Euler Polyominoes

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Abstract In this paper we introduce the notion of an Euler Polyomino based on those polyominoes whose outline admit an Euler path, i.e., a path that traces every edge at most once. We examine the subset of polyominoes this produces and discuss the number of such tiles, and we pose some tessellation problems related to these tiles.

Keywords: polyominoes, euler paths, tiling.

1 Introduction

Polyominoes as defined by Solomon Golomb in [Golomb], consist of one or more squares joined orthogonally. The term is meant to include the special cases of a monomino (one square), domino (two squares), tromino (three squares, and the first case that has more than one representation, straight and bent), and so on (tetromino, pentomino…). These pieces have given rise to any number of puzzles [Kadon].

For the purposes of this paper, we are interested in representing each polyomino by the graph consisting of the exterior of all the squares that make it up. Thus, for example, a domino would have six external edges and one internal one. More particularly we will be concentrating on those polyominoes whose graphs have a Euler path, or more colloquially whose graph can be drawn with a single continous line which does not trace any edge more than once.

2 Euler Paths

In a graph, an Euler (or Eulerian) Path is a path that traverses the edges of a graph once and only once. If the path furthermore starts and ends at the same point, we call it an Euler Circuit [MathWorld: Weisstein, Eric W. "Eulerian Path." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/EulerianPath.html]. We are interested in analyzing just those polyominoes whose graph permits an Euler Path.

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To find these graphs is relatively simple due to a theorem that states that a graph possesses an Euler Path if and only if there are at most two vertices with an odd number of edges. In the case of graphs induced by polyominoes, the only possible number of edges arises from a vertex are 2, 3 or 4, so it amounts to saying there are at most two vertices with degree 3.

We can further show that there cannot be a single such point. The sum of the degrees of all the vertices equals twice the number of edges since every edges is counted for each of its ends, and therefore the number of vertices of odd degree has to be even. Therefore the only cases a polyomino can satisfy our criterion is to have either 0 or 2 vertices of degree 3.

A graph admits an Euler Circuit if and only if there are no vertices of odd degree, or in our case if every vertex has degree 2 or 4.

3 Enumerating Tiles

Our representation of polyominoes via their skeleton has an interesting side effect. Generally polyomino enumerations are distinguished by deciding which polyominoes are allowed and which are considered distinct. For example, Golomb excluded polyominoes with internal holes presumably because holes put a damper on the shapes that can be tessellated but later authors do not tend to respect this limitation.

Another property taken into account in counting polyominoes concern the transformations allowed. *Fixed* polyominoes are considered equivalent if they are related by a translation. In this version, for example, the domino occurs in two forms, vertical and horizontal. *One-sided* polyominoes are equivalent if they are related by a translation or rotation. In this case one has to look at a tetromino to find a polyomino not invariant under reflection.

Free polyominoes allow translation, rotation and reflection and represent what is normally meant when counting polyominoes. For example there are 12 free pentominoes whereas there are 18 one-sided pentominoes.

Our definition which focuses on the graph formed by tracing the outside of each constituent square has one unusual property which makes it a bit different from other counting methods, which is the fact that by doing so, any polyomino with an isolated hole is equivalent to the pentomino with the hole filled in, however this is not true of polyominoes surrounding holes of size 2 or greater. To make the distinction clearer we will refer to *graph* polyominoes to mean polyominoes counted in this manner. Note that the minimum number of squares to surround a hole of size 2 is 9.

The first thing that comes to mind given this definition is to ask how many there are. We only expect approximate bounds as there is no known formula for the number of polyominoes of a given degree. The best known bounds are [insert] and the number of polyominoes of each degree are represented as sequence XXX in the OECD [Ref].

4 Tiling Forms

The next thing we can attempt to do with these tiles is to see what shapes can be made. We restrict ourselves to all tiles with six or fewer squares and search determines there are 14 such tiles which have been provided as the other part of our exchange gift. The total number of squares adds up to 65 and so the first problem we set ourselves was trying to determine if the tiles excluding the first one could tile an 8x8 board (initial results suggest that the answer is “no” but analysis has not been completed, however we have managed some intriguing near misses.

Figure XXX shows some other symmetric shapes we have attempted (TODO attempt them!).

3.1 Path Tilings

We also wish to pose problems that make use of the Euler property. We define a group of tiles to define an Euler Pattern, if one can trace a path that traces out each tile separately before moving on the the next tile. In other words, a tiling such that the Euler Points (vertices of degree three) are adjacent. Since tiles that exhibit an Euler Circuit have no distinguished vertices, we treat them as wild and they can be introduced anywhere.

Tilings with this rule tend to be somewhat sparse and so we propose two questions:

1) What is the smallest bounding rectangle into which one can form an Euler Pattern employing all fourteen tiles?
2) What is the smallest number of internal holes possible in an Euler Pattern employing all fourteen tiles?

3) How symmetric can one make an Euler Pattern?

5 Software

Our initial explorations have been performed in Python and the ongoing effort is coded in [http://www.github.com/lymanhurd/euler-tiles]. Here you will find code defining the fourteen tiles of our initial set and code that successively tries to tile a region via a depth-first search.

Each tile has at most eight dihedral symmetries [Ref] consisting of four rotation and four reflections, however given symmetries some tiles will exist on only 1, 2, 4 different forms. The software expands the initial tile set to include the set of symmetries for each tile. Also, to speed search the software arbitrarily chooses one of the tiles with eight distinct forms and artificially restricts it to occur in only one of these forms, which forces the solution to be unique up to transformation.

6 Conclusions

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Acknowledgements

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References

[1] Entry 1

[2] Entry 2