Problem.
Peter and Betty take turns in a game with n stones in each of m piles. In his turn, Peter must take 1, 2 or 3 stones from any one pile. In her turn, Betty must take one stone from 1, 2 or 3 piles. Whoever takes the last stone overall is the winner. Determine for each n and m who has a winning strategy, according to who moves first.

First, let's consider the case when $n = m$.

Observation 1.
For $n = 1, 2, 3$, it is easy to prove that whoever moves first wins.

Observation 2.
For $n = 4$, whoever moves first loses.

(1) Suppose Peter moves first.

  (a) After Peter's first move, he leaves behind (4442), (4442) or (4441).
  (b) Betty responds by leaving behind (3333), (3332) or (3331).
  (c) After Peter's second move, he leaves behind (3332), (3331), (333), (3322), (3321), (332) or (3311).
  (d) Betty responds by leaving behind (2222), (2221), (222), (2211) or (221).
  (e) After Peter's third move, he leaves behind (2221), (222), (2211), (221), (22), (2111), (211) or (21).
  (f) Betty responds by leaving behind (1111), (111) or (11).
  (g) After Peter's fourth move, he leaves behind (111), (11) or (1).
  (h) Betty responds by leaving behind (0) and wins.

(2) Suppose Betty moves first.

  (a) After Betty's first move, she leave behind (4443), (4433) or (4333).
  (b) Peter responds by leaving behind (444), (443) or (433).
  (c) After Betty's second move, she leaves behind (443), (442), (433), (432), (422), (333), (332) or (322).
  (d) Peter responds by leaving behind (44), (43), (42), (33) or (32).
  (e) After Betty's third move, she leaves behind (43), (42), (41), (33), (32), (31), (22) or (21).
  (f) Peter responds by leaving behind (4), (3) or (2).
  (g) After Betty's fourth move, she leaves behind (3), (2) or (1).
  (h) Peter responds by leaving behind (0) and wins.

Conjecture.
For $n \geq 5$, Betty wins no matter who moves first.

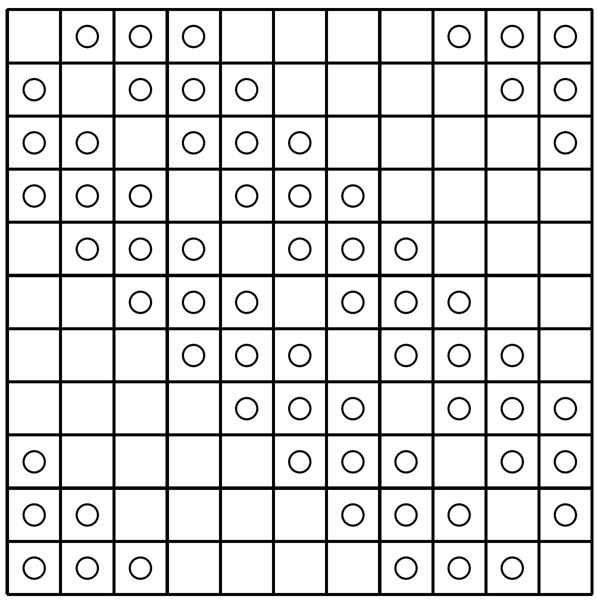
Generalizing as $m > n$:

Solution:

We first assume that Peter goes first. We consider three cases.

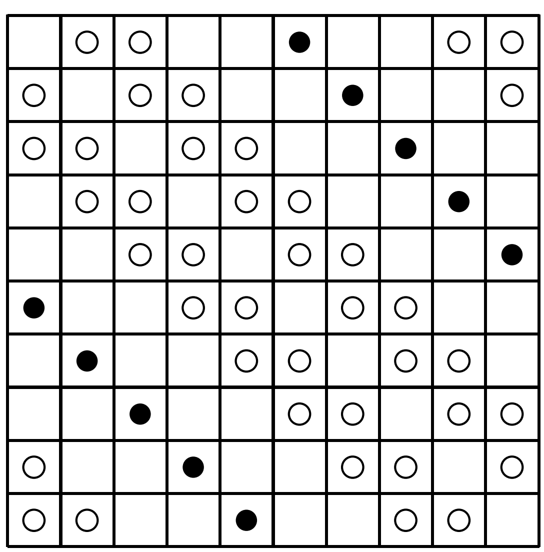
Case 1. n is even.

Draw an $m \times m$ grid, with each row representing a pile. Place stones on (i, j) if and only if $|i-j| \leq \frac{n}{2}$ or $|i-j| \geq m - \frac{n}{2}$. Then there are n stones in each of the m rows, none lying on the main diagonal where $i = j$. The entire configuration is symmetric about this main diagonal. Betty's strategy is to take (j, i) whenever Peter takes (i, j). By symmetry, Betty gets the last stone and wins. The diagram below illustrates the case $m = 11$ and $n = 6$.

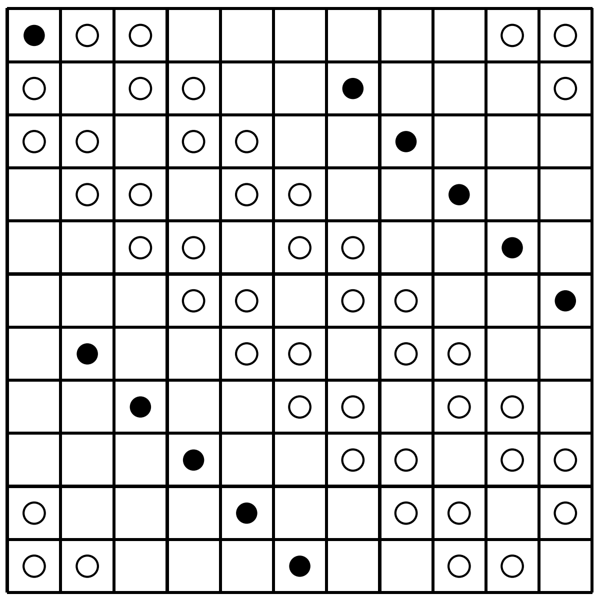


Case 2. n is odd but m is even.

Here $n - 1$ is even. So we temporarily replace n by $n - 1$, and draw an $m \times m$ grid as in Case 1. We now add a stone to each row at (i, j) where $|i - j| = \frac{m}{2}$. The argument is then the same as in Case 1. The diagram below illustrates the case $m = 10$ and $n = 5$.



Case 3. n is odd and m is also odd.
Here $n-1$ is even. So we temporarily replace n by $n-1$, and draw an $m \times m$ grid as in Case 1. We now add a stone to each row except the first at (i, j) where $|i-j| = \frac{m-1}{2}$, and another stone at (1,1). As in Case 1, the overall configuration is symmetric about the main diagonal, even though there is now a stone at (1,1). Since each pile has the same number of stones at the start, Betty may assume that Peter takes stones from the first pile, and that they include the one at (1,1). If Peter takes at least two stones, Betty can take the stones in matching positions except for (1,1). If Peter takes (1,1), Betty can take (1,2) and (2,1). After this, she can use the symmetry strategy and wins. The diagram below illustrates the case $m = 11$ and $n = 5$.



Suppose Betty goes first. In Case 3, she just removes (1,1) the lone stone on the main diagonal. In Cases 1 or 2, she removes any pair of stones symmetric about the main diagonal. Thereafter, she uses the symmetry strategy as before.

All that is left is to consider when $m < n$

Computer Search Results by Brian Chen

Black circle means Peter wins. White circle means Betty wins. Rows are piles. Columns are stones.

Peter moves first:

P	1	2	3	4	5	6	7	8	9	10	11	12
1	●	○	○	○	○	○	○	○	○	○	○	○
2	●	●	○	○	○	○	○	○	○	○	○	○
3	●	●	●	○	○	○	○	○	○	○	○	○
4	●	●	●	○	○	○	○	○	○	○	○	○
5	●	●	●	●	○	○	○	○	○	○	○	○
6	●	●	●	●	●	○	○	○	○	○	○	○
7	●	●	●	●	●	○	○	○	○	○	○	○
8	●	●	●	●	●	●	○	○	○	○	○	○
9	●	●	●	●	●	●	●	○	○	○	○	○
10	●	●	●	●	●	●	●	○	○	○	○	○
11	●	●	●	●	●	●	●	●	○	○	○	○
12	●	●	●	●	●	●	●	●	●	○	○	○

Betty moves first:

B	1	2	3	4	5	6	7	8	9	10	11	12
1	○	○	○	○	○	○	○	○	○	○	○	○
2	●	○	○	○	○	○	○	○	○	○	○	○
3	●	●	○	○	○	○	○	○	○	○	○	○
4	●	●	●	⦿	○	○	○	○	○	○	○	○
5	●	●	●	●	○	○	○	○	○	○	○	○
6	●	●	●	●	○	○	○	○	○	○	○	○
7	●	●	●	●	○	○	○	○	○	○	○	○
8	●	●	●	●	●	○	○	○	○	○	○	○
9	●	●	●	●	●	○	○	○	○	○	○	○
10	●	●	●	●	●	●	○	○	○	○	○	○
11	●	●	●	●	●	●	●	○	○	○	○	○
12	●	●	●	●	●	●	●	○	○	○	○	○

Closing Remarks

The motivation for this problem is from a problem presented in the mathematics contest Tournament of the Towns. It's origin is from the Fall 2013, A-level contest. It is a unique variation from the well known game of Nim.

Upon first inspection, it seems as though there is some symmetry between the players Peter and Betty, however it turns out not to be so simple. By the computer search results by Brian Chen, it seems as though Betty actually has an advantage over Peter, but the unsolved problem is to what extent does Betty have an advantage. In other words, what happens when $n > m$?

Another point of curiosity is when Betty goes first, and $n = m = 4$. This point is certainly an anomaly in the usual pattern. It is labelled with a dark dot and a circle.

We can come up with a solution for $n = m, n < m$, so all that is left is when $n > m$. This remaining case is much more difficult than the first two, however we can make some guesses about the trend, based off of the computer search results by Brian Chen.