An Introduction to Gilbreath Numbers

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Abstract

We begin with a "magic trick." The trick works as a result of the Gilbreath Principle, which can be proven using mathematical induction. In this paper we use the second version of the Principle to develop a new classification of number, the Gilbreath Continued Fraction. Once they are defined we then go on to describe Gilbreath Numbers and their place in the unit interval. Lastly, we look at some generalizations.

1 Introduction

1.1 A Magic Trick and Gilbreath Permutations

In 1958 an undergraduate math major at UCLA named Norman Gilbreath published a note in *The Linking Ring*, the official publication of the International Brotherhood of Magicians [6], in which he described a card trick. Stated succinctly, this trick can be performed by handing an audience member a deck of cards, letting him or her cut the deck several times, and then dealing N cards from the top into a pile. The audience member takes the two piles (the cards in hand and the set now piled on the table) and riffle-shuffles (in a riffle shuffle, the deck is split into two halves, one in each, hand, and the cards are released by the thumbs so that they fall on the table interwoven) them. The magician now hides the pile of cards (under a cloth, behind the back) and proceeds to produce pairs of cards where one card is black and the other red, claiming this is proof of the magician's powers.

The key to this trick is that the cards are pre-arranged in black/red order before the deck is handed out. Cutting the deck does not change this arrangement. When the top N cards are dealt into a pile, the black/red pattern is still there, but the order is reversed. It is then a mathematical induction argument to show that however the riffle-shuffle is performed, consecutive pairs of cards still maintain opposite colors. Let us look at a small example of this with eight cards and the subscripts just referring to first, second, third, etc. card in the original deck with R for red, B for black. The process is illustrated below.

Original order	After cutting	Two piles	Shuffled
B_1	B_5	B_1	R_8
R_2	R_6	R_2	B_7
B_3	B_7	B_3	B_1
R_4	R_8	R_4	R_6
B_5	B_1		R_2
R_6	R_2	R_8	B_5
B_7	B_3	B_7	B_3
R_8	R_4	R_6	R_4
		B_5	

If cards are taken two at a time from either the top or bottom of the deck, then we get pairs consisting of one of each color.

As is somewhat obvious, the colors do not matter as much as we have two types of cards. In fact, having a pattern of *two* types of characteristics does not matter. This trick also works if the cards are arranged by suit (e.g., Clubs, Hearts, Spades, Diamonds) and, after shuffling, dealt off four at a time.

Now let us turn this mathematical. For any nonempty set, S, a permutation on S is a bijection from S onto itself. We are only concerned with permutations on sets of numbers, beginning with $\{1, 2, 3, \ldots, N\}$, and eventually moving onward to $\{1, 2, 3, \ldots\}$, with a particular property (this is seen in the Ultimate Gilbreath Principle below). Notationally, we will denote our permutation using π and for any j, $\pi(j)$ refers to the number in the jth place of the permutation, not the placement of the number j in the permutation. So with $\{3, 4, 5, 2, 1\}$, $\pi(1) = 3$ and $\pi(5) = 1$.

Let us repeat the figure above, but this time with numbers rather than cards. Note that now that we are not performing a trick, the cutting, which gives the audience the idea of some randomness, is unnecessary.

Original order	Two "piles"	"Shuffled"
1	5	5
2	6	4
3	7	6
4	8	7
5	9	3
6	10	8
7		2
8	4	9
9	3	1
10	2	10
	1	

Now we write this as a permutation on $\{1,2,3,\ldots,N\}$. The permutation for the example above is

$$\{5,4,6,7,3,8,2,9,1,10\}.$$

This is what is known as a *Gilbreath Permutation.*

Not every permutation is a Gilbreath Permutation. In fact, while there are $N!$ permutations on $\{1,2,3,\ldots,N\}$, there are only 2^{N-1} Gilbreath Permutations. This fact and the *Ultimate Gilbreath Principle* below, which tells us which permutations are Gilbreath Permutations, are from [2]. It is there the reader can find the proof of this.

Theorem 1 (The Ultimate Gilbreath Principle)
For a permutation π of $\{1,2,3,\ldots,N\}$ the following are equivalent:

1. *π is a Gilbreath Permutation.*
2. *For each j, the first j values*
$$\{\pi(1),\pi(2),\ldots,\pi(j)\}$$
are distinct modulo j.
3. *For each j and k with $jk \le N$ the values*
$$\{\pi((k-1)j+1),\pi((k-1)j+2),\ldots,\pi(kj)\}$$
are distinct modulo j.
4. *For each j, the first j values are consecutive in $1,2,3,\ldots,N$; that is, if you take the first j values from the permutation, they can be rearranged as j consecutive numbers less than or equal to N.*

Just for the sake of pointing things out, it is Part Three that makes the magic trick work. It says whether you are taking cards from the deck in groups of two (red/black) or four (Clubs, Hearts, Spades, Diamonds), if the deck is set up correctly, then the magician gets one card of each type. Our interest is in the fourth part. This says even if the j numbers are not currently written in order, they appear in order in $\{1,2,3,\ldots,N\}$. So, for example, the permutations $\{2,3,1,4\}$ and $\{3,4,2,1\}$ are Gilbreath Permutations of $\{1,2,3,4\}$, while $\{3,4,1,2\}$ is not.

1.2 A Quick Look at Continued Fractions

Continued fractions have a long history with many interesting results. Even a quick look must contain lots of ideas. We present this subsection's theorems without proof, as they can be found in any text on continued fractions, such as [11]. To keep our investigations easier, we will keep with the simple continued fractions.

Definition 1 *Let a_i, $i = 0, 1, 2, 3, \ldots$ denote a collection (possibly finite) of integers*[1] *with a_i is a positive for $i \geq 1$. A* simple continued fraction *is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

To simplify our notation we will write this number as

$$[a_0; a_1, a_2, a_3, \ldots].$$

The individual $a_0, a_1, a_2, \ldots$ are referred to as the partial quotients *of the continued fraction expansion.*

Since our work on this is only concerned with numbers in the unit interval, we will refer the interested reader to [11] to learn more about all types of continued fractions.

Example 1 *The finite simple continued fraction* $[2; 3, 6]$ *represents*

$$2 + \frac{1}{3 + \frac{1}{6}} = 2 + \frac{1}{\frac{19}{6}} = 2 + \frac{6}{19} = \frac{44}{19}.$$

In the other direction,

$$\frac{32}{15} = 2 + \frac{2}{15} = 2 + \frac{1}{\frac{15}{2}} = 2 + \frac{1}{7 + \frac{1}{2}}.$$

So $32/15 = [2; 7, 2]$.

The technique above for turning a rational number into a continued fraction is guaranteed to terminate. Notice that in this dividing process ($32 \div 15$, then $15 \div 2$) the remainders, which in the next step become the denominators, are strictly decreasing. Thus the remainder must at some point become the number 1 finishing our continued fraction expansion in the rational numbers.

If x is a positive *irrational number*, then there exists a largest integer a_0 such that $x = a_0 + \frac{1}{x_1}$ where $0 < (x_1)^{-1} < 1$. Note

$$x_1 = \frac{1}{x - a_0} > 1$$

and is irrational (after all, x is irrational and a_0 is an integer).

1. Some references use complex numbers with integer coefficients.

We repeat this process, starting with x_1, find a_1, the greatest integer such that

$$x_1 = a_1 + \frac{1}{x_2}$$

where $0 < (x_2)^{-1} < 1$. As we continue we generate the continued fraction for x

$$[a_0; a_1, a_2, a_3, \ldots]$$

where this time the sequence of a_i does not terminate.

Example 2 *An example of this is* $\sqrt{3}$ *as*

$$\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots] \text{ or, more conveniently, } [1; \overline{1, 2}].$$

This continued fraction expansion can be verified by solving the following equation

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + (x - 1)}},$$

where $(x - 1)$ *is the repeating part of our continued fraction.*

A quadratic irrational is a number of the form

$$\frac{P \pm \sqrt{D}}{Q},$$

where P, Q, and D are integers, $Q \neq 0$, $D > 1$ and not a perfect square. Lagrange, in 1779, proved that the continued fraction expansion of any quadratic irrational will eventually become periodic. It also goes the other direction. Euler showed that if a continued fraction is eventually periodic, then the value can be expressed as a quadratic irrational number.

There are also non-repeating, infinite, simple continued fractions. As an example,

$$\pi = [3; 7, 15, 1, 292, 1, 1, \ldots]$$

and we know this cannot end or repeat as π is a transcendental number (not the solution to a polynomial with integer coefficients). We will revisit transcendental numbers later.

We now turn our attention to *convergents*.

Definition 2 *Let* x *have the simple continued fraction expansion (finite or infinite) of* $[a_0; a_1, a_2, a_3, \ldots]$. *The* convergents *are the sequence of finite simple continued fractions*

$$c_0 = a_0, \quad c_1 = a_0 + \frac{1}{a_1}, \quad c_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \quad \ldots.$$

and, in general, $c_n = [a_0; a_1, a_2, a_3, \ldots, a_n]$.

Typically, the next step is to represent these convergents as the rational numbers that they are. For example, $c_0 = a_0 = \frac{p_0}{q_0}$, $c_1 = a_0 + \frac{1}{a_1} = \frac{p_1}{q_1}$, $c_2 = a_0 + \frac{1}{a_1+\frac{1}{a_2}} = \frac{p_2}{q_2}$, and so on.

These convergents approach a limit. If we start with two convergents c_i and c_{i+1}, the literature on continued fractions then shows that

$$c_{i+1} - c_i = \frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} = \frac{p_{i+1}q_i - p_i q_{i+1}}{q_{i+1}q_i} = \frac{(-1)^{i+1}}{q_{i+1}q_i}$$

From the definition of convergent it can be seen that q_i is always positive and increasing. Thus we have

Theorem 2 *For any simple continued fraction* $[a_0; a_1, a_2, a_3, \ldots]$ *the convergents,* c_i *form a sequence of real numbers where for all* i

- $c_{2i-1} < c_{2i+1} < c_{2i}$, *and*
- $c_{2i+1} < c_{2i+2} < c_{2i}$.

Thus the sequence of convergents has the property

$$c_1 < c_3 < c_5 < \cdots < c_{2i-1} < \cdots < c_{2i} < \cdots < c_4 < c_2 < c_0.$$

This brings us, finally, to a theorem that says any infinite simple continued fraction has meaning as a unique point on the real number line.

Theorem 3 *Let* $[a_0; a_1, a_2, a_3, \ldots]$ *represent an infinite, simple continued fraction. Then there is a point* x *on the real line such that* $x = [a_0; a_1, a_2, a_3, \ldots]$

2 Gilbreath Numbers

2.1 Creating Gilbreath Continued Fractions

We now wish to take a Gilbreath Permutation such as $\{3, 4, 2, 1\}$ and turn it into a continued fraction which then represents a real number in the unit interval. Our method is to write the entries in the permutation as a *Gilbreath Continued Fraction* whence $\{3, 4, 2, 1\}$ will become

$$0 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1}}}}.$$

We can check that the simplified form of this rational number is 13/42. Now rather than write out the continued fraction we will take advantage of the fact that all the numerators are 1 and write the number in bracket notation noting the whole number part (0) and the denominator values in order, thusly

$$0 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1}}}} = [0; 3, 4, 2, 1].$$

In the practice of continued fractions, we do not allow a representation to end with a 1. That is because

$$[0; 3, 4, 2, 1] \text{ is the same as } [0; 3, 4, 3].$$

If the terminal digit is not allowed to be 1 then we get the property that continued fraction representations are unique, unlike decimal representations where $1.000\ldots = 0.999\ldots$. This is one of the perquisites in dealing with continued fractions rather than decimals. However, not ending in 1 creates issues with Gilbreath Permutations, so we will allow the expansion to terminate with a 1.

If we begin with a Gilbreath Permutation of length N,

$$\{\pi(1), \pi(2), \ldots, \pi(N)\},$$

we may increase its length ad infinitum by inserting the numbers $N+1, N+2, N+3, \ldots$ to make a permutation on $\mathbb{N}$ and keep the property of being a Gilbreath Permutation. This must be done at the end, and with the numbers in order. Placing a new number anywhere else would mean we no longer have a Gilbreath Permutation. This is easiest to see with Part 4 of Theorem 1. Thus we can extend $\{4, 3, 5, 2, 1, 6\}$ making the sequence

$$\{4, 3, 5, 2, 1, 6, 7\}, \quad \{4, 3, 5, 2, 1, 6, 7, 8\}, \quad \{4, 3, 5, 2, 1, 6, 7, 8, 9\}, \quad \ldots$$

which are each consecutive in $\mathbb{N}$ and the goal is the infinite ordered set of numbers

$$\{4, 3, 5, 2, 1, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\}.$$

Another method of making an infinite string using Gilbreath Permutations is in Section 3. We will refer to the part of the expansion where $a_k = k$ as where the permutation straightens out.

Of course, each of these finite strings can be the terms in a finite simple continued fraction (representing a rational number) and this sequence of numbers converges to an irrational number that is the infinite simple continued fraction

$$[0; 4, 3, 5, 2, 1, 6, 7, 8, 9, 10, 11, 12, \ldots].$$

2.2 An Analysis Point of View for Gilbreath Continued Fractions

To make the notation easier to read, we will refer to our continued fractions using the typical notation $[0; a_1, a_2, a_3, \ldots, a_n]$, keeping in mind each $a_j = \pi(j)$. Let $\mathcal{G}$ represent the set of finite and infinite Gilbreath Continued Fractions. Moreover, let $\mathcal{G}_F$ be the set of Gilbreath Numbers whose representation is a finite string. That makes them the set of rational numbers in $\mathcal{G}$, and $\mathcal{G}_I$ the set of Gilbreath numbers whose representation is an infinite string, the irrationals in $\mathcal{G}$. These form a subset of the unit interval $[0, 1]$ and the natural question to ask is, "How much of the unit interval is taken up by these numbers?" We will show that $\mathcal{G}$ is a countably infinite set and a very sparse set in terms of category.

Theorem 4 *The cardinality of the set of Gilbreath Continued Fractions, $\mathcal{G}$, is $\aleph_0$, the cardinality of the natural numbers.*

Proof: For any fixed N, there are 2^{N-1} possible Gilbreath Permutations of $\{1, 2, 3, \ldots, N\}$. Thus the set of numbers in $\mathcal{G}_F$ that look like $[0; \pi(1), \pi(2), \ldots, \pi(N)]$ for each fixed natural number N is finite and the cardinality of $\mathcal{G}_F$ is $\aleph_0$. Now if $x \in \mathcal{G}_I$ there is a $k \in \mathbb{N}$ such that if we write $x = [0; a_1, a_2, \ldots, a_j, \ldots]$, then for $j \geq k$ we have $a_j = j$. In fact, this k is where $\pi(k-1) = 1$. So if we fix $N \in \mathbb{N}$ and insist $a_j = j$ for all $j > N$, how many prefixes are there for this continued fraction that are still Gilbreath? The answer is, of course another 2^{N-1}. Thus there are finitely many sequences for each place where the continued fraction straightens out. So $\mathcal{G}_I$ is countable, too, which means $\mathcal{G}$ is a countable set.

An immediate consequence of this is that the set $\mathcal{G}$ must be a first category set, the countable union of nowhere dense sets. However, we claim there is even more. The set $\mathcal{G}$ is in fact a *scattered* set. For the definition of this, we go to Freiling and Thomson [3].

Definition 3 *Let $S \subset \mathbb{R}$. We say S is scattered if every nonempty subset of S contains an isolated point.*

This idea of scattered has appeared in papers by Cantor, Young & Young, Denjoy, Hausdorff, and others, but without such colorful nomenclature. Scattered sets are different from countable sets and nowhere dense sets. The Cantor Set is nowhere dense (and uncountable), but not scattered The rational numbers are countable, but not scattered. Of course a scattered set cannot be dense, but could be first category. In [3], the authors prove that any countable G–delta set of real numbers is scattered.

Theorem 5 *The set of Gilbreath Continued Fractions, $\mathcal{G}$, is a scattered set in $\mathbb{R}$.*

Proof: For any $x = [0; a_1, a_2, a_3, \ldots, a_n] \in \mathcal{G}_F$ where n is fixed, this number must be isolated. For any $t \in \mathcal{G}_F$, let us assume t has length at most n. There are finitely many of these, so there exists a $\varepsilon_1 > 0$ so that the open ball (interval) centered at x with radius ε_1 does not intersect $\mathcal{G}_F$. If we were to append more numbers onto x, creating

$$y_k = [0; a_1, a_2, a_3, \ldots, a_n, n+1, n+2, \ldots, n+k]$$

these y_k are converging toward some irrational number $y = [0; a_1, a_2, a_3, \ldots, a_n, n+1, n+2, \ldots, n+k, \ldots]$. Thus there is an $\varepsilon_2 > 0$ such that the open ball centered at x with radius ε_2 intersects only finitely many y_k. This argument also explains why there is an ε_3 so that the open ball around x of that radius must miss $\mathcal{G}_I$. Thus the point is isolated.

If $x \in \mathcal{G}_I$, then x is obviously not an isolated point in $\mathcal{G}$ as there is a sequence of points in $\mathcal{G}_F$ that converge to x. However, there is an $\varepsilon > 0$ so that

$$B(x, \varepsilon) \cap \mathcal{G}_I = \emptyset,$$

where $B(x, \varepsilon)$ is the open ball with center x and radius ε. This argument has to do with the place k where the continued fraction expansion "straightens out"; that is, for $j \geq k$ we have $a_j = j$.

This result is powerful as it implies previous results. In [1] it is stated that any scattered set is necessarily both countable and nowhere dense.

Topologically, it is obvious that $\mathcal{G}$ is not open (no scattered set in $\mathbb{R}$ can be). It is, in fact, a closed set in the unit interval.

Theorem 6 *The set of Gilbreath Numbers is a closed set in* $[0, 1]$.

Proof: Pick an x in the complement of $\mathcal{G}$. This x has a continued fraction expanion that is *not* Gilbreath. Let N represent the least index where we see it is not Gilbreath; that is, $a_1, a_2, \ldots, a_{N-1}$ is ordered in $\{1, 2, 3, \ldots, N-1\}$. We create the following sequence of continued fractions

$$\begin{aligned} y_N &= [0; a_1, a_2, \ldots, a_{N-1}, b_N] \\ y_{N+1} &= [0; a_1, a_2, \ldots, a_{N-1}, b_N, b_{N+1}] \\ y_{N+2} &= [0; a_1, a_2, \ldots, a_{N-1}, b_N, b_{N+1}, b_{N+2}] \\ &\vdots \end{aligned}$$

where each y_n is a Gilbreath Continued Fraction.

Let

$$\varepsilon = \frac{1}{2}\left(\inf_{n \geq N}\{|x - y_n|\}\right).$$

Such an infimum must be positive since there is a unique limit for the y_n (which is not x). Then the open ball with center x and radius ε is contained in the complement of $\mathcal{G}$. Thus $\mathcal{G}$ is a closed set.

3 Generalizations

A Generalized Gilbreath Permutation is a permutation of $\{k, k+1, k+2, \ldots, k+(N-1)\}$ that has the form

$$\{\pi(1)+(k-1), \pi(2)+(k-1), \ldots, \pi(N)+(k-1)\},$$

where $1 < k$ and π is a Gilbreath Permutation. For example, the sequence $\{5,6,4,7,8,3\}$ is the Gilbreath Permutation $\{3,4,2,5,6,1\}$ with 2 added to all of the entries. These can be turned into simple continued fractions just as easily as with the ordinary Gilbreath Permutation. There are, however, some differences. The second way to turn a Generalized Gilbreath Permutation into an infinite string of numbers is to append the as yet unused natural numbers in such a way that the "Gilbreath-iness" of the string is maintained (that is; Theorem 1 is not violated). Given the Gilbreath Permutation $\{\pi(1), \pi(2), \ldots, \pi(N)\}$ either the number

$$\min\{\pi(1), \pi(2), \ldots, \pi(N)\} - 1$$

or

$$\max\{\pi(1), \pi(2), \ldots, \pi(N)\} + 1$$

can be put at the end of the string to maintain a Gilbreath Permutation.

When $k = 1$ the extension of a finite Gilbreath Permutation to an infinite Gilbreath Permutation has only one solution. This is not true for a Generalized Gilbreath Permutation since any or all of the numbers $\{1, 2, 3, \ldots, k-1\}$ can be placed in the permutation after the Nth entry (or course, not in just any order). As a quick example, the finite permutation $\{4,3,5,2,6\}$ can be extended to $\{4,3,5,2,6,7,8,9,\ldots\}$ or $\{4,3,5,2,6,1,7,8,9,\ldots\}$ or $\{4,3,5,2,6,7,1,8,9,\ldots\}$ or $\{4,3,5,2,6,7,8,1,9,\ldots\}$ and many others.

Let us look at how this changes things with generalized, infinite Gilbreath Continued Fractions. Suppose x is the number given by

$$x = [0; 4,3,5,2,6,7,8,9,\ldots] = 0 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{5 + \ddots}}}.$$

Since we can slip in a 1 at any space after the 2 and NOT lose the "Gilbreath-iness" of the situation we get a sequence of numbers

$$\begin{aligned} y_1 &= [0; 4,3,5,2,6,1,7,8,9,\ldots], \\ y_2 &= [0; 4,3,5,2,6,7,1,8,9,\ldots], \\ y_3 &= [0; 4,3,5,2,6,7,8,1,9,\ldots], \\ &\vdots \end{aligned}$$

of generalized, infinite Gilbreath Continued Fractions that converge to x. Hence unlike their counterpart, these points are not isolated points, but limit points. Although in this example the y_n's use all the natural numbers in their continued fraction form, this is not necessary. If x had been missing the values 1 and 2, then the 2 could have be snuck in and the 1 left out and still we would have a converging sequence.

This can become more complicated if even more terms, 1, 2, ..., $k-1$, are missing. Let

$$x = [0; 5, 6, 4, 7, 3, 2, 1, 8, 9, 10, 11, \ldots].$$

Create the new value

$$y = [0; 5, 6, 4, 7, 8, 9, 10 \ldots].$$

This is x with S, the set of missing numbers, equal to $\{3, 2, 1\}$, the values between the 8 (the k where where $a_j = j$ for all $j \geq k$) and 7 (that is $k-1$). Some subsets of S, with order intact, can be slipped in to the right of the 4 (because this now straightens out from the 7 onward to create Gilbreath Continued Fractions that converge to y). These subsets are $\{3\}$, $\{3, 2\}$, and $\{3, 2, 1\}$.

4 Other Gilbreath Numbers

We can create numbers another way by starting with the Gilbreath Permutation $\{3, 4, 2, 1\}$ and placing the numbers, in order, as the digits in the (finite) decimal expansion, 0.3421. We shall call such a number a *finite Gilbreath Decimal.*

There is the "cut-and-paste" method. This way $\{3, 4, 2, 1\}$ can become the decimal

$$0.34213421342134 21 \ldots.$$

This, by virtue of being a repeated decimal, is really a rational number which is equal to

$$\frac{3421}{9999} = \frac{311}{999}.$$

Admittedly, there seems to be nothing exciting there. Cut-and-paste with simple continued fractions gives us

$$[0; 3, 4, 2, 1, 3, 4, 2, 1, 3, 4, 2, 1, \ldots].$$

This gives us something a little more interesting than the rational number version. Recall that continued fractions with repeated patterns are quadratic irrationals.

Example 3 *Turning our attention to our repeating Gilbreath Continued Fraction* $x = [0; 3, 4, 2, 1, 3, 4, 2, 1, 3, 4, \ldots]$ *becomes the equations*

$$x = 0 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1 + x}}}}.$$

This simplifies to the quadratic equation

$$29x^2 + 33x - 13 = 0$$

whose (positive) solution is

$$x = \frac{-33 + \sqrt{2597}}{58} \approx 0.309668434913\ldots.$$

It is easy to generalize the repeating decimal representation:

$$x = 0.\pi(1)\pi(2)\ldots\pi(N)\pi(1)\pi(2)\ldots\pi(N)\ldots$$

has rational number representation

$$x = \frac{\pi(1)\pi(2)\ldots\pi(N)}{10^N - 1}.$$

Things are much uglier with repeating Gilbreath Continued Fractions. As a general example, $x = [0; a, b, c, d]$ leads us to the equation

$$x = \frac{bcd + bcx + b + d + x}{abcd + abcx + ab + ad + cd + ax + cx + 1}$$

and an even worse looking solution.

In decimal form, the "limit" of this repeating continued fraction is the irrational number $0.435216789101112\ldots$, which is reminiscent of the better-known number

$$0.12345678910111213\ldots.$$

Lastly, in lots of places (e.g. [12]) one finds that $[0; 1, 2, 3, 4, 5, \ldots]$ is a *transcendental* number. Hence the infinite Gilbreath Continued Fractions (regular and generalized) must also be transcendental. The number $[0; 1, 2, 3, 4, 5, \ldots]$ is related to Bessel Functions[2]. The nth Bessel Function $J_n(x)$ is the solution to the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2) = 0.$$

2. Thanks to Stephen Lucas of James Madison University for pointing this out to us.

One of the myriad of relations involving Bessel Functions is that

$$[0; 1, 2, 3, 4, \ldots] = -i\,\frac{J_1(2i)}{J_0(2i)} = 0.697774657964\ldots,$$

where $i = \sqrt{-1}$.

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