The Composer

Glenn Hurlbert*

Abstract

A composer attempts to create a musical language from which to build a new form of compositional structure.

1 Rimsakovic Triads

Nick Rimsakov decides one day to see if he can write pieces of music based on chord progressions involving only those triads (triples of notes) whose notes are well separated. He settles on disallowing half and whole steps, meaning that each pair of notes is at least a minor third apart. He considers, for example, G and A to be too close, regardless of how high or low each note is played. That is, while particular instances of the two notes may be separated by several octaves, they still differ by a whole step on the 12-tone scale. This reduces the choices to diminished (gaps of two minor thirds), minor (minor third gap then major third gap), major (major then minor third gaps), and augmented (two major thirds) triads in each of the twelve keys. Since these are fairly consonant chords, he should be able to create nice sounds from various sequences of them. There are $3 \cdot 12 + 4 = 40$ such *separated* triads in all, reduced from the 220 triads that exist without restriction.

Furthermore, Nick would like to choose a special collection from this set of 40 that would have the extra property that any one triad can be followed by any other triad in a chord progression, thus dispensing with more complicated music theory. He feels that this, in some way, generalizes the pentatonic scale, a collection of 5 of the 12 notes (typically thought of as the black keys on a piano) that can be played in any order with pleasing results. Maybe his set of *Rimsakovic* triads can be played in any order nearly as pleasantly.

He reasons that to accomplish this he should require that every pair of Rimsakovic triads share a note. One way to realize this is to choose every separated triad that contains a C, for instance. This produces the following 10 triads.

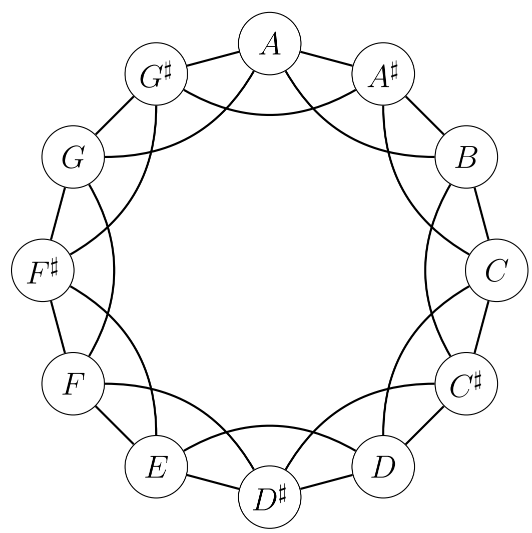
inversion	diminished	minor	major	augmented
root	$C\ E^\flat\ G^\flat$	$C\ E^\flat\ G$	$C\ E\ G$	$C\ E\ G^\sharp$
first	$C\ E^\flat\ A$	$C\ E\ A$	$C\ E^\flat\ A^\flat$	
second	$C\ F^\sharp\ A$	$C\ F\ A^\flat$	$C\ F\ A$	

*Virginia Commonwealth University, Richmond, VA 23284 ghurlbert@vcu.edu

To Nick, this collection doesn't seem reasonable because every single composition will have the C droning throughout. Nothing against drones, but this appears to be a fairly restricted vocabulary. So he wonders if it is possible to conceive of Rimsakovic triads that don't all contain some fixed note. If so, how many triads could there be in such a collection? More than 10?

2 A Graph Model

To represent the situation visually we can refer to the *graph* (set of *vertices* [points] with *edges* [lines] joining some pairs of them) below. Its vertices are the notes of the 12-tone scale, and two vertices are joined by an edge when they are too close together, as discussed above.



The forbidden pairs in Rimsakovic triads.

In graph theoretic terms, the 12 vertices with just the outer 12 edges is called a *cycle*, and is denoted $\mathcal{C}_{12}$ to indicate its number of vertices. For any graph $\mathcal{G}$ we can create a new graph called its *square* (written $\mathcal{G}^{(2)}$) by simply adding edges between pairs of vertices that are otherwise two edges apart from each other. For example, if we start with just the outer edges in the above graph, we see that B and $C^\sharp$ are two edges apart (through C), and so we would add the inner edge between B and $C^\sharp$. Thus the Rimsakovic graph above is $\mathcal{C}_{12}^{(2)}$.

Notice that the separated triads are precisely those triples of vertices that have no edges among them; e.g. the D minor triad $D\ F\ A$. Graph theorists call such things *independent sets*. The Rimsakov condition that each pair of triads share a note means that each pair of independent sets *intersects*. So a Rimsakovic collection of triads is an intersecting collection of independent sets of $\mathcal{C}_{12}^{(2)}$.

The intersecting collection above (with all Cs) is an example of what is referred to as a *star* — a collection of sets for which some element is in each of them. What Nick is wondering is whether there is an intersecting non-star collection of more than 10 independent triples of $\mathcal{C}_{12}^{(2)}$.

3 Some Background

In 1961 Paul Erdős, Chao Ko, and Richard Rado introduced the study of intersecting collections of sets, all of which have the same size r (the situation above has $r = 3$). Without any restrictions between the members of sets, they discovered that no intersecting collection is larger than the biggest star, provided $r \leq n/2$, where n represents the number of elements available ($n = 12$ above).[1] If $r < n/2$ then the biggest star is the unique maximum collection, but when $r = n/2$ there is one other, namely any collection formed by picking exactly one set from each pair $\{X, \overline{X}\}$, where $\overline{X}$ denotes the set of elements not in X (its *complement*). An example with $n = 6$ and $r = 3$ is below, the choices on the top, their complements on the bottom; both of the resulting collections are intersecting. Note that if instead we always choose the set that contains a 1, we construct a star.

123	124	346	126	134	246	245	236	146	156
456	356	125	345	256	135	136	145	235	234

Because the star is the unique best when $r < n/2$ it is interesting to wonder, then, what is second best. One idea is to fix a set of size 3 and always choose at least two from it. An example with $n = 7$ and $r = 3$ is below — the triples correspond to the columns and the 'x's identify their elements (i.e. the first triple is 123 and the last is 237); the lines merely highlight the structure. This *2-of-3* collection has 13 sets, compared to 15 for the biggest star.

1	x	x	x	x	x	x	x	x	x				
2	x	x	x	x	x					x	x	x	x
3	x					x	x	x	x	x	x	x	x
4		x				x				x			
5			x				x				x		
6				x				x				x	
7					x				x				x

One can imagine constructions such as *3-of-5*, *4-of-7*, and so on. If you know about the binomial coefficients $\binom{m}{k}$ (the number of ways to choose k objects from a set of m objects), then it is not so tricky to count the number of sets in such collections: for sets of size r in a t-*of*-$(2t-1)$ construction with n elements, there are $\sum_{j \geq t} \binom{2t-1}{j} \cdot \binom{n-2t+1}{r-j}$ sets. It can be a challenge to determine which of these is largest for various values of n, r, and t.

But it turns out that there is another construction, due to Anthony Hilton and Eric Milner, which is at least as big as (and usually bigger than) all of these. To describe it, let's use the elements $\{1, 2, \ldots, n\}$. We start with the set $12 \cdots r$ and then include every r-element set that contains n and at least one of 1, 2, …, or r (see below for the case $n = 7$ and $r = 3$).

1	x	x	x		x	x	x						
2	x	x		x				x	x	x			
3	x		x	x							x	x	x
4					x			x			x		
5						x			x			x	
6							x			x			x
7		x	x	x	x	x	x	x	x	x	x	x	x

In this instance, the Hilton-Milner collection has the same size as the 2-of-3 collection above it. But in general, it has $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$ sets, which is not only bigger than the above constructions but also bigger than every other possible non-star construction.

[1]If $r > n/2$ then every pair of sets intersect, so the entire collection is intersecting. For example, when $n = 5$ and $r = 3$ the collection of all ten triples of $\{1, 2, 3, 4, 5\}$ is 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345.

4 A New Paradigm

Because of the various instances in which there might be prohibitive relations that prevent some pairs of elements from being in the same set (such as making committees that avoid antagonistic pairs of people), we can model more general situations by graphs whose edges represent such prohibitions, as we've done for separated triads. In such graphs, we're looking for the largest intersecting collection of independent sets of size r.

Fred Holroyd first introduced these general investigations, which gave rise to a very challenging conjecture that is still unresolved today. The *independence number* of a graph $\mathcal{G}$ is the size of the largest independent set of $\mathcal{G}$ and is denoted $\alpha(\mathcal{G})$. For example, $\alpha(\mathcal{C}_{12}) = 6$ (one cannot do better than choosing every other vertex of a cycle) and $\alpha(\mathcal{C}_{12}^{(2)}) = 4$ (consider $A\ C\ D^\sharp\ F^\sharp$). Holroyd proved that if $r \leq \alpha(\mathcal{C}_n)$ then the biggest star is the largest intersecting collection of independent sets of size r. John Talbot then proved the same result for $\mathcal{C}_n^{(2)}$ when $r \leq \alpha(\mathcal{C}_n^{(2)})$. (He actually proved this for $\mathcal{C}_n^{(k)}$, the graph that joins each vertex to its k closest neighbors on the cycle.)

More subtle is the *minimax independence number* of $\mathcal{G}$, denoted $\mu(\mathcal{G})$. An independent set is *maximal* if it is not contained in some larger independent set. For example, $A\ C^\sharp\ F$ is maximal, though not maximum, in $\mathcal{C}_{12}^{(2)}$. Then $\mu(\mathcal{G})$ is the size of the smallest *maximal* independent set. Holroyd and Talbot conjectured that, for any graph $\mathcal{G}$, if $r \leq \mu(\mathcal{G})/2$ then the size of the largest intersecting collection of independent sets of size r is achieved by a star. This is a subject of very active current research. It is almost completely unknown, however, what kind of structure the largest non-star has; that is, there is nothing known about a Hilton-Milner type version for independent sets in graphs.

5 Rimsakovic Solution

Because of Talbot's result, we now know that the largest collection of Rimsakovic triads is the star collection in the opening chart (or other like it containing a fixed note different from C). But since Nick wants to avoid drones, maybe we should consider some of the other constructions we've learned.

For Hilton-Milner, we start with some independent triad. Since the construction needs a fourth independent element, the triad must be diminished, say $A\ C\ D^\sharp$, which makes the fourth element $F^\sharp$. Thus the collection consists of the 4 triads $A\ C\ D^\sharp$, $A\ C\ F^\sharp$, $A\ D^\sharp\ F^\sharp$, and $C\ D^\sharp\ F^\sharp$.

For the 2-of-3 construction we again start with the diminished triad $A\ C\ D^\sharp$ (other triads take up too much space). This yields the 8 triads $A\ C\ D^\sharp$, $A\ C\ E$, $A\ C\ F$, $A\ C\ F^\sharp$, $A\ D^\sharp\ F^\sharp$, $C\ D^\sharp\ F^\sharp$, $C\ D^\sharp\ G$, and $C\ D^\sharp\ G^\sharp$.

Now, this doesn't mean that there isn't some yet unthought of construction of 9 Rinsakovic triads. Is there? While you're pondering that question, Nick will start to experiment with various sequences of those 8 triads on his piano. Some are kind of haunting.

References

[1] A.J.W. Hilton and C.L. Spencer, A generalization of Talbot's theorem about King Arthur and his Knights of the Round Table, *J. Combin. Theory*, Ser. A **116** (2009), no. 5, 1023–1033.