Solving Puzzles Backwards

Anany Levitin

In addition to the entertainment value of puzzles, some of them can be very effective in demonstrating general problem solving strategies. For example, puzzles can be used to illustrate solving problems backwards, which is the subject of this short paper.

Backward Reasoning

Solving problems backwards is usually understood as *regressive reasoning*, which is one of the oldest approaches to problem solving. George Polya has traced it back to mathematicians of ancient Greece. He paraphrases Pappus, for example, who lived around 300 A.D., as follows: “In analysis, we start from what is required, we take it for granted, and we draw consequences from it, and consequences from the consequences, till we reach a point that we can use as starting point in synthesis… This procedure we call analysis, or solution backwards, or regressive reasoning.” [7, p. 142]

I'm going to give several examples of puzzles illustrating this strategy. Given the occasion — the eleventh Gathering for Gardner — it is appropriate to start with a puzzle from Martin Gardner's *The Colossal Book of Short Puzzles and Problems*:

Interrupted Bridge Game A telephone call interrupts a man after he has deal about half of the cards in a bridge game. When he returns, no one can remember where he had dealt the last card. Without learning the number of cards in any of the four party dealt hands, or the number of cards yet to be dealt, how can he continue to deal accurately, everyone getting exactly the same cards he would have had if the deal had not been interrupted? [3, p.229]

Fred Schuh, a prominent Dutch researcher of puzzles, has also recognized regressive reasoning as an important general technique. As an example of its applications, he gives the following coin-row transformation puzzle:

Coin-Row Transformation Puzzle Transform in as few moves as possible a row of three dimes separated by two quarters to a row with all the dimes to the left of both quarters. On every move, a dime and an adjacent quarter are to be moved as one whole to remain adjacent, but one is not allowed to reverse their order [8, p. 17].

Another puzzle in which regressive reasoning leads to an easy solution was provided recently by Dick Hess:

Trapping the Knight What is the minimum number of moves on an infinite chessboard needed for a knight to reach a position from which it can move only to a previously visited square? [5, p. 28]

A more challenging puzzle where regressive reasoning helps not only to determine the puzzle's instances for which it has a solution but also helps to design an algorithm to achieve it is W. Lloyd Milligan's version of an old puzzle proposed by Henry Dudeney:

Crowning the Checkers An even number of checkers, n, are placed in a row. First, move a checker over a checker to make a king, then move a checker over two checkers, then a checker over three checkers, and so on, each time increasing by one the number of checkers to be passed over. The objective is to form $n/2$ kings in $n/2$ moves.” [3, p. 271]

My last example is a minor variation of the puzzle invented by James Propp:

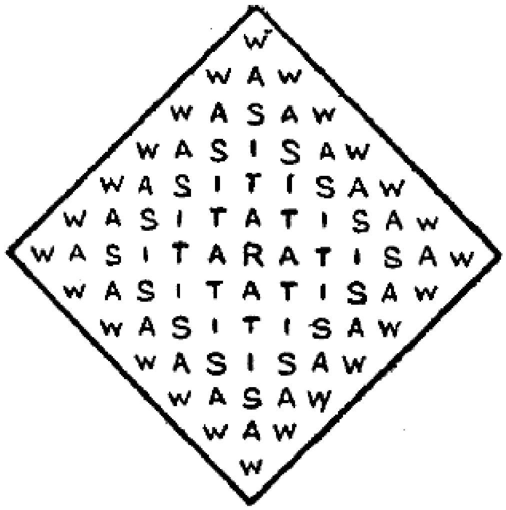
Penny Distribution Machine The "machine" consists of a row of boxes. To start, one places n pennies in the leftmost box. The machine then redistributes the pennies as follows. On each iteration it replaces a pair of pennies in one box with a single penny in the next box to the right until there is no box with more than one coin. Is it true or false that the final distribution of pennies never depends on the order in which the machine processes the coin pairs? [6, p. 62].

Topological Reverse

Topological reverse was introduced by David Ginat and Michal Armoni as follows. “There is a natural direction associated with several basic data structures, such as strings, lists, and directed graphs. In such data structures, it is natural to devise a computation based on the given topological ordering, e.g., from sources to sinks. However, sometimes it may be much more efficient to perform the computation in a direction opposite to that of the topological ordering; e.g., from last to first, or from sinks to sources.” [4]

The following classic puzzle by Dudeney perfectly illustrates this variety of solving problems backwards:

Palindrome Counting In how many different ways can the palindrome WAS IT A RAT I SAW be read in the diamond-shaped arrangement of letters shown below? You may start at any W and go in any direction on each step — up, down, left, or right — through adjacent letters. The same letter can be used more than once in the same sequence [2, Problem 30].



Problems with a given result

The third variety of puzzles related to backward problem solving are puzzles in which the objective is to find an input for which a given algorithm produces a given output. Here is the classic representative of such recreational problems:

The Josephus Problem Place *n* men round a circle so that if every *m*th man is killed, the remainder shall be certain specified individuals [1, pp. 32–36].

Note that in such problems solving backwards is not a solving strategy but rather a problem type.

Finally, we should mention as a related problem type so-called retro puzzles. These are chess problems to determine which moves were played leading up to a given position. For examples of such problems, see two books by Raymond Smullyan [8], [9].

References

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