The Leaning Tower of Pingala

Richard K. Guy
Department of Mathematics & Statistics, The University of Calgary.

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As Leibniz has told us, from 0 and 1 we can get everything :

$$0$$
$$1$$

Multiply the previous line by a — a

and add b times the line before that — $a^2 + b$

$$a^3 + 2ab$$
$$a^4 + 3a^2b + b^2$$
$$a^5 + 4a^3b + 3ab^2$$
$$a^6 + 5a^4b + 6a^2b^2 + b^3$$
$$a^7 + 6a^5b + 10a^3b^2 + 4ab^3$$
$$a^8 + 7a^6b + 15a^4b^2 + 10a^2b^3 + b^4$$
$$a^9 + 8a^7b + 21a^5b^2 + 20a^3b^3 + 5ab^4$$
$$a^{10} + 9a^8b + 28a^6b^2 + 35a^4b^2 + 15a^2b^3 + b^5$$
$$a^{11} + 10a^9b + 36a^7b^2 + 56a^5b^3 + 35a^3b^4 + 6ab^5$$

If you tip your head on one side

						1
					1	
				1		1
			1		2	
		1		3		1
	1		4		3	
1		5		6		1
	6		10		4	
		15		10		1
			20		5	
				15		1

you'll see that the coefficients (A011973)
form the Leaning Tower of ~~Pascal, 1653AD~~
~~Omar Khayyam, 1100AD~~
~~Al Karaji, 1000AD~~
Pingala, 200BC

There are infinitely many particular cases:

$$\begin{array}{l} 1 \\ a \\ a^2 + b \\ a^3 + 2ab \\ a^4 + 3a^2b + b^2 \\ a^5 + 4a^3b + 3ab^2 \\ a^6 + 5a^4b + 6a^2b^2 + b^3 \end{array}$$

For example, $a = 2$, $b = -1$ gives the natural numbers (A000027)

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$$

$$\begin{array}{l} 1 \\ a \\ a^2 + b \\ a^3 + 2ab \\ a^4 + 3a^2b + b^2 \\ a^5 + 4a^3b + 3ab^2 \\ a^6 + 5a^4b + 6a^2b^2 + b^3 \end{array}$$

$a = 3$, $b = -2$ gives the Mersenne numbers (A000225)

$$0,\ 1,\ 3,\ 7,\ 15,\ 31,\ 63,\ 127,\ \ldots 2^n - 1$$

Everyone believes that infinitely many of them ($n =$ 2,3,5,7,…) are prime

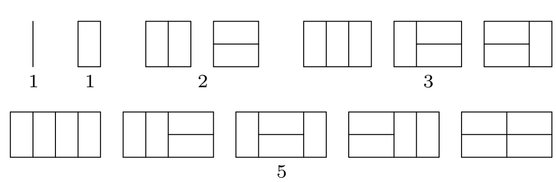
…but no-one can prove that!

$$\begin{array}{l} 1 \\ a \\ a^2 + b \\ a^3 + 2ab \\ a^4 + 3a^2b + b^2 \\ a^5 + 4a^3b + 3ab^2 \\ a^6 + 5a^4b + 6a^2b^2 + b^3 \end{array}$$

$a = 1$, $b = 1$ gives the Fibonacci numbers (A000045)

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \ldots$$

about which whole books have been written.
They are the numbers of ways of packing dominoes in a $2 \times (n-1)$ box:



$a = 2$, $b = 1$ gives the Brahmagupta-Pell numbers (A000129)

$$0, \; 1, \; 2, \; 5, \; 12, \; 29, \; 70, \; 169, \; 408, \; \ldots$$

These were probably known to the Babylonians nearly 4000 years ago.

They are the denominators of good approximations

$$\frac{1}{1} \qquad \frac{3}{2} \qquad \frac{7}{5} \qquad \frac{17}{12} \qquad \frac{41}{29} \qquad \frac{99}{70} \qquad \frac{239}{169} \qquad \frac{577}{408}$$

(convergents to the continued fraction) to the square root of 2

Suppose that you want to know if there's a number whose square is 2.

1 is too small, and 2 is too big.

So take the average, $\frac{3}{2}$.

Divide it into 2, giving $\frac{4}{3}$.

$\frac{3}{2}$ is too big, $\frac{4}{3}$ is too small.

We've already learned that the arithmetic mean is greater than the geometric mean!

Take the average of $\frac{3}{2}$ and $\frac{4}{3}$: $\frac{17}{12}$.

Then the average of $\frac{17}{12}$ and $\frac{24}{17}$: $\frac{577}{408}$.

$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}$, are the 1st, 2nd, 4th, 8th of the convergents;

and $3^2 = 2 \cdot 2^2 + 1$, $17^2 = 2 \cdot 12^2 + 1$, $577^2 = 2 \cdot 408^2 + 1, \ldots$.

The process doesn't stop!! $\sqrt{2}$ **is irrational!!**

In Babylonian $\sqrt{2} = 1; 24, 51, 10, 7, 46, 6, 4, \ldots$

compared with $577/408 = 1; 24, 51, 10, 35, 17, 38, 4, \ldots$

They knew that 1; 24, 51, 10 is better than 1; 24, 51, 11
and, that if they had enough clay tablets, they could get as close as they liked.

$a = 1$, $b = 2$ gives the Jacobsthal numbers (A001045)

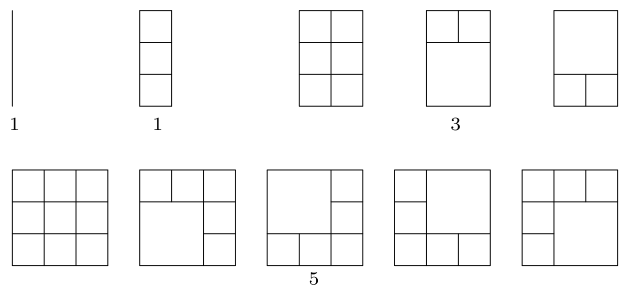
$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ \ldots$$

which, apart from the zeroth, are all odd. In fact

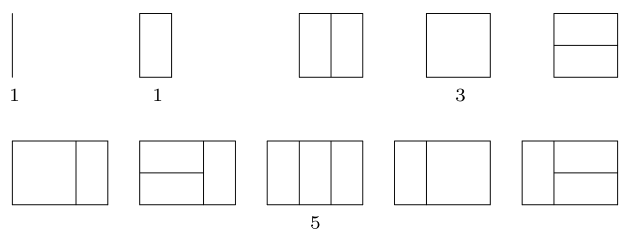
$$J_{n+1} = 2J_n + (-1)^n$$

They were useful to us when we analyzed Conway's "subprime Fibonacci sequences" [*Math. Mag.*, Dec. 2014.]

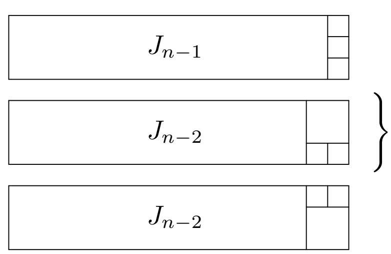
They are also the number of ways of tiling a $3 \times n-1$ rectangle with 1×1 and 2×2 square tiles.



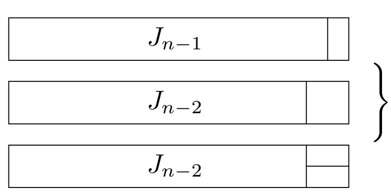
Or the number of ways of tiling a $2 \times n-1$ rectangle with 2×1 dominoes and 2×2 squares.



These facts follow from the following diagrams:



$$J_n = J_{n-1} + 2J_{n-2}$$



$$J_n = J_{n-1} + 2J_{n-2}$$

$a = x$, $b = -1$ gives the Chebyshev polynomials of the first kind, $T_n(x)$,

$a=2x$, $b = -1$ gives Chebyshev polynomials of the second kind, $U_n(x)$; (A049310 and A039599),

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_2(x) = 4x^2 - 1$$
$$U_3(x) = 8x^3 - 4x$$
$$U_4(x) = 16x^4 - 12x^2 + 1$$
$$U_5(x) = 32x^5 - 32x^3 + 6x$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$$
$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$$
$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$$
$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$$
$$U_{10}(x) = 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1$$
$$U_{11}(x) = 2048x^{11} - 5120x^9 + +4608x^7 - 1792x^5 + 280x^3 - 12x$$

which satisfy the following formulas:

$$(1-x^2)U_n'' - 3xyU_n' + n(n+2) = 0, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

Let's factor our original polynomials:

$$P_2 = \utilde{a}$$
$$P_3 = \utilde{a^2 + b}$$
$$P_4 = a(\utilde{a^2 + 2b})$$
$$P_5 = \utilde{a^4 + 3a^2b + b^2}$$
$$P_6 = a(a^2 + b)(\utilde{a^2 + 3b})$$
$$P_7 = \utilde{a^6 + 5a^4b + 6a^2b^2 + b^3}$$
$$P_8 = a(a^2 + 2b)(\utilde{a^4 + 4a^2b + 4b^2})$$
$$P_9 = (a^2 + b)(\utilde{a^6 + 6a^4b + 9a^2b^2 + b^3})$$
$$P_{10} = a(a^4 + 3a^2b + b^2)(\utilde{a^4 + 5a^2b + 5b^2})$$
$$P_{11} = \utilde{a^{10} + 9a^8b + 28a^6b^2 + 35a^4b^3 + 15a^2b^4 + b^5}$$
$$P_{12} = a(a^2 + b)(a^2 + 2b)(a^2 + 3b)(\utilde{a^4 + 4a^2b + b^2})$$

The underwaved polynomials are "primitive parts", analogous to the **cyclotomic polynomials**.

This illustrates that our sequences are **divisibility sequences**, that is:

$$m \mid n \text{ implies that } u_m \mid u_n$$

Here's how to see that: $u_n = au_{n-1} + bu_{n-2}$. Guess that $u_n = Ax^n$.

$$Ax^n = aAx^{n-1} + bAx^{n-2}$$

$$x^2 = ax + b$$

$$x = \frac{-a \pm \sqrt{D}}{2} \quad \text{where} \quad D = a^2 - 4b$$

say $x = \alpha$ or β, so that $u_n = A\alpha^n + B\beta^n$ and $u_0 = 0$ and $u_1 = 1$ give

$$0 = A + B, \quad 1 = A\alpha + B\beta, \quad A = -B = 1/(\alpha - \beta)$$

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and the divisibility is clear.

The Lucas-Lehmer theory tells us that a prime p divides

$$u_{p-\left(\frac{D}{p}\right)}$$

where $\left(\frac{D}{p}\right)$ is the Legendre symbol: ± 1 according as D is, or is not, a quadratic residue (square) mod p (or is zero if $p \mid D$).

For example, for the Fibonacci numbers the discriminant $D = 5$.
So u_{p-1} is divisible by p if p is of shape $10k \pm 1$,
and u_{p+1} is divisible by p if p is of shape $10k \pm 3$,
and u_{5n} is divisible by 5.

But this is not "only if" !!

For example $13 \mid u_{14}$ $(= 377)$ and hence $13 \mid u_{14k}$ for all k,

but in fact $13 \mid U_{7k}$ for all k.

We know that the "rank of apparition" of p is a divisor of $p - \left(\frac{D}{p}\right)$,

but we don't know which!

Here's something else we don't know!
A member, u_n, of one of these sequences can be prime only if n is prime;
since u_{pq} is divisible by u_p and by u_q.
But if p is prime, then u_p is not necessarily prime!
Among the Fibonacci numbers
$u_3 = 2$, $u_5 = 5$, $u_7 = 13$, $u_{11} = 89$, $u_{13} = 233$, $u_{17} = 1597$ are all prime,
but $u_{19} = 4181 = 37 \times 113$ is not!
We do not even know if there are infinitely many Fibonacci primes, ...
or infinitely many Mersenne primes, . . .
or infinitely many Brahmagupta-Pell primes, . . .
or infinitely many Jacobsthal primes, . . .
. . .
There are infinitely many things we don't know !!

but there are infinitely many things WE DO KNOW !!

That's the beauty of Mathematics !!