Black or White

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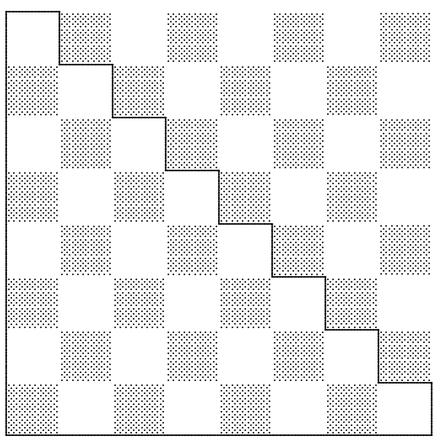
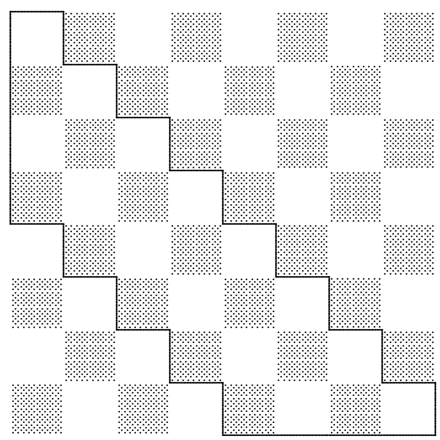
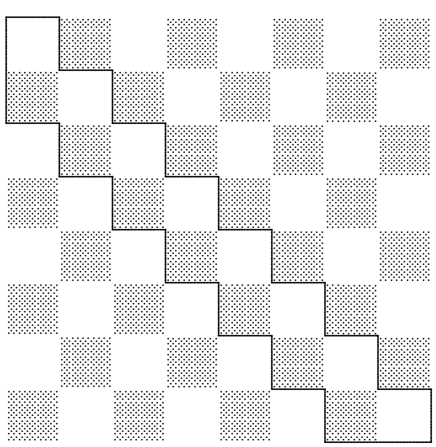
Anna chooses a cell of a standard 8×8 chessboard. She challenges Boris to deduce whether the chosen cell is black or white. He may choose a subboard consisting of one or more cells, bounded by a closed polygonal line with no self-intersections. She will then announce whether her cell is inside or outside this subboard. Naturally, Boris wants to minimize the number of subboards needed for accomplishing his objective.

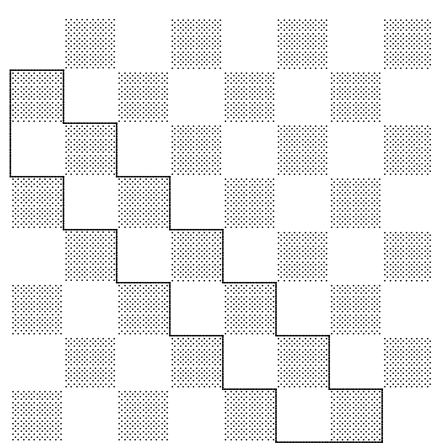
His first approach takes advantage of the fact that all cells along any diagonal are of the same colour. He comes up with the following method which requires four subboards. The first subboard is shown in the diagram below on the left. We may assume that Anna's announcement is "Inside". This narrows down the chosen cell to eight diagonals. Then the second subboard is shown in the diagram below on the right.





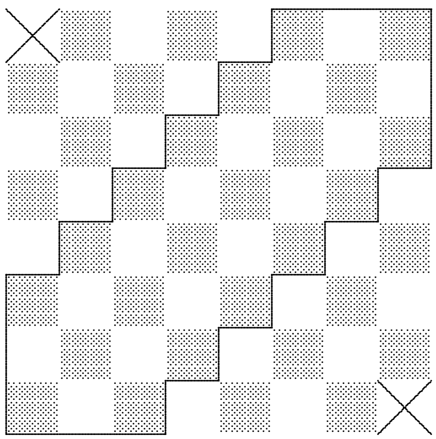
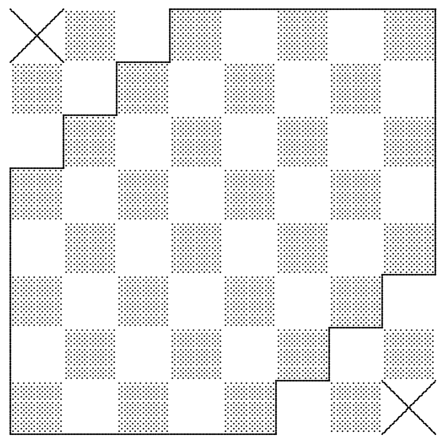
Again, we may assume that Anna's announcement is "Inside". This narrows down the chosen cell to four diagonals. Now the third subboard is shown in the diagram below on the left. We may assume that Anna's announcement is still "Inside". This narrows down the chosen cell to two diagonals. Finally, the fourth subboard is shown in the diagram below on the right.



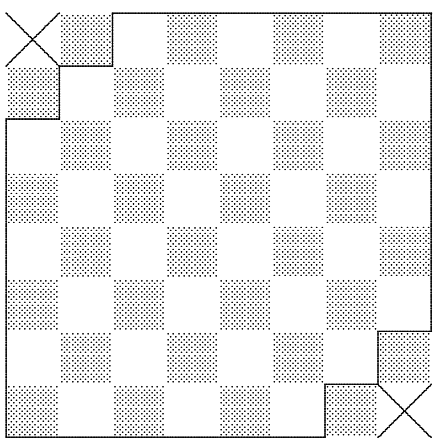
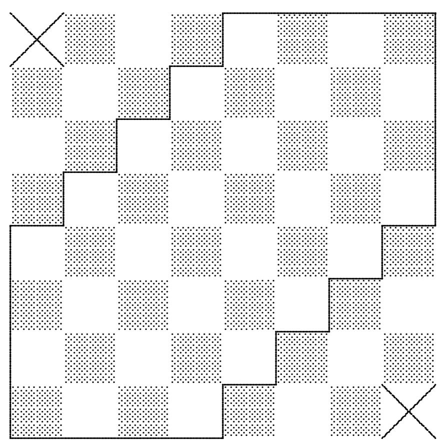


If Anna's announcement is "Inside", then her cell is black. Otherwise, it is white. In any of the preceding subboards, had Anna's announcement been "Outside", a similar and simpler situation arises.

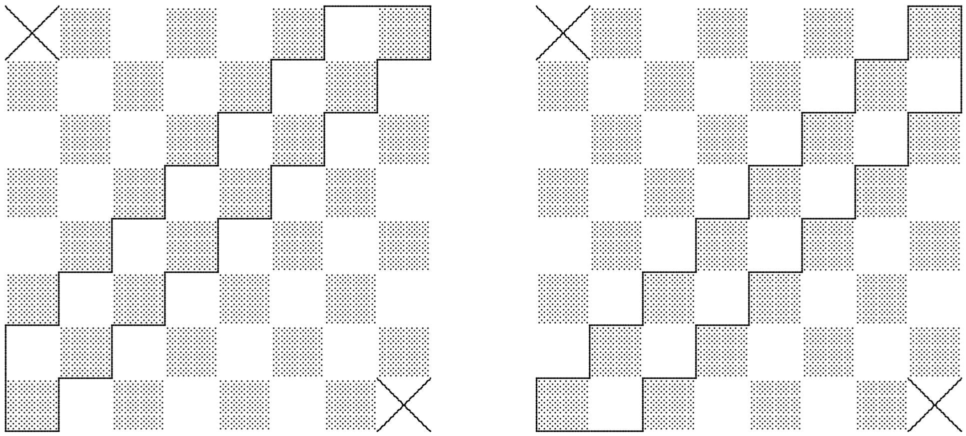
Rather pleased with what he has accomplished so far, Boris tries to simplify his approach and reduce the number of subboards necessary down to three. To his chagrin, he is unable to do so. His method only works if two cells in opposite corners are deleted from the chessboard. Then the first subboard is as shown in the diagram below on the left. If Anna's first announcement is "Outside", then the second subboard is as shown in the diagram below on the right.

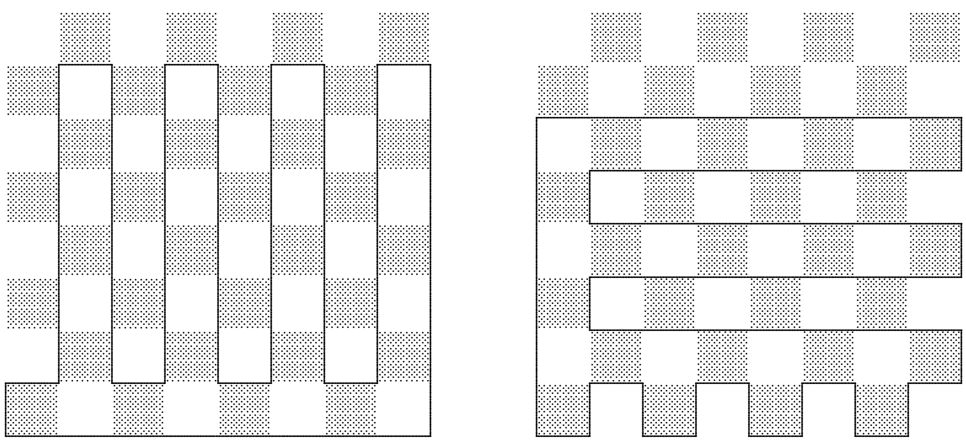
If Anna's second announcement is still "Outside", the third subboard is as shown in the diagram below on the left. If Anna's second announcement is "Inside", the third subboard is as shown in the diagram below on the left. In either case, "Outside" means black and "Inside" means white.

Suppose Anna's first announcement is "Inside". Then the second and the third subboards are as shown in the diagram below. If Anna's second and third announcements are the same, the cell is black. If they are different, the cell is white.

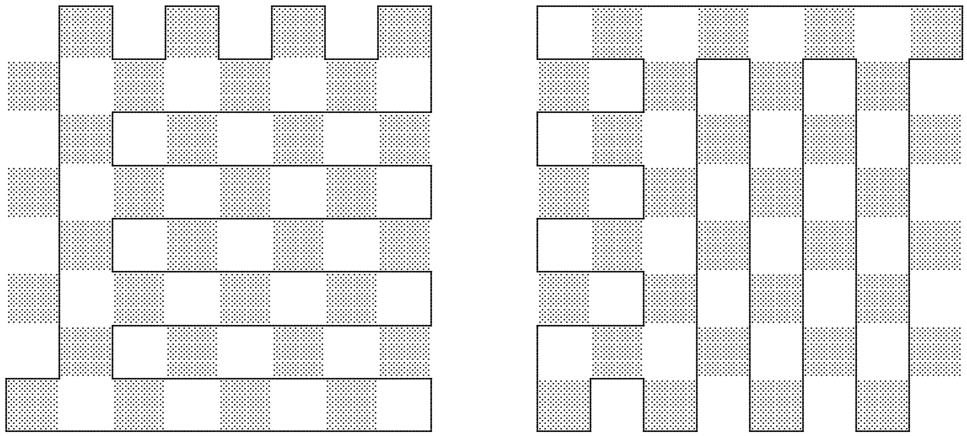


This is not entirely satisfactory. So Boris thinks of taking advantage of the fact that the black cells and white cells are symmetrically situated. In his new approach, the first subboard is shown in the diagram below on the left. By symmetry, we may assume that Anna's announcement is "Inside". Then the second subboard is shown in the diagram below on the right. If Anna's announcement is "Inside", then her cell is black. Otherwise, it is white.

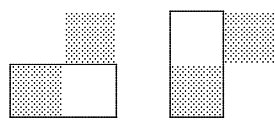


Clearly, one subboard will not be sufficient, since such a subboard must separate all the white cells from all the black cells. So the best possible result has been achieved. However, Boris is still not satisfied. In both approaches so far, he must wait for Anna's announcement before he can choose his next subboard. This is known as an *adaptive solution*.

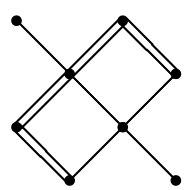
What Boris would like is a non-adaptive solution, in which he would present all subboards to Anna at the same time, and make the deduction upon receiving all the announcements simultaneously. After a while, he comes with one. The two subboards are as shown in the diagram below. If Anna's announcements are the same, the cell is black. If they are different, the cell is white.



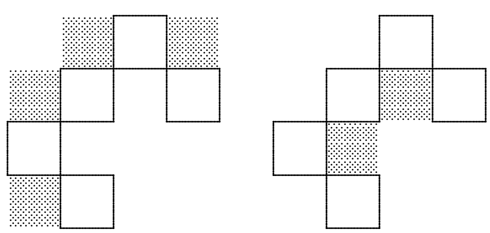
When Boris finally gets around to answering Anna's challenge, Anna is duly impressed. She has not thought that the task can be done using less than four subboards. So she tries to see if she can duplicate the accomplishment of Boris. She starts with the 2×2 chessboard, and there is indeed a non-adaptive solution using only two subboards. They are shown in the diagram below. If the two announcements are the same, the cell must be black. Otherwise, it is white.



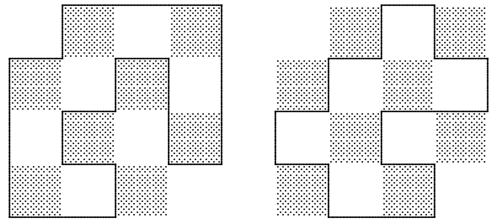
Anna now moves onto the 4×4 chessboard. There are eight white cells. She constructs a graph where each vertex represents a white cell, and two vertices are joined by an edge if a Bishop can move directly between the two white cells they represent. The longest Bishop path without including four vertices forming a square is indicated by a doubled line with three segments. The five white cells represented by the vertices on this path will be inside both subboards. The reason why a square is forbidden is because the four white cells represented by the vertices forming a square will enclose a black square, which must then appear inside both subboards also.



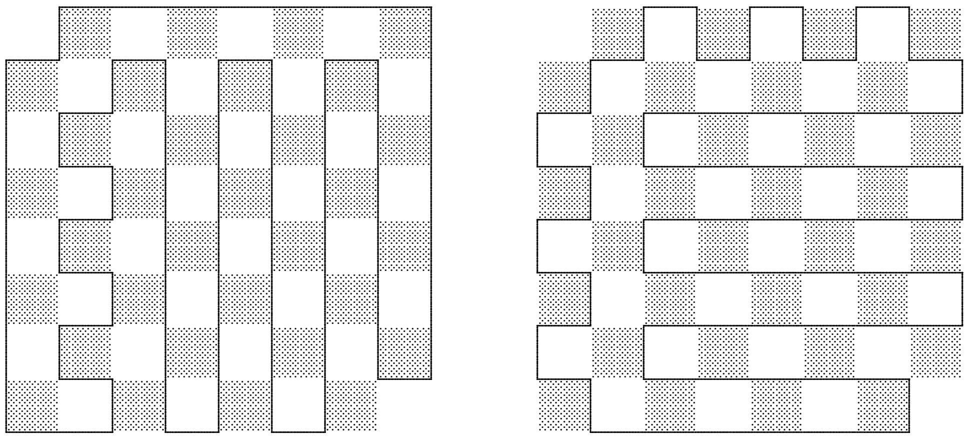
These five white cells may be connected by two disjoint sets of black squares, as shown in the diagram below.



The two subboards are shown in the diagram below. They have been extended so that each includes one of two black squares near the bottom right corner. If the two announcements are the same, then the chosen cell must be white. If the announcements are different, the cell must be black.



This scheme may be generalized to work for any $2n \times 2n$ chessboard. The diagram below shows the solution for the 8×8 chessboard.



The Fall 2013 A-Level paper in the International Mathematics Tournament of the Towns is the source of the challenge from Anna.