Three Pairs of Problems

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We present three pairs of problems.

Problem 1A.
Kim and his friend Abdullah were going from Lahore to Benares along the scenic Grand Trunk Road, which would take them through Umballa, Delhi and Alighur. Excited about their adventure, they went through Umballa and reached Delhi in a week. Then Abdullah wanted to go back while Kim wanted to push on. Nevertheless, they stayed together. However, the dispute had slowed them down, so that they only made it to the next stop each week. Whenever they were in Umballa, Delhi or Alighur, the argument about which way to go would flare up again. Eventually, they reached Benares, but only after having changed directions ten times. How many different week-by-week itineraries could they have followed?

Problem 1B.
Kim, Abdullah and their friend Chota Lal were going from Lahore to Alighur. Kim and Abdullah went along the scenic Grand Trunk Road which would take them through Umballa and Delhi. Chota Lal left Lahore at the same time, but along a straight road directly to Delhi. At Umballa, Abdullah went on a striaght road directly to Alighur while Kim continued on the Grand Trunk Road. At Delhi, Chota Lal rejoined Kim as they both arrived at the same time, and they pushed on without stopping. Finally, all three of them arrived at Alighur simultaneously. Kim travelled at the same constant speed, whether alone or in company. Each of Abdullah and Chota Lal travelled at a constant speed when alone. There was a statue of Rudyard Kipling at the intersection of the two straight roads. Was it possible for Abdullah and Chota Lal to have met each other there?

Problem 2A.
Forty Thieves, ranked 1 to 40, were trying to cross a river in a boat which took two of them to row. However, if the ranks of two of them differ by more than 1, they would refuse to be in the boat together. This meant that only two could cross at a time, but the same two must bring he boat back. Their leader, whose rank was 1, appealed to Ali Baba for assistance. As it happened, Ali Baba also wanted to cross the river. "I can get you guys over," he said, "if you make me an honorary thief with same rank as you." After some hesitation, the leader accepted Ali Baba as his equal, and Ali Baba master-minded the operation. What was the minimum number of one-way crossings required to get everyone across

Problem 2B.
Forty Thieves, ranked 1 to 40, had some gold coins in their possession equal to their respective ranks. Someone else had stolen 41 gold coins from the Royal Mint. If any group of the Forty Thieves had exactly 41 gold coins among them, they would all be hanged. Ali Baba was ordered to round them up and put them in prison, entering two at a time. One of the two could surrender all his gold coins to the other, and then Ali Baba could release him immediately. No trading of gold coins could occur in prison as the thieves were held in individual cells. Naturally, no thief was willing to yield his gold coins, unless he was ordered to do so by Ali Baba. What was the maximum number of thieves Ali Baba could have in prison and yet save them all?

Problem 3A.

An Evil Witch had imprisoned Princess Anna. When Prince Boris got to the Evil Castle, he found fifteen veiled ladies being formed into a line by the Evil Witch. “One of these veiled ladies is your Princess Anna. Seven of the others were transformed by me from poisonous frogs. The remaining seven were poisonous toads. After I have opened the door to my castle, they will go inside and sit down in order. Then you can come in and kiss any of them. If she is indeed the Princess, the two of you are free to go. If not, you will find out whether you have just kissed a poisonous frog or a poisonous toad. This information will not be of much use to you,” she laughed, “because you will die from the kiss.” While the veiled ladies followed the Evil Witch into the Evil Castle, the Fairy Godmother materialized beside Prince Boris. “I will help you in two ways. First, here are three life-saving pills,” she whispered as she handed them to him. “Second, I will give you a divine revelation. The first seven veiled ladies were poisonous frogs and the next seven were poisonous toads. Princess Anna is last in line.” Then she vanished. Feeling that he did not even need the pills, Prince Boris stepped confidently into the Evil Castle. To his dismay, the veiled ladies were in clockwise order at a round table, with no indication of who was the first to sit down. Could he still rescue Princess Anna?

Problem 3B.

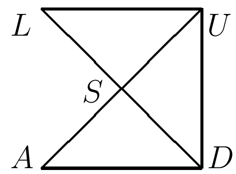
After he had imprisoned Prince Boris, an Evil Wizard granted Princess Anna an audience. “Here are thirty-five cards, each with a number on the back,” he said as he dealt them on the table so that they formed a five by seven array. For each card, the side with the number faced down, so that Princess Anna could not see it. Then he continued, “You may turn over one of the cards and read the number on it. You must then tell me the sum of all thirty-five numbers. If it is correct, you may take your Prince Boris with you. If not, you will suffer a horrifying death.” While he closed his eyes in glee as he planned Princess Anna’s demise, the Mafia Godfather materialized beside Princess Anna. “I will give you a divine revelation. In any two by three or three by two array, the sum of the numbers on the six cards is always 336.” He vanished just as the Evil Wizard opened his eyes and fixed his gaze at Princess Anna. Could she rescue Prince Boris?

Solution to Problem 1A.

Let k_n denote the number of possible itineraries for Kim and Abdullah if they changed directions n times. We claim that $k_n = k_{n-1} + k_{n-2}$. Consider first the case where n is even. This means that they would go onto Benares. Hence the last change in directions must occur in Delhi or Umballa. If it occured in Delhi, this means that he must have returned from Alighur, and had gone there earlier from Delhi. If we shorten the last segment “Delhi — Alighur — Delhi” to just “Delhi”, we will get a path counted in k_{n-2}. On the other hand, if the last change in directions occured in Umballa, this means that he could have gone back to Lahore had he not done so. If we replace the segment “Delhi — Alighur — Benares” at the very end by “Lahore”, we will get a path counted in k_{n-1}. The case where n is odd can be handled in an analogous manner. Just interchange “Lahore” with “Benares” and “Umballa” with “Alighur”. This justifies the claim. It follows that k_n are just the Fibonacci numbers, with $k_0 = 1$ and $k_1 = 2$. Hence $k_2 = 3$, $k_3 = 5$, $k_4 = 8$, $k_5 = 13$, $k_6 = 21$, $k_7 = 34$, $k_8 = 55$, $k_9 = 89$ and $k_{10} = 144$.

Solution to Problem 1B.

Let L, U, D, A and S be the locations of Lahore, Umballa, Delhi, Alighur and the statue of Rudyard Kipling. If Abdullah and Chota Lal do meet each other at S, this must occur after Abdullah left U and before Chota Lal arrived at D. Let the lengths of these two time intervals be x and y respectively. Let the constant speeds of Kim, Abdullah and Chota Lal be a, b and c respectively. Since $LU + UD > LD$, $a > c$. Since $UD + DA > UA$, $a > b$. Now $UD = a(x + y)$, $US = cx$ and $SA = by$. We have $US + SA = cx + by < a(x + y) = UA$, which is a contradiction. It follows that Abdullah and Chota Lal cannot have met each other at S.



Solution to Problem 2A.

We first show that 153 crossings are sufficient. In the first 8 crossings, we get thieves 39 and 40 over to the far shore, as shown in the chart below.

Crossing Number	Ranks of thieves on Near Shore	in Boat	on Far Shore
First	3,4,...,40	1,1,2	1,1,2
Second	1,2,...,40	1,2	1
Third	1,4,5,...,40	2,3	1,2,3
Fourth	1,1,2,4,5,...,40	1,2	3
Fifth	4,5,...,40	1,1,2	1,1,2,3
Sixth	2,3,...,40	2,3	1,1
Seventh	2,3,...,38	39,40	1,1,39,40
Eighth	1,1,2,...,38	1,1	39,40

Note that apart from thieves 39 and 40, everybody is back on the near shore. Using the procedure, we can get thieves 37 and 38 across in another 8 crossings, and so on. Finally, one more crossing will accomplish the task. The total number of crossings is indeed $8 \times 19 + 1 = 153$. We now show that 153 crossings are necessary. Consider the crossings in pairs, one to the far shore and the very next one back to the near shore. A gain for this pair is defined as an increase of the number of thieves on the far shore after this pair of crossing is completed. When two thieves cross over, the gain is obviously 0 as two thieves must come back. The gain is 1 when three thieves cross over together. The very last crossing, which is to the far shore, generates a gain of 3. To get $n+1$ thieves across, we must of course gain $n+1$. So three thieves must cross over together $n-1$ times. Obviously, this cannot happen in every crossing from the near shore, because in that case, thieves 1, 1 and 2 must all come back. Now two of them can come back right away, and one of them can cross over next time to fetch the third. So this can happen every other crossing from the near shore. So there must be at least 3 other crossings between two crossings from the near shore with three thieves. This means that we have to add $n-2$ sets of 3 crossing to the total. So the minimum number of one-way crossings is $(n-1) + 3(n-2) = 4n - 7$. For $n = 40$, we have $4n - 7 = 153$.

Solution to Problem 2B.
For any pair of thieves who have 41 gold coins between them, at least one of them must either give up his gold coins or receive the gold coins from another thief. Since there were twenty such pairs, gold coins must change hands at least ten times. Ali Baba could have the twenty thieves with odd numbers of gold coins enter the prison in pairs. One thief in each pair gave all his gold coins to his cellmate and was released. Each of the thirty thieves in priosn had an even number of gold coins in his possession, and no group of them could have exactly 41 gold coins among them.

Solution to Problem 3A.
Number the veiled ladies 1 to 15 in clockwise order. Prince Boris kisses number 8. There are three cases.
Case 1. Number 8 is Princess Anna.
Then the mission is accomplished.
Case 2. Number 8 is a poisonous frog.
Then number 15 must be a poisonous toad and Princess Anna is not between these two. Prince Boris takes the first life-saving pill and kisses number 4. There are three subcases.
Subcase 2(a). Number 4 is Princess Anna.
Then the mission is accomplished.
Subcase 2(b). Number 4 is another poisonous frog.
Then number 5 to number 7 are all poisonous frogs. Prince Boris takes the second life-saving pill and kisses number 2. If it is Princess Anna, the mission is accomplished. If it is another poisonous frog, then number 1 is Princess Anna. If number 2 is a poisonous toad, then number 3 is Princess Anna. In either case, Prince Boris takes the last life-saving pill and accomplishes his mission.
Subcase 2(c). Number 4 is a poisonous toad.
Then number 1 to number 3 are all poisonous toads. Prince Boris takes the second life-saving pill and kisses number 6. The analysis is analogous to Subcase 2(b).
Case 3. Number 8 is a poisonous toad.
Then number 1 must be a poisonous frog and Princess Anna is not between these two. Prince Boris takes the first life-saving pill and kisses number 12. The analysis is analogous to Case 2.

Solution to Problem 3B.
Princess Anna partitions the 5×7 table into six 2×3 or 3×2 rectangles, two of which overlaps at the central square. She knows that the sum of all 35 numbers is 6×336 minus the number on the card in the central square. Confidently, she turns over that card and accomplishes her mission by annoucing the correct sum.