SAINT MARY'S MATH CONTEST QUALIFYING PROBLEMS

HIGH SCHOOL DIVISION

INTRODUCTION BY NANCY BLACHMAN AND
PROBLEM SETS PROVIDED BY NICK BAXTER

While some of you anxiously awaited the latest issue of Scientific American, I anxiously awaited Saint Mary's Math Contest (SMMC) qualifying problem sets.

I found the SMMC problems easy to understand, intriguing, and challenging to solve. I have fond memories of my father asking questions that helped me to figure out ways of approaching problems and solving some of them. When I solved a Saint Mary's math problem, I felt a sense of accomplishment.

Students were given a set of 8-10 problems four times a year. We were given about a month or two for each set, ample time for experimentation and research. To get credit for solving a problem, not only did we need to find a reasonable answer, but we also had to write up our logically defensible process for solving the problem.

About ten years ago, I searched on the Internet for Saint Mary's Math Contest and was disappointed to find little of significance. While attending the East Bay Community Foundation Math/Science Fair at the Chabot Space and Science Center in Oakland, California, I asked Jim Sotiros if he knew anything about SMMC. Jim asked Hugo Rossi, the Deputy Director of the Mathematical Sciences Research Institute (MSRI). Hugo, in turn, asked a half a dozen or so of his math circle buddies. Joshua Zucker learned about the contest from the book *Saint Mary's College Mathematics Contest Problems* (published by Creative Publications, Inc. in 1972), which he won as a prize when he was in high school in Los Angeles. He later came across files of problem sets from the contest when he was a math teacher at Gunn High School in Palo Alto. The Saint Mary's Math Contest was discontinued in 1985.

Being a great development director at the Mathematical Science Research Institute (MSRI), Jim Sotiros asked me if I wanted to resurrect the Saint Mary's Math Contest. With Joshua Zucker and Jim Sotiros' encouragement, I decided to create a festival rather than a competition and name it after Julia Robinson (1919-1985), who was renowned for her role in solving Hilberts tenth problem, a conundrum that had baffled the worlds finest minds for half a century, and was the first woman president of the American Mathematical Society (1983-1984).

At the Festival, attendees may work together. There is at least one facilitator at each table who strives to ask more questions than she (or he) answers. The Festival problems offer diverse accessible entry points arithmetic, hands-on puzzles, card tricks, patterns, coloring so that K-12 students can find activities that grab their attention. Participants are encouraged to seek logical approaches to problem solving, not just answers. The first Julia Robinson Mathematics Festival was hosted by Google in April 2007. Our second Festival was hosted by Pixar in 2008. There have been over a hundred of these festivals in the US and abroad. Unlike math competitions, these non-competitive Julia Robinson Mathematics Festivals equally attract girls and boys (check out the Huffington Post article http://tinyurl.com/JRMFcolm.)

Saint Mary's Math Contest

In 1975, Saint Mary's College announced that it was discontinuing the competition. Many teachers expressed regret. Lyle Fisher and William Medigovich, teachers at Redwood High School in Larkspur, California, took over the program, renamed it, and compiled the book *Brother Alfred Brousseau Problem-Solving and Mathematics Competition Senior Division* (Dale Seymour Publications, 1984). The following introduction and history is excerpted from that book.

Introduction to the Saint Mary's Math Contest

We believe that problem solving can and should be stimulating, challenging, and fun. Problem solving is an art; it requires a certain feel, or touch.

The problem solvers of the future will be those who can examine information and look for patterns that suggest solutions. The ability to generalize from specific data may be a better preparation for a student than the repetition of arithmetic skills. It is important to make mathematics vital and exciting, and problem solving is the approach.

Though mathematics may be a paper and pencil sport, when approaches to solutions demand long and tedious calculations, we encourage the use of calculators and computers.

History of the Saint Mary's Math Contest Qualifying Problems

In 1959, long before others saw the importance of teaching problem solving, Brother Alfred Brousseau of Saint Mary's College, Moraga, California, recognized the need. He began a problem-solving and mathematics competition for junior and senior high students.

The competition was unique in that the core of the program extends over the entire school year. Students were given a set of eight to ten problems four times during the year and were allowed ample time for experimentation and research. They were instructed not only to find a reasonable answer, they also were encouraged to develop a logically defensible process for solving the problem. Answers weren't accepted unless accompanied by a fully developed solution.

When Brother Alfred Brousseau and his colleague at Saint Mary's, Brother Brendan Kneale, initiated the program, its purpose was to stimulate math achievement in the nine high schools of the Christian Brothers of California. A statement as to the nature of the competition was made that year. "The ratings of problems will take into account the following factors: (1) correctness of the solution; (2) its brevity and elegance; (3) neatness of presentation."

Four problem sets were distributed to participating schools on specific dates and were returned to Saint Mary's and Holy Names' staff members for grading.

Solving the Saint Mary's Math Contest Qualifying Problems with my father, Nelson Blachman, inspired me to to found the *Julia Robinson Mathematics Festival* (see jrmf.org) in 2007.

At the ninth Gathering for Gardner (G4G9), I learned that Nick Baxter also participated in Saint Mary's Math Contest and that he saved his problem sets. I borrowed them, made copies, and typeset the problems so you won't need to decipher my mimeographed copies.

Problem Set #2, November 15, 1972, Due December 15, 1972

11. A die is thrown until one of the number previously obtained comes up again. What is the average number of throws for which the throws are different?

12. In the Fibonacci sequences, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, etc., where each number after the first two is the sum of the two preceding numbers, find pairs of numbers whose product is 9 greater than the square of a Fibonacci number. As a result, make a conjecture regarding an infinity of such pairs.

13. There are four tangent spheres of radius r (each sphere is tangent to the other three externally). Find the radius of the sphere between the four spheres and tangent to each of them.

14. Study the Pascal triangle and determine which rows consists entirely of numbers *not* divisible by 3.

15. An idealized version of the trunk of a tree might be taken as a cone. If the growth of the tree continues to give a cone similar to the original cone and if the amount of substance added yearly is the same, find an expression for r_{n+1}/r_n where r_n is the bottom radius of the tree at the end of n years.

16. Given the following algebraic operations along with the commutative and associative laws of addition and multiplication and the distributive law of multiplication over addition:

$$a + a = a \qquad aa = a \qquad (ab)' = a' + b' \qquad (a + b)' = a'b'$$

$$a + ab = a \qquad b + b' = 1 \qquad bb' = 0$$

Simplify the expression $(a + bc)'(b + b'c)'$
(Prime is an operation of complementation.)

17. Find all the right triangles with integral sides for which the radius of the in-circle is 3. Explain your method of arriving at your results.

18. Find the cube root of $-46 + 9i$ in the form $a + bi$ where a and b are integers an i is the square root of -1. Show the method employed.

19. A police car traveling at 120 ft/sec is following another car going at the same speed at a distance of 1320 ft. The top speed of the police car is 150 ft/sec. At a given time (call this $t = 0$), the police car starts to accelerate. What acceleration should the police car have if it is to overtake the other car just when it reaches maximum speed? Equations of motion:

$$v = v_0 + at$$

$$s = v_0 t + \frac{1}{2}at^2$$

where v_0 is the speed at time $t = 0$, v is the speed at time t, a is the acceleration and s is the distance covered since $t = 0$.

20. A recursion operation is performed as follows. A three-digit integer in base ten abc has its digits operated on as follows:

$$20c + b + a$$

to form a new integer $a'b'c'$. Is there any three digit integer that are respectively equal to a', b', c'? Derive.

Problem Set #3, January 5, 1973, Due February 15, 1973

21. Given an $n \times n$ square (a checkerboard would be such with $n = 8$). Determine a formula for the following. We wish to know in how many ways 4 squares can be in sequence either horizontally, vertically, or diagonally (as 45° to the horizontal in either direction). Find a formula that gives this result. Explain.

22. Use a tabular method analogous to the Pascal Triangle to find the coefficients in the expansion of
$$(1 + x + x^2 + x^3 + x^4)^5$$

23. The midpoints of a parallelogram (P_0) are connected in sequence to give a quadrilateral P_1. Then the midpoints of P_1 are connected in sequence to give a quadrilateral P_2. And so on. Express the area of quadrilateral P_n in terms of the area of P_0.

24. Consider the sequence of values formed as follows:
$$(5 + 1)/6 = 1$$
$$(5^3 + 1)/6 = 21$$
$$(5^5 + 1)/6 = 521$$
$$(5^7 + 1)/6 = 13021$$
etc.

Find the recursion relation for this sequence in the form
$$T_{n+1} = AT_n + BT_{n-1}$$
Show that the relationship holds in general.

25. An equilateral triangle has sides s. Each side is divided into three equal parts and on the middle segment an equilateral triangle of side $s/3$ is constructed. Consider now the outside perimeter of the resulting figure. The process is repeated five times in all, at each step each line segment in the figure is divided into three parts and an equilateral triangle constructed on the middle segment. What is the perimeter of the resulting figure in terms of s?

26. The corners of a cube are cut off so that the cuts do not interfere with or touch each other. Then the same process is repeated for the resulting figure. How many space diagonals does the final figure have?

27. The difference of the cubes of two consecutive odd primes is 31106. What are those primes?

28. Objects are in positions $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$. Their positions are changed according to the linear transformations $2k + 1$. For example, 1 goes to 3, 2 goes to 5; 11 goes to 23 reduced by 13 or 10. However 6 goes to 13 (not zero). After the positions have been changed the same operation is performed again. How many such transformations are required to return the objects to their original positions?

29. If in a given interval $|x - x_0| \leq d$ around a fixed point x_0, the maximum of the absolute value of $f(x)$ is M, while in the same interval the maximum of the absolute value of $g(x)$ is M', while

$$|f(x) - f(x_0)| < e$$

and

$$|g(x) - g(x_0)| < e'$$

in the interval, show that

$$|f(x)g(x) - f(x_0)g(x_0)| < Me' + M'e$$

30. Analyze the following equation and plot its graph.

$$|y| + |x - 3| + |x - 7| = 10$$

Problem Set #4, February 25, 1973, Due March 15, 1973

31. Cut a 10×10 square into two pieces that can be fitted together to form a $60/7$ by $35/3$ rectangle. Explain.

32. According to Fermat's last theorem[1], there is no solution in positive integers to the equation

$$a^3 + b^3 = c^3$$

Find the minimum value that may be taken $|a^3 + b^3 - c^3|$ where a, b, c are positive integers, and $a < b < c$. (*Hint:* Look for a numerical example that will give the minimum.)

33. A straight line connects the points $(20, 0)$ and $(0, 30)$. If a line is drawn from $(4, 0)$ to $(14, 9)$ on this line, what is the equation of the reflected line (reflected as if the given line were a mirror)?

34. There are three mutually tangent circles of radius r. External tangents to the circles in pairs form an equilateral triangle. What is the area inside this triangle, but outside the circles?

35. Given an $n \times n$ tic-tac-toe structure in space where each level consists of an $n \times n$ board and there are n such boards. Spaces can be arranged in sequence n at a time horizontally, vertically, and diagonally, including space diagonals. What is the formula for the number of ways n spaces in this structure can be aligned? Explain.

36. Factor $x^{30} - 1$ into eight factors with integer coefficients. Explain.

[1] At the time this problem set was written, Fermat's last theorem was generally believed to be true. It was proved by Euler for cubes and finally proved for all larger exponents by Andrew Wiles in 1994.

37. Given that the sequence $7, 27, 111, 483, 2199, 10347, \ldots$ is the sum of two geometric progressions, namely
$$ar^{n-1} + bs^{n-1}$$
determine the values of a, r, b, and s by means of algebra.

38. A table of numbers
$$\begin{array}{lllll} 1 & 1 & & & \\ 1 & 3 & 2 & & \\ 1 & 5 & 8 & 4 & \\ 1 & 7 & 18 & 20 & 8 \end{array}$$
is built up as follows. There are multipliers 2 and 1 that operate on successive elements of one row to give an element in the next row. Thus to obtain the latest given row from the preceding row, we multiply 2 and 1 by 0 and 1 respectively, add the results and obtain the first element 1. Then we multiply 2 and 1 and 5 respectively, add the results, and obtain 7. Multiply 2 and 1 by 5 and 8 respectively, add the results, and obtain 18. Multiply 2 and 1 by 8 and 4 respectively, add, giving 20. Multiply 2 and 1 by 4 and 0 respectively, add, giving 8.
Find the sum of the numbers in the fifteenth row.

39. Prove that $x^2 + 3x + 1$ does not divide $x^{24} - 9x^{12} - 1$.

40. Prove that the determinant
$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$
is non-negative if $a, b, c,$ etc. are real.

Problem Set #1, October 1, 1973, Due November 1, 1973

1. A table

1	1	1	1						
1	2	3	7	4	3	2	1		
1	3	6	10	12	12	10	6	3	1

   is built up after the manner of a Pascal triangle, but adding four consecutive terms instead of two. What is the sum of all the numbers in the first ten rows of this table?

2. Find the equation whose roots are $2 \pm i$, $4 \pm 2\sqrt{3}$, 5, -6.

3. A bullet of radius $3/16''$ is shot into a block of styrofoam at an angle of 30° to the vertical to the surface and comes out at the same angle through a parallel surface, the length of the track being 30 inches. What is the volume of the hole?

4. Two regular hexagons, one of side 6 inches and the other of side 4 inches are similarly placed with their centers 8 inches apart. What is the difference of the non-overlapping areas in the two polygons?

5. Find the equations of the plane through the intersection of the two planes
$$3x - 7y + 4z = 5 \qquad 2x + 5y - 8z = 11$$
   and the point $(4, -7, 13)$.

6. If b is 25, find a value of a satisfying $a^b = b^a$ to the nearest thousandth.

7. Find the smallest base in which
$$15^2 + 21^2 = 630$$
   (all these numbers are in the required base).

8. Find the forms of all integers in terms of prime factors that have exactly 24 divisors. One example would be:
$$p^5 q^3.$$

9. A square is inscribed in the ellipse
$$x^2/a^2 + y^2/b^2 = 1$$
   with its sides parallel to the axes of the ellipse. Find its area.

10. There are ten questions in an examination. A student feels that he knows the answers to five of them perfectly. For two, he thinks he has a 50% chance of being correct and for the other three, a 25% chance of being correct. What is his probability of getting 80% or better in the test?

Problem Set #2, November 15, 1973, Due December 15, 1973

11. There are 100 tickets in a lottery and five prizes are to be given. Prove or disprove the following: Buying two tickets doubles the probability of winning at least one prize.

12. What is the side of a regular tetrahedron for which the volume is numerically equal to the surface area? Show derivation.

13. The expression
$$\frac{1+2x}{1-x-x^2}$$
expanded into a finite series. Find the first ten coefficients in the expansion and the law governing the relation of these coefficients.

14. Find the quadratic equations whose roots are the square of the roots of
$$x^2 + ax + b = 0$$

15. An airplane is traveling at 500 miles an hour at an angle of 7.5° to the horizontal. How long will it take to rise 40,000 ft vertically?

16. Express
$$\frac{40!}{16!24!}$$
as the product of primes of powers beginning with the smallest prime.

17. A number N (positive integer in base ten) is divided by three and the integer quotient found. Then this quotient is divided by three; etc. Find a relation between N, the sum of the quotients (Q), and the digit in the base three representation of the number. Example of the process. $N = 632$. Quotients are $210, 70, 23, 7, 2$. Their sum is 312. The base three representation is 212102. The required relation involves the sum of the digits of the base three representation.

18. For the following equations, find the number of solutions in positive integers.
$$x+y+z=2$$
$$x+y+z=4$$
$$x+y+z=6$$
$$x+y+z=8$$
$$x+y+z=10.$$
For example, for $x+y+z=8$, we could have $x=4, y=2, z=2$. But these quantities may be permuted to give $x=2, y=4, z=2$ and $x=2, y=2, z=4$.
Part Two: Find an expression for the number of solutions of $x+y+z=2n$.

19. Given a line segment $1''$ long. Points above the line are taken and joined to the ends of this line segment so as to form an angle of 60°. What is the measure of the area enclosed by the given line segment and the curve formed by these points?

20. Dice are constructed with two 1's, two 2's, a 3 and a 4. Show the number of ways the various sums can come up with four of these dice.

Problem Set #3, January 5, 1974, Due February 15, 1974

21. Amounts are put in the bank as follows: At the end of 1 year, \$2048; at the end of the second year, \$1024; at the end of the third year, \$512; ...this continues with the amount being put in at the end of each year being one-half what was put in at the end of the previous year. The process continues for twelve years ... \$1 being put in the bank at the end of the 12th year. If interest is at 5% compounded annually, how much money is in the bank at the end of the twelfth year? *Note:* The formula for compound amount is $P(1+i)^n$, where P is the principle, i is the interest rate per conversion (compounding) period and n is the number of conversion periods.

22. A car starts from rest accelerating at 5 ft/sec/sec. Two seconds later, another car starts from rest accelerating at 7 ft/sec/sec. How far will each of the cars have traveled by the time the second car catches up with the first? Distance covered from rest at acceleration a in t seconds is $\frac{1}{2}at^2$.

23. Find all the primes (report them in base ten notation) whose reciprocals give a decimal period of length five in base seven.

24. For the sequence $T_1 = 1, T_2 = 3, T_3 = 10, T_4 = 33, T_5 = 109, \ldots$ with the general recursion relation
$$T_{n=1} = 3T_n + T_{n-1}$$
find the continued fraction representation of $T_2/T_1, T_3/T_2$, etc. and conjecture the form of the continued fraction representation of T_{n+1}/T_n.

25. A parabola with its axis parallel to the x-axis has a general form
$$(y-a)^2 = bx + c$$
Find the equation of such a parabola going through $(3,4), (-1,7)$ and $(4,-6)$.

26. Find the value of
$$\sum_{k=76}^{99} k^3$$

27. Find five unit fractions (numerator is 1, denominator is positive integer) whose sum is a unit fraction for which the denominators are in the ratios $3:4:5:6:7$.

28. Two triangles have two sides in proportion, i.e., $a : b < a' : b'$ while their areas are such that
$$A : A' = a^2 : a'^2 = b^2 : b'^2$$
Prove or Disprove: The triangles are similar.

29. There is a series of concentric spherical shells such that the volume inside the first sphere is to the volume between the first two spheres as 1:2 the volume between the first two spheres is to the volume between the second and third spheres as $2:3$; etc. the successive ratios being $1:2:3:4:5:6:\ldots$. Let the radii of the spheres be $r_1, r_2, r_3, \ldots$. Find the ratio:
$$r_1 : r_2 : r_3 : \ldots : r_n$$

30. The opposite sides of a quadrilateral are 120° and 80°. On measuring the diagonals, it was found that the diagonal was 1.093 ft more than the other. By how much (in minutes and hundredths of a minute) do the angles differ from 90°?

Problem Set #4, February 25, 1974, Due March 25, 1974

31. Find the value of $\cot 15°$ in closed radical form.

32. Find all the primes whose reciprocal gives a decimal with a period length of five.

33. A line segment is drawn from $(-3, -7, 4)$ to $(18, 42, 18)$ in space. How many points with integral coordinates are on this line segment? Explain.

34. Tangents are drawn to the circle
$$x^2 + y^2 = 25$$
at $(3, 4), (3, -4), (-3, 4)$, and $(-3, -4)$. What is the area enclosed by the quadrilateral formed by these tangents?

35. If the roots of the equation $x^3 - 7x^2 + 5x - 8 = 0$ are each increased by 2, what is the equation with these altered roots?

36. A spiral ramp goes around a cylinder of radius 20 ft with an angle of rise of 10°. What distance is covered in 20 circuits of the cylinder?

37. In a game you throw two dice, the house betting that you will get a seven in a certain number of throws. What is the minimum number this might be if the house is to make a profit?

38. Two lines are drawn from the centroid of a regular tetrahedron to two of the vertices of the tetrahedron. Find the cosine of the angle between these lines.

39. Form a "Pascal" triangle with two numbers a, b in the first row and the other numbers building up in the usual way a "Pascal" triangle builds. Select a and b so that the sum of the numbers in the upward slanting diagonal (left-justified table) gives the sequence: $1, 4, 5, 9, 14, \ldots$ with each term after the first two the sum of the two preceding terms. Show the first six lines of the table.

40. If $f_1 = x, f_2 = 1/1 + x, f_3 = (1 + x)/(2 + x)$, where at each step x is replaced by $1/(1+x)$, what is an expression for f_n with coefficients given in terms of a well known sequence of numbers?

Problem Set #1, October 1, 1974, Due November 1, 1974

1. Find the equation of the line passing through the intersection of
$$3x - 7y = 4$$
$$2x + 6y = 5$$
and the point $(7, -12)$.

2. Build an equilateral triangle on each side of a square facing into the square. Connect the four vertices inside the square. Find the ratio of the area of this figure to the area of the square.

3. Form a sequence of squares consisting of six terms such that the sum of the first two is a square, the sum of the first three is a square, etc. Explain your method of arriving at this sequence.

4. Alice in Wonderland came to a hole that was 1/5 of her height. How to get through. The Rabbit began nibbling on a fungus and at each nibble decreased in volume by 10%. Alice made a quick calculation: How many nibbles will it take before I reach 1/5 of my height?

   So she got through the hole. But on the other side people seemed to be enormous. But again the Rabbit found a fungus such that each nibble increased one's volume by 10%. Another quick calculation: How many nibbles to get back to normal height?

   Your answers and calculations, please.

5. Solve for x in degrees and minutes.
$$\sin(x - 12^\circ) = 3\sin(60^\circ - x)$$

6. Prove or disprove: An equilateral triangle can be inscribed in any regular polygon. (Inscribe means that its vertices are on the sides of the polygon.)

7. The circle
$$x^2 + y^2 = 144$$
is drawn on graph paper. How many complete squares on the graph paper are inside the circle? (If the circle cuts through a square, that square is not counted; if the circle goes through the upper corner of a square, the square is counted.)

8. For the equation
$$x^2 - 3x - 5 = 0$$
with roots r and s, find the value of
$$r^7 + s^7.$$

9. What is the probability of getting three or more 1's when throwing six dice?

10. Given a set of n points in a plane (n is even). Is it possible to find a line such that the projection of the n points on the line gives n distinct points? Explain.

Problem Set #2, November 15, 1974, Due December 15, 1974

11. Find the dimensions of a box with integral edges such that the surface area is twice the volume.

12. Prove that for the infinite sequence $1, 2, 4, 8, 16, 32, \ldots$ consisting of the powers of two, no three terms are in arithmetic progression.

13. A tunnel 13 ft. in diameter is discharging water at the rate of 10 ft/sec over the entire area of the tunnel. If a lake is 6 miles long and 3/4 miles wide on the average, how long will it take to raise the level of the water 1 inch? Express the answer in hours, minutes, and seconds.

14. Find a rule of divisibility by 4 in base seven and prove that this rule holds in general. (A rule of divisibility enables one to determine whether the number is divisible by 4 using the digits of the number without actually carrying out the division.)

15. There are three containers with balls as follows:
    A. 5 white, 3 black.
    B. 4 white, 4 black.
    C. 3 white, 7 black.
    What is the probability that on transferring a ball from A to B, then from B to C, and finally from C to A, the contents of the three containers remains unchanged?

16. Given an ellipse with its major diameter in place. Find a ruler and compass construction for determining the foci.

17. n equally spaced points are marked on a circle. Starting with any point, every k^{th} point is connected (k is less than $n/2$). In this way a star is formed. (If k and n have a common factor, it will be necessary to have more than one starting point to cover all the points.) What is the angle at the vertex of the star? Prove your result.

18. Given the first n integers. A particular arrangement of these integers will have a certain number of inversions. An inversion is defined as a situation where a larger number precedes a smaller number. Thus, for the arrangement $3, 1, 4, 2, 5$ of the first five integers, 3 is larger than 1 or 2 and 4 is larger than 2. Hence there are three inversions. Find a rule for determining the number of inversions when the sequence of integers is reversed. Prove this rule.

19. Two men take four days to do a job when working together (call them A and B); B and C take five days when working together; C and A take 6 days when working together. How long does it take for all three working together to do the job?

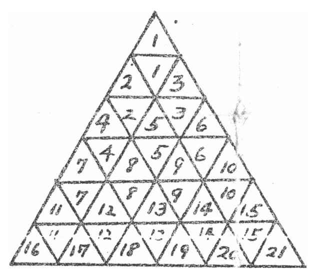
20. Find the value of:
$$\sum_{a=1}^{\infty} 1/(a^2 + 16a + 63)$$

Problem Set #3, January 5, 1975, Due February 15, 1975

21. Two teams A and B are playing a series of five games in a playoff. The team that wins three games first wins. A has sixty percent probability of winning a single game; B has a forty percent probability of winning a single game. What is the overall probability that team A wins the series?

22. The corners of a square of sides s are lopped off and rearranged with the remaining figure to form a square of the same size. What is the length of the sides of the right triangles removed from the corners?

23. In a triangular diagram where the side of the triangle is of length n and the side is divided into n unit parts, the upward pointing triangles are numbered and the downward pointing triangles are numbered as in the figure (Case of $n = 6$ is shown). Find a formula for the sum of the numbers in the triangle. (For $n = 6$, the sum is 351.)



24. The length of the period of 1/7 in base two is 3, of 1/13 is 12 and of 1/17 is 8. What is the length of the period of

$$(1/7)(1/13)(1/17) = 1/1547?$$

Explain.

25. A freight train traveling 45 miles/hour decelerates at the rate of 1/2 ft/sec/sec. How far does it go before stopping?
Equations of motion for uniformly accelerated motion are:

$$d = v_0 t + (1/2)at^2$$

$$v = v_0 + at$$

$$v^2 = v_0^2 + 2ad$$

where v_0 = initial velocity at time taken as $t = 0$, v = final velocity at time t, a = uniform rate of acceleration and d = distance traveled in time t.

26. Find the equation whose roots are the square of the roots of the equation:

$$x^3 + px^2 + qx + r = 0.$$

27. Six dice are thrown. What is the probability that three pairs will show?

28. If the probability that people ar born in a given month is the same for all months, how many people should there be in a group so that the probability that two people are born in the same month is greater than 75%?

29. Find the value of the infinite sequence of cube roots.

$$\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\dots}}}$$

30. Three circles of radius r are mutually tangent in pairs. What is the radius of the circle that is tangent to all three circles?

Problem Set #4, March 1, 1975, Due April 3, 1975

31. A circle of radius 10 inches has ten equal circles tangent to it and tangent to each other placed around it. What is the radius of this circle?

32. The triangular number $T_n = n(n+1)/2$. Find four instances of three consecutive triangular numbers adding up to a perfect square.

33. Prove that if $A+B+C = 180°$, then

$$\tan(A)+\tan(B)+\tan(C) = \tan(A)\tan(B)\tan(C)$$

and if

$$A+B+C = 90°,$$

then

$$\tan(A)\tan(B)+\tan(B)\tan(C)+\tan(C)\tan(A) = 1.$$

34. Find an expression for the side of a regular polygon of n sides for which the perimeter is numerically equal to the area.

35. In the *Mathematics Magazine* of January 1975, p. 51, the following problem is proposed. Eight points are taken on a circle and connected consecutively to form an inscribed octagon. Prove that if the angles at vertices $1,3,5,7$ are added, the sum is the same as for the angles at vertices $2,4,6,8$ and the sum is 3π.

    Discover and prove the following. If $2n$ points are taken on a circle and connected consecutively to form an inscribed polygon of $2n$ sides, then the sum of the angles at vertices $1,3,5,\dots,2n-1$ is the same as the sum of the angles at the vertices $2,4,6,\dots,2n$ and the sum is equal to $(n-1)\pi$.

36. Given n quantities. Determine a formula for the number of sums that can be formed with these quantities. (A sum involves at least two quantities.)

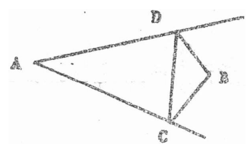
37. An integer of the form $aabb$ has its digits permuted in all possible ways. Prove that no matter in what base the integer is interpreted, it may not represent a prime in any of its permutations.

38. Let z equal the infinite continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\ddots}}}$$

where the denominators are alternative 1 and 2. Find the value of z in closed radical form.

39. A mirror consists of two vertical strips, making an angle A between them. If a point source of light at B has the light reflected at C and D and return to B, find the angle CBD in terms of angle A.



40. There are six cats, each of which is pursuing one of the other cats and each of which is being pursued by one of the other cats. In how many different ways may this be done? (Cats may form several groups.)

Problem Set #4, March 15, 1979, Due April 15, 1979

31. A two-digit number consisting of two different non-zero digits has its digits reversed. The two numbers are then squared and the difference taken. What is the largest integer that always divides this difference?

32. If a vertex angle on a square with side 4 is trisected, the following figure results:

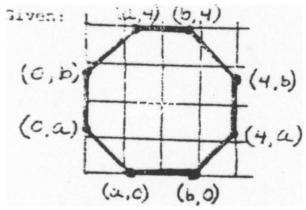
Find the area of the shaded portion.

33. An automobile comes a distance d_1 at speed s_1, then a distance d_2 at a speed s_2, and finally a distance d_3 at speed s_3. Write a formula for its average speed over the entire distance.

34. A spherical balloon 5 inches in diameter has a thickness of .05 inches. What is the thickness when the balloon has a diameter of 9 inches? Find the answer correct to four decimal places.

35. Sue Harris was drilling for water in Death Valley and hit pay sand. She noticed the spot was 10,000 meters from one corner of a rectangle plot, 15,000 meters from the opposite corner, and 6,000 meters from a third corner. How far is it to the fourth corner?

36. Given



What is the value of a and b if the shaded figure is a rectangular octagon? Use

$$\sqrt{3} \approx 1.732$$
$$\sqrt{2} \approx 1.414$$

37. Find a set of whole numbers when the product of each by the sum of all the others is known. You are to find six number a, b, c, d, e, f if you are given that:

$$a(b+c+d+e+f) = 184$$
$$b(a+c+d+e+f) = 225$$
$$c(a+b+d+e+f) = 301$$
$$d(a+b+c+e+f) = 369$$
$$e(a+b+c+d+f) = 400$$
$$f(a+b+c+d+e) = 525$$

38. How many ways could one make \$2.43 with 5-cent and 8-cent stamps? What are the possible combinations?

39. It is possible to select four grid points on a coordinate system so that the joins of all points (six in all) do not contain any other grid points. The points $A(2,3)$ and $B(11,17)$ have a join with no other grid point on it. Find two other points C and D so that no three of the points are on the same line and the joins of any of the four points, A, B, C, D, do not contain any other grid point.

40. In what bases (less than or equal to 12) is 2101 a perfect square?