The 100 Prisoners Puzzle Revisited

Yossi Elran

The Davidson Institute of Science Education, The Weizmann Institute of Science

Introduction

The 100 prisoners problem was first posed by Peter Bro Mitersen in 2003 and a few versions of the puzzle were subsequently published [1-3]. Being a probabilistic puzzle, it is not easy to derive the general solution for n prisoners. Even though the solution itself is a simple, straightforward expression that converges to ln 2, a horrible intermediate expression is used:

$$1-\frac{1}{(2n)!}\sum_{l=n+1}^{2n}\binom{2n}{l}(l-1)!(100-l)!=1-\sum_{l=n+1}^{2n}\frac{1}{l}$$

Eq. 1 General solution to the 100 prisoners problem

In this paper, we present the problem as a soccer team puzzle and show a direct way to derive and explain the solution without using the intermediate formula.

The soccer team problem:

A new coach has been appointed to your favorite world cup soccer team, or football team if you happen to be British. The coach has decided that the 18 player team has good control of the ball, but unfortunately constantly loses games by making stupid decisions on the field, so he decides to exercise their mental skills. He rounds up the players, who are wearing t-shirts numbered 1 through 18, and shows them 18 identical boxes neatly lined up in a row.

“In each of these 18 boxes there is a slip of paper marked with a number between 1 and 18. In a moment, each of you will be sent to an isolated room. I will then call each and every one of you one by one and ask you to open any nine boxes, read the slips of paper inside each box, return them, close the boxes and return to your room. You can open any nine that you please, but you are not allowed to communicate with each other in any manner once you’ve entered your isolated rooms, nor are you allowed to mark the boxes in any way. If, after you have all visited the boxes, each of you has found within one of the nine boxes the same number that is written on his shirt, you will all get a $100,000 bonus. If, however, even one of you fails to reveal his own t-shirt number you will all be fined $100,000 and sent to play soccer for one month on an isolated island where they use coconuts as the ball.

One of the brighter players objected, “ Wait a minute! If I randomly open nine out of eighteen boxes, then I have a fifty percent chance of finding my t-shirt number. So does John here, and so does Sam. The probability that all of us succeed is the multiplication of these probabilities, which is 50% times 50% times 50% - 18 times - in other words 1/2 to the power of 18, which comes out less than four ten-thousandths of a percent?! There is no chance we’ll beat those odds?! You might as well send us off right now!”

“It’s true” retorted the coach “that if each player chooses nine boxes randomly, then the chances are negligible, however, I’m giving you now 5 minutes to come up with a strategy that will greatly improve the chances of success to over 30%!”
The players huddled together and after 5 minutes were sent to isolation. One by one, the coach took the players from their rooms and each opened nine boxes. Amazingly enough, they all found their numbers! What was their strategy?

Solution to the soccer team problem:

When huddled together, the players decided to mentally number the outside of the boxes with numbers from 1 to 18: The first in the row is mentally labeled 1, the second 2, and so on, until 18 for the last box in the row. The boxes had now a mental number on the outside and a number on the slip inside the box. Each player was to first open the box whose number on the outside was the same as the number on his own t-shirt. Then, he should read the number inside the box and go to the box whose number on the outside is the same as that number. He opens that box, reads the number on the inside and then opens the box that is numbered with that number, and so on. If within 9 boxes he finds the number on his t-shirt, then he is done because he has found his number.

For example: suppose the 18 boxes labeled on the outside are
1-2-3-4-5-6-7-8-9-10-11-12-13-14-15-16-17-18, and the slips of paper on the inside labeled:
13-8-5-18-3-15-11-7-9-12-2-10-4-14-6-16-17-1. The player with the number 7 on his shirt first opens the box labeled 7 on the outside. He then reads the note inside the box. On it is written the number 11, so he then goes to box number 11, opens it and reads the note inside which is numbered 2. He then goes to box number 2, opens it, reads the note which is numbered 8, goes to box 8, opens it and reads the note which is numbered 7. Here the player can stop because he found his number - the number on his shirt. Surprisingly, if every player uses this strategy, the odds are about 33%!

Explanation of the solution

To understand this strategy, we first note that the players have created a set of cyclic chains of numbers. Each chain depicts the route of a few players. Using the example above, there are the chains: 7-11-2-8-7, 1-13-4-18-1, 3-5, 6-15, 9-9, 10-12, 14-14, 16-16, 17-17.
For the players to succeed, there must be no chain larger than 9. A chain larger than 9 means that more than nine boxes have to be opened for each player whose number is included in this chain to find his number. This cannot happen since only nine boxes are opened. Moreover, if there is a chain larger than 9, then there is only one such chain. It may be of length 10 or 11 or 15 or whatever - but it is the only one with a length greater than 9.

We need to calculate the probability that within a random permutation of 18 numbers, we won’t find a cycle or ‘chain’ of length 10, 11, 12 etc. up to 18

Let’s side step a bit and look at a simpler calculation. Suppose there are four players and four boxes and each player is allowed to open only two boxes. In this case we need to calculate the probability that there are no chains of length 4 or 3.

First, we calculate that there are 4! (4 factorial = 24) possible arrangements of 4 numbers:

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

From these, six are *distinct* chains of length 4: 1234, 1342, 1423, 1243, 1324, 1432. The rest are just the same chains starting from a different number. A nice way to show this is to write the four numbers 1-4 at the corners of a square and draw all possible lines connecting them. We can trace six different cyclic paths along the lines, where each path visits each vertex once. These correspond to the six chains: 1234, 1342, 1423, 1243, 1324, 1432 - each direction (clockwise, anti-clockwise) is considered a separate path. Such cycles are called Hamiltonian cycles after the Irish mathematician William Rowan Hamilton who discovered them.

Fig. 1 Hamiltonian cycles on a square

Since 6 out of the 24 permutations are distinct length-4 chains, the possibility that a chain of length 4 is created is 6 in 24 - or - 1/4.

Now, we calculate the probability that a length-3 chain is created from the 4 numbers: 1,2,3 and 4. The number of possibilities of arranging the 4 numbers is still the same: 24. The number of Hamiltonian cycles on the triangle is 2. Since there are four possible triangles like this, depending on which number of the 4 you leave out - 1,2,3 or 4 - we multiply the number of Hamiltonian cycles - 2 - by 4 and get 8. So there is a chance of 8 in 24 or 1/3 to create a length-3 chain.

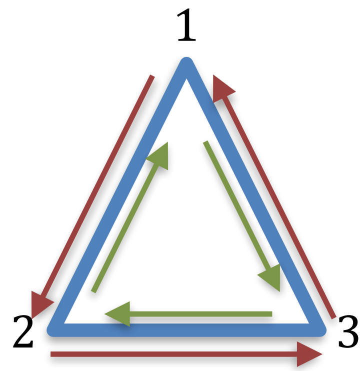

Fig. 2 Hamiltonian cycles on a triangle

This means that with 4 players and 4 boxes, the probability of creating a chain larger than 2 is one third plus one quarter, which equals 7/12 or about 58%. So, the complimentary probability of creating chains of length 2 or less is 100%-58%=42% - which is a fair chance for the players.

This argument can be generalised to *n* players and boxes to get the simple expression:

$$\lim_{n \to \infty} \left(1 - \sum_{l=n+1}^{2n} \frac{1}{l} \right) = ln(2)$$

Eq. 2 General solution to the soccer team problem

It is well-suited to the occasion, to note that the connection between Hamiltonian cycles and chains was first brought to the attention of the wider public by Martin Gardner who wrote about it in an article on the binary gray code[4].

References

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