On the Border Between Recreational and "Serious" Mathematics: Rectangle Free Coloring of Grids

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1 Introduction

Martin Gardner wrote about *recreational mathematics.* However, if you browse through the *Gathering for Gardner proceedings* you will see some *serious mathematics* inspired by him. The line between recreational and serious mathematics is thin indeed. We give an example of a *serious* theorem in Ramsey Theory and some work in *recreational* mathematics that it inspired. We do not rigorously define the terms *recreational* or *serious* but leave that serious problem in sociology to the reader.

Notation 1.1 If $n \in \mathbb{N}$ then $[n] = \{1, \ldots, n\}$. If $n, m \in \mathbb{N}$ then $G_{n,m}$ is the grid $[n] \times [m]$.

The Gallai-Witt theorem[1] (also called the multi-dimensional Van Der Waerden theorem) has the following corollary: *For all c, there exists $W = W(c)$ such that, for all c-colorings of $[W] \times [W]$ there exists a monochromatic square.* The classical proof of the theorem gives very large upper bounds on $W(c)$ and is somewhat difficult. It is in the *serious* camp. Despite some improvements [1] the known bounds on $W(c)$ are still quite large.

What if we relax the problem to seeking a *monochromatic rectangle*? Then we can obtain far smaller bounds. Of more importance, the proofs are easily understood and fun. I would call the area *recreational* at least in the case of two colors.

Def 1.2 A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a, b), (a + c_1, b), (a + c_1, b + c_2), (a, b + c_2)\}$ for some $a, b, c_1, c_2 \in \mathbb{N}$. A grid $G_{n,m}$ is *c-colorable* if there is a c-coloring of $G_{n,m}$ with no monochromatic rectangles.

By two applications of the the pigeonhole principle $G_{c+1,c^{c+1}+1}$ does not have a c-coloring. We leave that *fun* exercise to the reader.

[1]It was attributed to Gallai in [7] and [8]; Witt proved the theorem in [12].

Our Main Question:

Fix c. For which values of n and m is $G_{n,m}$ c-colorable?

Def 1.3 Let $n, m, n', m' \in \mathbb{N}$. $G_{m,n}$ *contains* $G_{n',m'}$ if $n' \le n$ and $m' \le m$. $G_{m,n}$ *is contained in* $G_{n',m'}$ if $n \le n'$ and $m \le m'$. Proper containment means that at least one of the inequalities is strict.

Clearly, if $G_{n,m}$ is c-colorable, then all grids that it contains are c-colorable. Likewise, if $G_{n,m}$ is not c-colorable then all grids that contain it are not c-colorable.

Def 1.4 Fix $c \in \mathbb{N}$. OBS_c is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not c-colorable but all grids properly contained in $G_{m,n}$ are c-colorable. OBS_c stands for *Obstruction Sets*.

We leave the proof of the following theorem to the reader.

Theorem 1.5 *Fix $c \in \mathbb{N}$. A grid $G_{n,m}$ is c-colorable iff it does not contain any element of* OBS_c.

By Theorem 1.5 we can rephrase our main question as:

What is OBS_c*?*

In the remainder of this paper we state tools used to obtain colorings and proof of lack of colorings, and describe what we know. There are no proofs. For the full version of the paper see

`https://arXiv.org/abs/1005.3750`

or Google *arXiv* and search for *Gasarch*.

Cooper, Fenner, and Purewal [2] generalize our problem to multiple dimensions. Molina, Oza, and Puttagunta [4] look at some variants of our questions.

2 Tools to Show Grids are Not c-colorable

A *rectangle-free subset* $A \subseteq G_{n,m}$ is a subset that does not contain a rectangle.

Theorem 2.1 *If $G_{n,m}$ is c-colorable, then it contains a rectangle-free subset of size* $\lceil \frac{nm}{c} \rceil$.

Theorem 2.2 *Let $a, n, m \in \mathbb{N}$. Let $q, r \in \mathbb{N}$ be such that $a = qn + r$ with $0 \le r \le n$. Assume that there exists $A \subseteq G_{n,m}$ such that $|A| = a$ and A is rectangle-free.*

1. *If $q \ge 2$ then*
$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$
2. *If $q = 1$ then*
$$r \le \frac{m(m-1)}{2}.$$

We leave it to the reader to show that, for all $c \ge 2$, G_{c^2,c^2+c+1} is not c-colorable. Use Theorems 2.1 and 2.2.

3 Tools for Finding c-colorings

Def 3.1 Let $c, n, m \in \mathbb{N}$ and let $\chi : G_{n,m} \to [c]$. A *half-mono rectangle with respect to* χ is a rectangle where the left corners are the same color and the right corners are the same color. χ is a *strong c-coloring* if there are no half-mono rectangles.

Table 1 is a strong $(4, 2)$-coloring of $G_{6,15}$.

1	1	1	1	1	3	3	3	2	3	3	2	2	2	2
1	2	2	2	2	1	1	1	1	4	4	3	3	3	2
2	1	3	3	2	1	2	2	2	1	1	1	4	4	3
2	2	1	4	3	2	1	4	3	1	2	2	1	1	4
3	3	2	1	4	2	2	1	4	2	1	4	1	2	1
4	4	4	2	1	4	4	2	1	2	2	1	2	1	1

Table 1: Strong $(4, 2)$-coloring of $G_{6,15}$

Theorem 3.2 *Let $c, n, m \in \mathbb{N}$. If $G_{n,m}$ is strongly c-colorable then $G_{n,cm}$ is c-colorable.*

In the full paper we generalize strong c-colorings to strong (c, c')-coloring and then use use finite fields, tournaments, and combinatorics, to obtain many strong (c, c')-colorings which leads to many c-colorings.

4 Which Grids Can be 2-Colored?

Using our tools we can show the following:

Theorem 4.1 $\text{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

5 Which Grids Can be 3-Colored?

We state a theorem about which grids can be 3-colored and which ones cannot. If we do not have a comment on the grid then the result follows from our tools. If we mention $G_{a,b}$ we omit saying the same holds for $G_{b,a}$.

Theorem 5.1 *The following are not 3-colorable:* $G_{19,4}$ $G_{16,5}$, $G_{13,7}$, $G_{11,10}$ *(special case). The following are 3-colorable:* $G_{19,3}$, $G_{18,4}$, $G_{15,6}$, $G_{12,9}$, $G_{10,10}$ *(program).*

We found the 3-coloring of $G_{10,10}$ by first finding a size 34 rectangle free subset of $G_{10,10}$ (by hand) and then wrote a program to find a coloring where that set was RED. Frankly we were trying to prove there was no such rectangle free set and hence $G_{10,10}$ would not be 3-colorable.

Theorem 5.2 $\text{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}\}$ *and their reversals.*

6 Which Grids Can be 4-Colored?

We state a theorem about which grids can be 4-colored and which ones cannot. If we do not have a comment on the grid then the result follows from our tools. If we mention $G_{a,b}$ we omit saying the same holds for $G_{b,a}$.

Theorem 6.1 *The following are not 4-colorable:* $G_{41,5}$, $G_{31,6}$, $G_{29,7}$, $G_{25,9}$, $G_{23,10}$, $G_{22,11}$, $G_{21,13}$, $G_{19,17}$ *(special case). The following are 4-colorable:* $G_{41,4}$, $G_{40,5}$, $G_{30,6}$, $G_{28,8}$, $G_{24,9}$ *(used a strong 4-coloring of* $G_{9,6}$*)*, $G_{22,10}$ *(Brad Larsen program)*, $G_{21,12}$ *(Bernd Steinback and Christian Posthoff program, and Tom Sirgedas program)*, $G_{20,16}$, $G_{18,18}$ *(Bernd Steinbach and Christian Posthoff program).*

Theorem 6.2 $\text{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\}$ *and their reversals.*

For a long time we didn't know OBS_4. We had a rectangle free set of $G_{18,18}$ of size $18 \times 18/4 = 81$ so we thought that 18×18 was 4-colorable. But we didn't even have a 4-coloring of $G_{17,17}$. On November 30, 2009 Bill Gasarch posted on his blog [3] *The* 17×17 *challenge:* the first person to email him a 4-coloring of $G_{17,17}$ gets $289.00. Brian Hayes, a popular science writer, put the problem on his blog [5] thus exposing the problem to many more people, including Brad Larsen and Tom Sirgedas who contributed as can be seen in Theorem 6.1. In February of 2012, Bernd Steinbach and Christian Posthoff emailed Bill their solution. They used sophisticated SAT-solvers and also wrote three brilliant papers [9, 11, 10] about their method. Bill happily paid them the $289.00. They then obtained a 4-coloring of 18×18 and emailed it to Bill free of charge.

7 Open Questions

The next step is to obtain OBS_5. It is likely that our tools will take of most of the way, but not all, and that the problems left are beyond today's computers. So the real open question is to develop better tools.

There is also an obstruction set for the set of grids that are 2-colorable without a monochromatic square. Obtaining this seems rather difficult.

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