 1. In these cases, termed "Φ-simple", all the magician has to do to determine the apex number is to sum the first and last card in the spectators row, using the correct modulus. Erhard Behrends and Steve Humble studied these cases in depth in an article that recently appeared in the "Mathematical Intelligencer". It turns out that the rule they give for "Φ-simple" rows is the same rule that we showed for the case of MOD (9). Indeed, this rule does in fact generate rows with the *minimal* number of non-zero elements for non-"Φ-simple" moduli.



MOD (9) Pascal Triangle Fractal

The MOD (10) case presents a "neat" magic trick. Write 11 numbers in a row. Add each pair of numbers, keeping only the units digit and create a triangle as before. Although the MOD (10) Pascal triangle is not "Φ-simple" - as can be observed by its corresponding fractal - the prediction is very easy. Multiply the value of the spectator's first and last cards by 1, his fourth and eighth card by 5 and his sixth card by 2 - and then sum the results MOD (10). If the fourth and eighth card have the same parity, they can be ignored, so summation of the first, last and twice the middle card is all that is needed and the prediction is made.



MOD (10) Pascal Triangle Fractal

We hereby invite many more magic tricks that can be based on the enigmatic Pascal Triangle.