GATHERING 4 GARDNER 13

G4G13 Exchange Book

Legacy, Puzzles, & Science

Atlanta, Georgia

APRIL 11 - APRIL 15, 2018

VOLUME 2

G4G13 Exchange Book

VOLUME 2

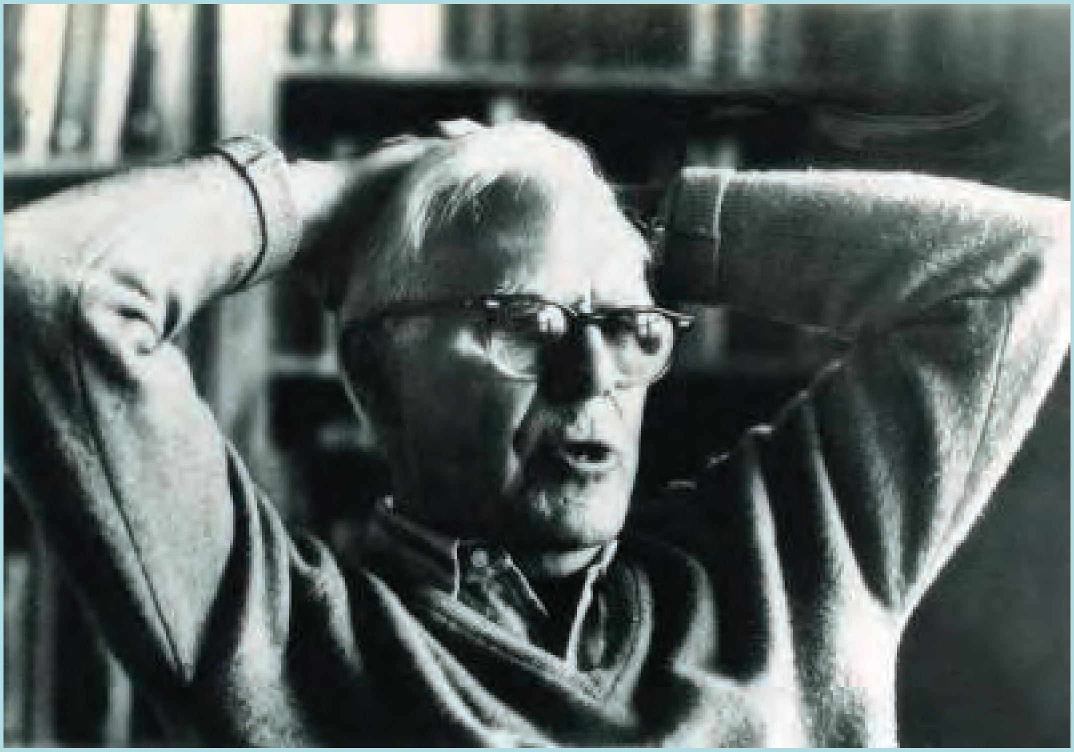
The Gift Exchange is an integral part of the Gathering 4 Gardner biennial conferences. Gathering participants exchange gifts, papers, puzzles and other interesting artifacts. This book contains gift exchange papers from the conference held in Atlanta, Georgia from Wednesday, April 11th through Sunday, April 15th, 2018. It combines all of the papers offered as exchange gifts in two volumes.

Table of Contents | VOLUME 2

LEGACY

PUZZLES

Table of Contents (cont.)



LEGACY

13 Parallels between Martin Gardner and Stan Freberg

John Miller (1949)
Portland Oregon
dialectrix.com/G4G

G4G 13
Atlanta

Gardner and Freberg had a curious number of co-incidences in their lives: Military service, magic, children's literature, popularization of subject material, and so on. This talk/paper will compare timelines based on their autobiographies and other biographic research.

These are parallels, more co-incidence than equivalence!

The Autobiographies

Martin Gardner's autobiography is *Undiluted Hocus-Pocus* (UH-P).

During the Centennial, I helped Colm Mulcahy glean U-HP for things to Tweet on @MGardner100th, so I noticed many details.

That same year, probably because Stan Freberg was one of my heroes, I discovered his autobiography, *It Only Hurts When I Laugh (*IOHWIL). I was struck by all the parallels between UH-P and IOHWIL.

The front jackets perfectly show the difference between these guys. Gardner, reserved vs Freberg, Leaning into you. Martin was 12 when Stan was born.

MAGIC || 1

Stan's live-in Uncle Raymond was "ConRay the Magician". Stan helped him set up. A rabbit was placed in ConRay's coat that was later pulled out of a hat and given to a member of the audience... always little Stan. Stan would exit the theater and then go in through the back door with the rabbit to use in the next show. When Stan's father objected to the trickery, Stan raised rabbits so that they could truly give them away. (There is a photo of ConRay in IOHWIL.)

Martin's father showed him a magic trick. (What was it?)

EARLY INFLUENCERS || 2

Martin's father gave him a copy of Sam Loyd's *Cyclopedia of 5000 Puzzles Tricks and Conundrums*.

On Oct 30, 1938, Stan's first time out eating alone in a diner, he heard Orson Welles' *War of Worlds* on the radio, live. He realized "it was just another marvelous use of radio". (Martin was in Tulsa, at Tulsa Tribune...)

The diner was The Root Beer Barrel, on Fair Oaks Avenue in South Pasadena, CA. IOHWIL P21. Freberg's other idols were Jack Benny and Fred Allen.

TEACHERS || 3

Young Stan Freberg's writing teacher said: *Creative, but Bombastic!*

Martin's math teacher: *You'll do Math in my class!* (Not Tic-Tac-Toe) P22 U-HP.

ARMY || NAVY || 4

Martin did PR for a RADIO signaling training school, stationed in Wisconsin.

Freberg entertained troops at a base hospital, stationed in Pasadena, CA. When off duty, he worked on radio, doing cartoon voices.

BASE NEWS in service || 5

Chosen for their aptitudes, both men edited newspapers as part of their military service.

Yeoman Gardner: *Badger Navy News, i*n Wisconsin before shipping out to USS Pope.

Corporal Freberg - *The Needle News, a*t Fort McArthur.

Freberg was first based at Fort McArthur, in Special Services, as a baker, till he got a mastoid infection, when he was taken to McCornack military hospital near Pasadena, where they put him to work locally.

TIME FOR BEANY || HUMPTY DUMPTY || 6

Gardner worked on a children's magazine while Freberg worked on a children's TV program!

Martin edited 10 issues/year of *HUMPTY DUMPTY* for 8 years, from 1952-1960 -- Obviously only part time from 1957 to 1960, once *Mathematical Games* began.

Freberg & others started *Time for Beany*, a TV puppet show in 1949.
Beany was the #1 Children's show, with a 70 share, for 5 years.
That's over one thousand 15-minute shows!

Anecdotes: After a stage hand dropped their cue cards during the Live show, Freberg's crew invented the first crude Teleprompter.

Einstein was a fan of Beany, and once left a meeting saying:
"You will have to excuse me, gentlemen. It's Time for Beany!"

JABBERWOCK(Y) SONG || ANNOTATED ALICE || 7

In 1960, Martin's *Annotated Alice* was a success. (Time warp ahead.)

In 1951, Walt Disney animated *Alice in Wonderland*. Freberg & Co. did The Jabberwock Song, but it was not included in final cut of the movie. The Chorus: Stan Freberg, Daws Butler, and the Rhythmaires.

Anecdote: The demo recording was included in the 2004 and 2010 DVD releases of the film. In the final release of the movie, the character of the Cheshire Cat (played by Sterling Holloway) sings the first few lyrics of the Jabberwocky poem when Alice encounters him.

DRAGONET || FLEXAGONS || 8

Martin's *Scientific American* article *HEXAFLEXAGONS* caused a lot of folding and flexing in December 1956 !

DRAGONET was an audio spoof of the TV show Dragnet, substituting a Knight for Detective Joe Friday. It sold a million copies in 3 weeks, in 1953. Freberg produced many such mini-radio comedies.

DRAGONET & *FLEXAGONS* put both men on career paths. They gained many followers.

JOHN & MARSHA || 8 continued

Freberg came up with a crazy vocal recording that went viral in 1957.
Some radio stations refused to play "John & Marsha", believing it was a secret recording of a passionate exchange between two real people.

Anecdote: The famous 1956 Snowdrift shortening commercial was a one-off of "John & Marsha". It was NOT Freberg's, but done by Hubley/Babbitt.

ORVILLE || DR MATRIX || 9

Freberg frequently appeared on *The Ed Sullivan Show* with another puppet, a moon man named Orville. They chatted about earthlings, smog, war, … Freberg was essentially banned from live TV for talking about the threat of Nuke War.

Martin often used Dr Matrix and others as a foil to discuss pseudo-science.

TYPEWRITERS || 10

Both men used Typewriters. (My talk showed Photos of each.)

STAN FREBERG PRESENTS THE UNITED STATES OF AMERICA || MATHEMATICAL GAMES || 11

Martin began *Mathematical Games* in 1957. The 4 Color Map Theorem was fun!

S*tan Freberg presents The United States of America* was a "Satirical Revue" with ten skits, from Columbus to the Battle of Yorktown, complete with voice actors, narration, a musical score, and sound effects.

It sold millions. I first heard this when I was in High School. I bought the vinyl album for $3.75 in the 1960's. A later CD edition contained Vol 1 & 2.

U.S. history teachers today play this in the classroom. It's very playful. There's a grain of truth in the skits. For example, 'Take an Indian to Lunch', was based on a 'Take a Negro to Lunch' campaign, if you can believe that.

The skits put you in the position of thinking about things like the folly of selling Manhattan for $26 worth of "Junk Jewelry", etc.

ADVERTISING || RECREATIONAL MATH, SKEPTICISM || 12

Gardner quote: "The best way, it has always seemed to me, to make mathematics interesting to students and laymen is to approach it in a spirit of play. Surely the best way to wake up a student is to present him with an intriguing mathematical game, puzzle, magic trick, joke, paradox, model, limerick, or any of a score of other things that dull teachers tend to avoid because they seem frivolous." -- Intro to Mathematical Carnival

Martin believed in the power of play. People were skeptical of having fun with Math.

Freberg quote: "The principle is the same on radio or television. My theory is, why should people be bored out of their skulls by advertising? If We have to live with it, why not make the commercials at least, if not more, entertaining than the show itself? What if people were afraid to leave the room for fear that they might have missed one of the entertaining commercials? Wouldn't that be the best of all possible worlds for the sponsor?"

Freberg poked fun at the "Hard Sell". He thought commercials should be Fun. You need to entertain people in exchange for their time.

Anecdote: Freberg's Radio/TV Mini-comedies were forerunners of SNL, et. al. Here's an example from 1962. Freberg made a group shot of 10 doctors in white tunics. Nine of them were obviously chinese. The one in front was a white guy. The caption or voiceover was:

9 out of 10 Doctors recommend CHUN KING chow mein.

This commercial/ad may have also appeared in print media.

Freberg probably created as many Ads as Martin did MG columns.

TELEVISION itself || 13

Gardner quote "Several years ago I read about a man who was so annoyed by the drivel on his television set that he blasted the screen with a shotgun." -- from MG's negative review of Jerry Mander's book *Four Arguments for the Elimination of Television* (p 361, Science - Good, Bad and Bogus)

Freberg was likewise incensed with TV programming, particularly commercials.

When CBS wanted Freberg to do a show, he wrote an opening with him holding a shotgun facing a shooting gallery with TVs moving along a conveyor belt…

ANNOUNCER (TV1): When Headache Strikes, you need fast Fast FAST relief!
BANG! CRASH!

ANNOUNCER (TV2): I tried everything for my constipation... BANG! CRASH!

ANNOUNCER (TV3): Stomach acid burned a hole right through this handkerchief, See? BANG! CRASH!

FREBERG TURNS TO FACE CAMERA: Good evening, I'm Stan Freberg.

CBS: You can't do that opening.

The Stan Freberg CBS show didn't last long, so Freberg turned to commercials.

Freberg refused to do commercials for cigarettes or alcohol, but he did for everything else.

My favorite: Cheerios, the 'Terribly Adult Cereal'. Naomi Lewis may be the actress in Cheerios.

Enjoy a smoked Esskay frank (Actor puts a lighter to a frank). Jesse White is in the ESKAY smoked franks commercial.

Martin appeared in 'Cigar and Rope' an early venture into TV, "MiniTrix".

Freberg's son Donavan got a nerdy reputation for his part in an encyclopedia commercial. The Encyclopedia Britannica commercial was EB's most successful ever.

THREE BONUS PARALLELS, IN CASE YOU DIDN'T LIKE SOME OF 1-13!!!

SCIENCE FICTION WRITER FRIENDS || 14

Science Fiction writer Ray Bradbury and Freberg were friends. Ray introduced Stan to Orson Welles. By then, Orson Welles was a Freberg Fan!

Isaac Asimov and Martin were members of the Trap Door Spiders group.

FATHERS || 15

SF: "My minister father believed that God intended us to find humour where we could in this over-serious world"

MG: His dad was christian in name only, a petroleum geologist, a kid at heart.

Similar fathers. Believed in humor, inquiry, honesty, not being too serious.

WIVES || 16

While doing a guest shot with Orville on a 1958 episode of The Frank Sinatra Show, Freberg met his wife-to-be Donna, who would also double as his producer until her death in 2000.

Wives Donna and Charlotte died in 2000, bisecting the men's lives.

Freberg however, promptly remarried.

TIMELINE

MG Born 1914 | SF Born 1926 | Depression era | MG Navy | Hiroshima |--- | SF Army | Beany/Humpty | Adman/MatGam | Wives Die 2000 |--- | MG Dies 2010 | SF Dies 2015.

Martin was ~12 when Stan was born.
Stan Freberg died three years ago, on April 7, 2015.

My awareness of both began in mid 1960's.

Book Reviews

It Only Hurts When I Laugh
His award-winning radio/TV commercials turned the advertising world askew. Convincing buttoned-down ad executives to allow him to poke fun at products was no joke, as detailed here in hilarious anecdotes.
https://www.publishersweekly.com/978-0-8129-1297-5

Undiluted Hocus-Pocus
Reviewed by Andy Magid
http://www.ams.org/notices/201403/rnoti-p281.pdf

Links

Freberg
https://en.wikipedia.org/wiki/Stan_Freberg
https://en.wikipedia.org/wiki/Time_for_Beany
Stan Freberg Presents the United States of America Volume One: The Early Years

Gardner
http://www.gamepuzzles.com/martin.htm
https://en.wikipedia.org/wiki/Martin_Gardner
https://simple.wikipedia.org/wiki/Martin_Gardner
https://en.wikipedia.org/wiki/Humpty_Dumpty_(magazine)

Miller (author)
http://dialectrix.com/G4G

Videos

Stan Freberg Commercials on YouTube (14 minutes)
https://youtu.be/_Bx4LBz8Xy4

Cheerios, the 'Terribly Adult Cereal'
https://www.youtube.com/watch?v=PauDwNFPucU

Jesse White - Esskay Franks commercial
https://youtu.be/vNjeIjuzliU

Betsy Ross and the Flag, audio starts here:
https://youtu.be/kCEE9pOkvQU?t=6m36s

Martin Gardner's 1950s "MiniTrix" films
http://martin-gardner.org/Film.html

Martin Gardner Cigar and Rope
https://youtu.be/uzr5jC93g-Q

Snowdrift Shortening Commercial 1956 (NOT FREBERG!) [Snowdrift]

Stan Freberg on Dick Cavett, McGovern-Hatfield amendment radio advertisment.
https://youtu.be/vt6iCTMp8xE

Awards

MG: Honorary doctorate from Bucknell University (1978)
MG: The American Institute of Physics science writer of the year award (1983)

Freberg is in the Radio Hall of Fame, the Animation hall of Fame, and the Songwriters Hall of Fame. Freberg won nearly two dozen CLIO awards, advertising's equivalent of the Oscar, and three Emmy Awards.

37 images used in presentation, not appearing here!

UH-P Cover, IOHWIL cover, ConRay the Magician, Sam Loyd's COP cover, Orson Welles, NAVY, ARMY, HUMPTY DUMPTY magazine covers, Beany and His Pals, Time 4 Beany puppetry, Beany Landing, Beany Telescope, Tenniel Jaberwocky, Disney's Jaberwocky, flexagon, DRAGONET, John-Marcia record label Snowdrift commercial, Orville, Martin Hard at Work, Stan in 1968 NYC, Four Color Map, SFPUSA Album Cover, 9 out of 10 doctors, Ducks with Three TVs, Cheerios, Esskay Franks, Cigar and Rope, Encyclopedia, Bradbury, Asimov, Stan Freberg flagwaver.

Stan Freberg

Lettered in Debate, performed a multi-voice solo play on stage in high school.
Creative writing, Critical Thinking.
He was an Audiomagician, creating radio-like programming in the TV era.
Did not play sports, but he could run fast!
Listed occupation on tax return: Guerilla satirist, marketeer.
Freberg had a boy and a girl, Donavan & Donna Jr.
Freberg did a radio ad for the McGovern-Hatfield amendment (withdrawl from Vietnam). An ad was criticized as being in Bad Taste. He publicly countered that on Dick Cavett show: "No, the war in Southeast Asia is Bad Taste."
Common Theme: Being told "You can't Do that".
Company: Freberg Limited (But Not Very)

Martin Gardner

Martin kept in touch with Milton Semer about shenanigans in Wash DC.
Martin's mother was a Methodist, liked colors, was K-Teacher.
Martin had two boys.
Martin: "I played a lot of tennis. My father was fairly wealthy, and we had our own tennis court. I also was on the high school tumbling team. I particularly liked the high bar."

Other Stuff

There are probably some deeper parallels, in Skepticism, or maybe in their early work (EG HUMPTY DUMPTY and BEANY episodes).

Martin Gardner: Annotator

Dana Richards

Martin Gardner was born to annotate. He only read what he was interested in, but he read with intensity. His library came to have tens of thousands of books and if you were to pick one at random from his shelf you would find the flyleaf contained a summary and it was copiously underlined with the occasional marginal remarks. The more philosophical the book the more comments to be found. He would take notes on cards about what he read and what he thought about it. He was constantly making notes on connections and then carefully filing them away. His filing system was legendary, both encouraging and rewarding correspondents.

In 1959 he wrote to Dennis Flannagan, his editor at *Scientific American*:

> "I had another idea, much earlier, for a different sort of magazine. I was going to call it *Marginalia*. It was to contain famous short stories with annotations by an expert, some professor. For example a short story by Fitzgerald."

He added that he would like to edit the magazine. The idea had intrigued him for over a decade. By this time he had already issued a lightly annotated *The Wizard of Oz and Who He Was* (1957), with a long discussion of L. F. Baum. More significantly, he had signed a contract early in 1958 to write *The Annotated Alice* (*AA*) (1960).

What is Annotation?

Annotation is an umbrella term covering many activities, each revolving around the central idea of "text." The text could be old and disputed. It could be unintelligible. It could be different things to different audiences. It could be one of many variants. In short, it could be misunderstood or have unappreciated significance.

The modern "textual scholarship" includes:

> systematic bibliography --- organized subject-based
> descriptive bibliography --- for the collector
> textual criticism ---definitive editions
> non-critical editing --- explication
> critical editing --- correcting and interpreting, and many others.

The original subject matter was incunabula, typically religious. But as texts multiplied and libraries bulged, all subjects invited scholarly guidance. Even so it was rare see the phrase "annotated edition" except for Talmudic/Biblical volumes. The "higher criticism" of nineteenth century German scholars was a prime example. "Critical editions" were common in the first half of the twentieth century; these featured light annotations (mostly glossary items) bound with scholarly articles.

Why is *The Annotated Alice* Different?

Martin Gardner did not model his book on any other prior work. He had a personal vision of what annotation should be. He was a free-lance writer raising a new family. Despite being loved by academics around the world he was not interested in adding to "the literature." He was interested in entertaining the public … by introducing them to the ideas, fascinating nuggets of gold, found in the scholarly literature. He would dig so they did not have to. He also cultivated the world of amateur scholars who, quite naturally, were interested in those aspects that the public would be too.

He began *AA* with "Let it be said at once that there is something preposterous about an annotated *Alice*." He explains that the modern reader needs help but that is not his main goal.

> "My task then was not to do original research but to take all I could find from the existing literature that would make the *Alice* books more enjoyable to contemporary readers."

The goal was enjoyment. He was guided by his own sensibilities.

> Yes, I often ramble, but I hope that at least some readers enjoy such meanderings. I see no reason why annotators should not use their notes for saying anything they please if they think it will be of interest, or at least amusing."

In his *The Annotated Thursday* (by G. K. Chesterton, 1999) Gardner says, "Many of my notes obviously tell much more than one needs to know to understand the novel. I hope they will be of interest nonetheless".

He had little interest in speculative academic exercises. He mainly did not imagine the public cared about academics arguing a thesis just to see if they could make it plausible.

> There are two types of notes I have done my best to avoid, not because they are difficult to do or should not be done, but they are so exceedingly easy to do that any clever reader can write them out for himself. I refer to allegorical and psychoanalytic exegesis. … Some learned commentaries of this sort are hilarious.

Vincent Starrett in a review stated, "I am certain of one thing: Nothing that ever can be discovered about *Alice* will make it a better story. Happily, Gardner feels the same way and has done his best to avoid inappropriate allegorical and psychoanalytic exegesis." In his *The Annotated Ancient Mariner* (1965) Gardner says, "The notes in this volume are intended to deepen the reader's understanding of the ballad as a straightforward narrative without going into more general questions of symbolic and moral intent." However he does discuss these in an afterword.

How Did He Do It?

The answer is research, research, and more research. The first type of research, as mentioned above, was a lifelong habit of careful reading. While he must have read for pleasure he never seemed to read to fill time. He was very fond of fantasy fiction (Dunsany, Chesterton, Cabell, etc.). He sought it out it, catalogued it and analyzed it, all while enjoying it. Everything was recorded on tens of thousands of file cards originally, and later, when he had the space, in a roomful of file cabinets. He was not necessarily researching a subject. His life seemed to be spent getting ready to write on a hundred subjects.

The second type of research was goal-oriented; when he had a book contract or when he was writing a column. We know when he was working on *In the Name of Science* and the *Annotated Alice* that he was a fixture at the New York Public Library. For many of us it is hard to imagine a time when research was not a click away. You had to read the footnotes, follow the notes, write to the authors (scores of them), and wait for the poor quality photostats. A shelf or two of reference works helped.

The third type is through cultivated correspondence. The follow-up to *AA* was *More Annotated Alice*. He said that he could put out a second volume without repeating any note from the first volume because he had accumulated a large box of letters from scholars and readers correcting, extending and adding to the existing notes. In addition there were decades of steady correspondence with a more focused set of experts who kept him abreast of the latest thing. He was a conduit more than a receiver inasmuch as every update he learned of he passed along in another letter.

Leslie Klinger, who has annotated many books (several with the same editor as Gardner, Robert Weil), reminded me that the true talent of the annotator lies in knowing when to ask, "What does that mean?" That is, knowing your reader and when something will be missed or misunderstood by that reader.

Why Is the Book So Successful?

This can only be speculated on. But the answer must lie with his successful tenure at *Scientific American*, where he delighted the public with monthly essays on mathematics for twenty-five years. He was successful in both ventures for the same reasons, I would argue. He did not write about math as a series of theorem-proofs. He made it come alive by analogies, parallels, and side-trips into magic, literature, art and other topics the public could relate to. Similarly, trusting his instincts, he knew that a popular annotated edition must be unfocused, wide-ranging and fun.

Recall that this is not patterned on prior work. Gardner single-handedly invented the genre. It was an immediate critical and financial success. His friend and editor Clarkson Potter, wrote to him, "[Your] fears for this book were groundless---for as I believed it would, it is as splendid as it has been successful." It was so successful that by 1962 Gardner's *The Annotated Snark* (Simon & Schuster) appeared.

Annotated editions of *Ancient Mariner* and *Casey at the Bat* soon followed. Further, Gardner introduced Potter to W. S. Baring-Gould who published the *Annotated Mather Goose* (1962) and the *Annotated Sherlock Holmes* (1967). He then advised Potter to have Michael Patrick Hearn produce the *Annotated Wizard of Oz* (1973). He encouraged Isaac Asimov and others. The number of annotated editions grew steadily in the 70's and 80's until the genre exploded (see appendix). The vast genre traces back *AA*, no further, and nearly all are patterned on Gardner's blueprint.

What Is the Future of Annotation?

Without a doubt, the future of annotation involves computers. However, as many have pointed out, the researcher who uses search engines lacks perspective. Search engines are remarkable, but they have "flattened" the landscape; you can go directly to something without the benefit of knowing how you got there. Annotation is the opposite. Annotation is all about the context.

Evan Kindley (*New Republic*, September 21, 2015) says it succinctly, "Not all rabbit holes are worth going down." He discusses the future of annotation and begins with the elephant in the room … crowd-sourcing. Consider Rap-Genius, now just Genius ("Annotate the World"). It started as a wiki-style website for rap lyrics. It now allows readers to annotate books. They even allow people to comment on *Alice*, but most of the "tates" are cribbed from Gardner. It is nice for people who have new insights to have an outlet for those. However, it should surprise no one that the signal-to-noise ratio is low on such sites. People without filters rarely say anything original and often are blithely wrong.

The legacy of *AA* is not just felt by Carrollians, it is that so many other books and communities have now bridged the gulf between scholarship and the public. I feel the world needs a new crop of "Martin Gardner"s. People that both research and filter, with humility and wisdom. We have many who have proven themselves, like Michael Patrick Hearn, Maria Tatar and Leslie Klinger, so there is hope.

Appendix

The point that *AA* was the root of a burgeoning endeavor is supported by this growing list of "annotated" editions.

- 1960 Alice, Potter
- 1962 Snark, Potter
- 1962 Mother Goose, Potter
- 1964 Uncle Tom's Cabin, Eriksson
- 1965 Ancient Mariner, Potter
- 1967 Sherlock Holmes, Potter
- 1967 Casey at the Bat, Potter

- 1970 Walden, Potter
- 1970 Lolita, McGraw Hill
- 1972 Don Juan, Doubleday
- 1973 Wizard of Oz, Potter
- 1974 Paradise Lost, Doubleday
- 1976 McGuffey Reader, Reingold
- 1976 Jules Verne, Crowell
- 1976 Christmas Carol, Potter
- 1977 Familiar Poems, Doubleday
- 1977 Frankenstein, Potter
- 1978 Shakespeare, Potter
- 1980 Gulliver's Travels, Potter
- 1981 Huckleberry Finn, Potter
- 1981 Poe (Tales), Doubleday
- 1982 Oscar Wilde, Potter
- 1986 Dickens, Potter
- 1987 Innocence of Father Brown, OUP
- 1988 Gilbert and Sullivan, Doubleday
- 1988 Ulysses, UCP
- 1988 Hobbit, Mifflin
- 1990 More Alice, Random House
- 1991 Night Before Christmas, Summit
- 1993 Sherlock Holmes, OUP
- 1994 Charlotte's Web, Harper
- 1995 Walden, Houghton Mifflin
- 1995 Jekyll and Hyde, Plume
- 1996 Gilbert and Sullivan, OUP
- 1997 Call of the Wild, UOP
- 1997 Lovecraft, Dell
- 1999 More Lovecraft, Dell
- 1999 The Man Who Was Thursday, Ignatius
- 2000 Definitive Alice, Norton
- 2000 (New) Wizard of Oz, Norton
- 2001 Huckleberry Finn, Norton
- 2001 Sherlock Holmes, Gasogene
- 2002 Classic Fairly Tales, Norton
- 2002 Flatland, Perseus
- 2004 Christmas Carol, Norton
- 2004 Brothers Grimm, Norton
- 2004 (New) Walden, YUP
- 2005 New Sherlock Holmes, Norton
- 2007 Secret Garden, Norton
- 2007 Uncle Tom, Norton

- 2007 Cat in the Hat, Random House
- 2008 New Dracula, Norton
- 2008 Hans Christian Anderson, Norton
- 2008 Turing, Wiley
- 2009 Origin (of Species), HUP
- 2009 van Gogh's Letters, Norton
- 2009 Wind in the Willows, Norton
- 2009 Maine Woods, YUP
- 2010 Pride and Prejudice, HUP
- 2010 Persuasion, Norton
- 2011 Phantom Tollbooth, Knopf
- 2011 Peter Pan, Norton
- 2011 Paradise Lost (Biblically), Mercer UP
- 2012 Frankenstein, HUP
- 2012 Emerson, HUP
- 2012 (New) Brothers Grimm, Norton
- 2012 Little Women, HUP
- 2012-2014 Sandman, DC
- 2014 New Lovecraft, Liveright
- 2014 Wuthering Heights, HUP
- 2014 Northanger Abbey, HUP
- 2014 Treasure Island, Fine & Kahn
- 2015 150th Alice, Norton
- 2015 Poe, HUP
- 2015 Importance of Being Earnest, HUP
- 2015 Malay Archipelago, NUS
- 2015 Emma, Anchor
- 2015 Grateful Dead, Simon & Schuster
- 2015 Little Women, Norton
- 2016 Mansfield Park, HUP
- 2016 Lincoln, HUP
- 2017 New Frankenstein, Liveright
- 2017 African American Folk Tales, Liveright
- 2017 Watchman, DC

While such a list contains biases it is fairly complete. I am aware of at least three additional editions that are in press; these 84 will soon be a hundred. Many "annotated" volumes have been excluded. For example *Bleak House* (Norton, 1977), *Green Gables* (OUP, 1997), *Uncle Tom's Cabin* (Norton, 2007), and *Frankenstein* (MIT, 2017) are more accurately described as critical editions. And the CUP edition of Catullus, is a scholarly translation. With the *Memoirs of Ulysses S. Grant* (Liveright, 2018) a new American History Annotated Series has begun.

Please contact me if you think this list needs to be updated.

The Animal Hunter through My Life

Tom Smith
Fair Haven Union High School

Everyone needs to start puzzle solving, or magic, somewhere, and so several of us started with “The Animal Hunter”. When we started, we had no idea it might be called a mental magic forcing device, that would come later. Nor did we probably think to work beyond the basic prop that we received. It was just a trick to fool our friends and family, but it is pretty clever!

My first experience with the prop came when I was seven or eight. I received an S. S. Adams magic set complete with three colored Cups and Balls, Magic Coin Box, Balancing Wand, Rice Bowls, Ball and Vase, and more. Several of them I could do, such as Ball and Vase, Balancing Wand, etc., but several ended up in the bottom of the toy box, including The Animal Hunter.

The Animal Hunter was a simple cream colored plastic disc about four inches in diameter. It had raised surfaces detailing seven animals and their names around the edge of the disc. These were printed in red. Below the picture of each animal, a hole was stamped through the disc. By today’s terms, it was a simple, durable, and somewhat attractive piece of magic. Unfortunately, with the direction booklet “gone with the wind”, that’s about all there was to it.

Time passed, as it always does, and the pieces of the magic set that could be found were retrieved from the abandoned toy box when a real interest in magic was renewed in ninth grade. With <u>The Amateur Magician’s Handbook</u> by Henry Hay in hand, many of the little tricks could be put to use. The Cups and Balls had new life, as did the black egg shaped Vanisher and The Rice Bowls. The Magic Coin Box could be enhanced by putting it into a ball of yarn, and The Three Shells could be used for the routine that was outlined in either MUM or Linking Ring. For someone on a very limited budget, the set turned into a gold mine, but that disc with the animals was still a mystery.

Moving ahead about ten years and I am teaching sixth graders about science, and in a unit on scientific method, we investigate ESP. To give students practice in following directions while exploring the topic, the VHS tape “Max Maven’s Mind Games” was put to good use. Students would follow Max’s instructions and be amazed at the outcome. One of the tests he presented dealt with astrology and involved a circle of symbols with a tail of four symbols outside the circle. Students would pick a number and begin counting that number starting on the first symbol of the tail, and then entering the circle on the count of five, and then continue their count, symbol by symbol until their secret number was reached. They then counted the symbols backward to the same number but avoiding the tail. Max then revealed the symbol on which they had landed. Once again, amazement, and then Max was off onto the next experiment. Little did I know, the methodology of The Animal Hunter had an inkling of a relationship with Max’s Astrology Experiment.

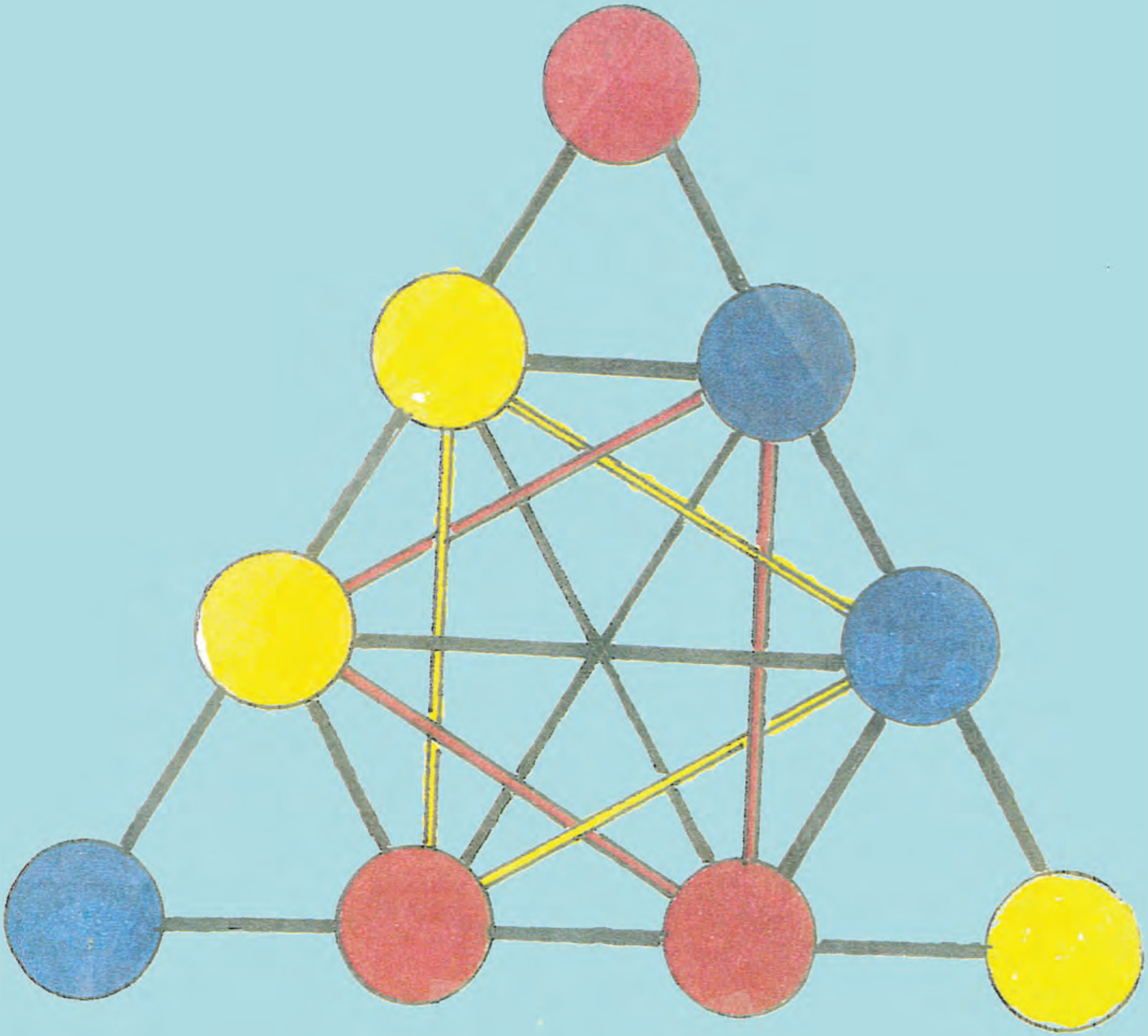
Last year at Magic Live 2017, I attended a session group about using magic as a tool to help children. Lo and behold, there it was… The Animal Hunter! Through the proceedings, we were shown the trick. Someone picked one of the animals and told no one. The performer tapped the various animals with a pencil while the participant silently spelled the animal's name. When the participant finished spelling, she said, "Stop!". The performer's pencil rested on the animal that was chosen. I had finally seen it! Continuing on, another person selected an animal and the routine was repeated, except different animals were tapped. Still, the performer ended tapping on the selected animal. Intriguing, indeed, to those of us not familiar with the workings of the chestnut.

We were then given a paper copy of the animal disc, which, in performance, wouldn't occur. Just as in the Astrology Experiment, show it and move on. Upon careful examination, the working became evident, especially with Max's Astrology Experiment, and it's tail. This was no random grouping of animals! They were carefully chosen! Now, the effect of the little plastic disc was clear, and it was clever!

While most folks stop with the animal disc, I wondered if the effect could be used in other ways. For one, since I teach chemistry, I could show students a wheel with the following elements: potassium, tin, tungsten, xenon, phosphorus, gold, helium, and mercury and the same effect, with a chosen element, would work beautifully. The effect could be repeated a few times and then the students could work in groups to try to discover the secret. Now they have something which they could show their parents and friends. This may seem like a tenuous connection to chemistry, but I believe that chemistry is all about patterns and problem solving, and that is what this problem would be.

How could you use "The Animal Hunter" in your field?

PUZZLES



Martin Gardner Puzzle-Games | Stephen Bloom, Lacey Echols, Jeremiah Farrell & Shannon Lieb | Page 39

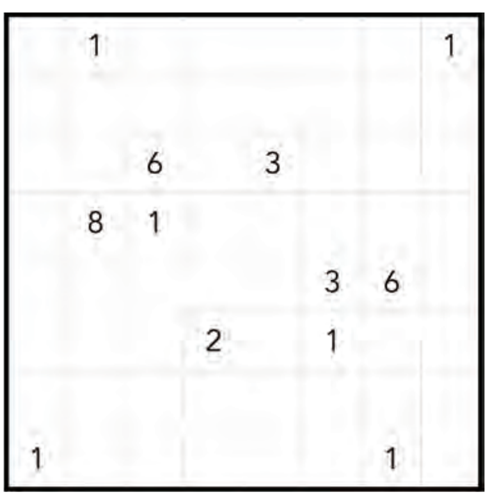
Shape Shikaku

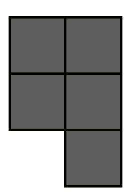
By Walker Anderson

Shikaku is a pencil-and-paper logic puzzle published by Nikoli. The goal of the puzzle is to partition the grid into rectangles along the grid lines. Each rectangle contains exactly one clue number which gives its area. The solution is unique. (rules from mellowmelon.wordpress.com)

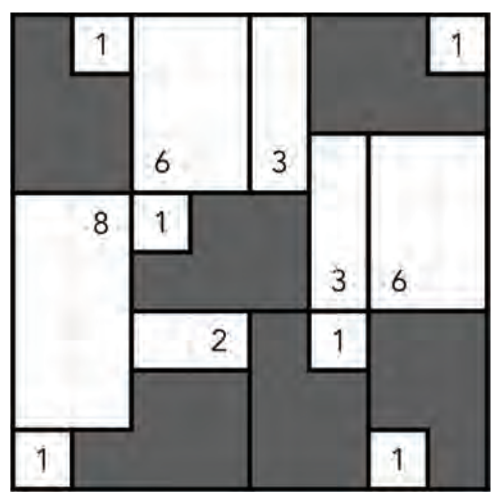
Shape Shikaku adds a change to these rules. Some of the squares in the grid will not be occupied by rectangles containing clue numbers. These squares must be copies of a shape that is provided to the right of the grid. The shape can be rotated and/or reflected, but it cannot overlap other shapes or rectangles in the grid. The number of shapes placed in the grid is given.

Here is an example puzzle, and its solution:



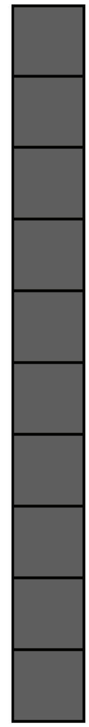


x 6



Long Division

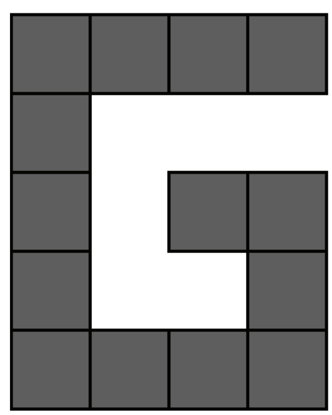
6									12
				4					
					8				
	10			4			6		
		6			4			4	
				4					
					6				
8									8



x 1

G for Gardner

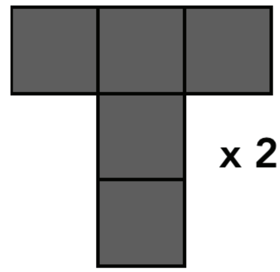
	9		9							
							8		9	
12										
										12
	2		2							
							12		4	



x 3

Zoom Lens

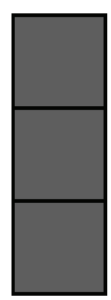
8									9
			5						
		5					5		
				3				2	
	3					2			
			2					3	
	4				4				
		6					5		
						6			
9									9



x 2

Hardwood Floor

							7	4	
5									
4									
									2
									8
	2	8							



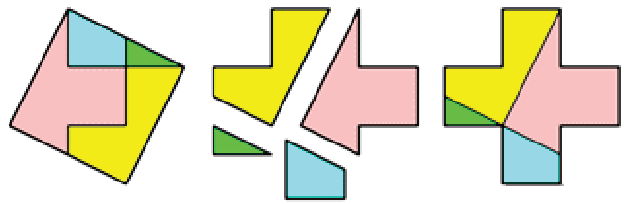
x 20

Development of the Loyd Polyominoes Puzzle

Donald Bell – **donald@marchland.org**

Summary

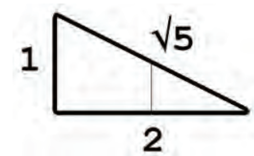
There are 5 tetrominoes and 12 pentominoes—17 polyomino shapes in total. The challenge is to find a group of only eight puzzle pieces that can make each of these 17 shapes. This design task is actually much more difficult than the ***Loyd Polyominoes Puzzle*** itself. There is a companion web site with downloadable files and other material at: **http://www.marchland.org/loyd**



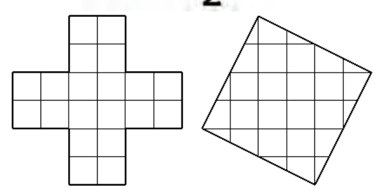
Background

About 100 years ago, Sam Loyd showed how to dissect a Greek Cross to a square in only four pieces. Note the small green triangle.

This is the starting point for quite a complex project, so it is necessary to give precise definitions to all the words being used, particularly those referring to assemblies of things. At one point we will have to consider collections of collections of collections. To be precise, seventeen "sets" of "groups" of "pieces". Each technical term will be highlighted in **CAPITALS AND BOLD** on first use. The word **SET** will be used in its mathematical sense of a collection of objects, no two of which are identical. But other technical words will just be defined as they are used. As Humpty Dumpty said in ***Alice in Wonderland***, "When I use a word, it means just what I choose it to mean."

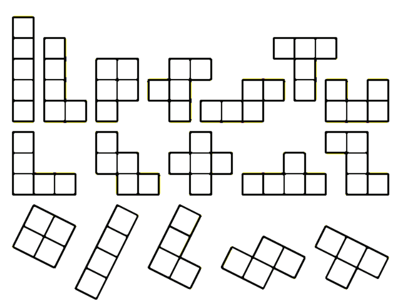


If the short side of the small green triangle is one unit, then its sides are 1, 2 and √5. It is one of two **BUILDING BLOCKS**. The other one is the unit square. Both of them have an area of 1 unit.



So, if the Greek Cross is 6 units wide and 6 units high as shown, then its total area is 20 units. This means that each side of the big square is √20 or 2√5 and its perimeter is 8√5.

For this project, a puzzle **PIECE** is made from one or more building blocks. There are well over 30 plausible shapes for puzzle pieces made in this way. The useful pieces have an area of 1 to 4 units. Several puzzle pieces can be put together to make a target **SHAPE**, like Loyd's Greek Cross or Square. The collection of pieces will be called a **GROUP.** The purpose of this project will be to identify a group of pieces that can make many target shapes. So, in the case of the Loyd Greek Cross and Square, the **group** of four **pieces** can make both of the two target **shapes**.

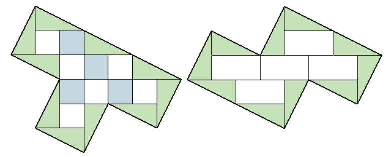


The Greek Cross is one of the **PENTOMINOES**, shapes made using five squares. There are 12 of them, usually known as I, L, P, R, S, T, U, V, W, X, Y and Z. Some people use the letter “F” for the R pentomino, and “N” for the S pentomino. The square is one of the **TETROMINOES** and there are 5 of them—square, I, L, skew and T. They are shown sloping to the right to match the Loyd dissection above. The skew may be "S"-shaped or "Z"-shaped.

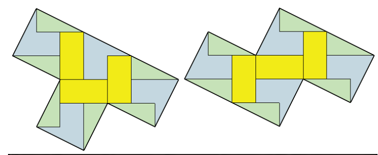
That makes a total of 17 target shapes, to be known as the **POLYOMINOES,** sometimes spelled "polyominos". The task is to find a group of the smallest number of puzzle pieces that can make each of these 17 shapes.

First Attempts

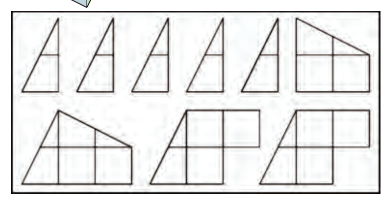
A group of 16 triangles and 4 squares can make all of the pentominoes and tetrominoes. For the tetrominoes, there are 10 triangles around the perimeter and the other pieces fill the interior. But 20 pieces is a big number. Can it be reduced?



By combining some building blocks in pairs, this number 20 can be reduced to about 13, as shown for the T and S tetrominoes. But it is difficult to get any lower using this method.

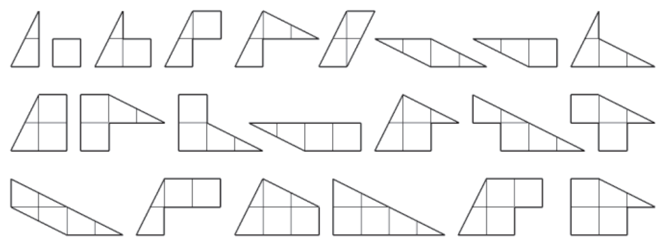


A research group at the Politecnico di Torino published this group of pieces that can make all 17 polyominoes. Web reference:
`http://www.iread.it/Poly/tepe_diss_en.php`
It has only nine pieces, five of them being the basic triangles.

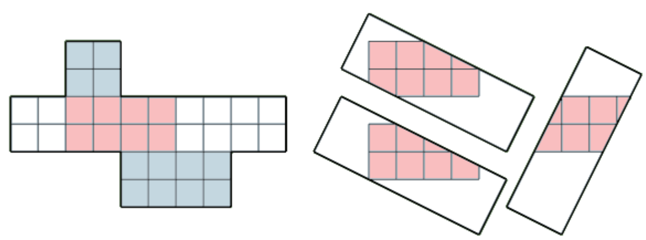


Analysis of the Problem

It is not easy to identify the most appropriate puzzle piece shapes to try. The unit square and small triangle can be glued together in many ways to make plausible puzzle pieces, ranging in area from one unit to six. Then each group of pieces that look promising must be tested against all 17 of the target shapes. A very tedious process!



No puzzle piece can be larger than 6 units. In the left diagram a white I pentomino is laid over a grey W pentomino. The pink rectangular area common to both has an area of 8 units. But when this is overlaid by an I tetromino, the pink area is reduced in one of the three ways as shown.

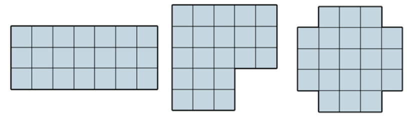


There are about 30 ways of combining small triangles and squares to make plausible puzzle pieces. These can be assembled into hundreds of groups and each group has to be tested to see if it can make all 17 of the target polyomino shapes. Obviously a computer is needed to do some of the computational "heavy lifting", and a program such as "Burr Tools" is called for. But, even then, a lot of human intervention is required, together with a rather sophisticated search procedure.

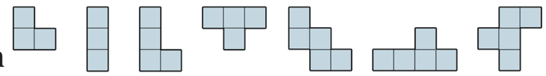
Using Burr Tools to Solve Puzzles

To illustrate how this can be done, we will set up an example problem and use Burr Tools to help in the search for a solution. This worked example is probably simple enough to be solved without a computer, but the real application, involving all the polyominoes, needs both Burr Tools and some new supporting computer programs, both for data preparation and for analysis. These will not be described in detail, but their main features should be fairly easy to follow.

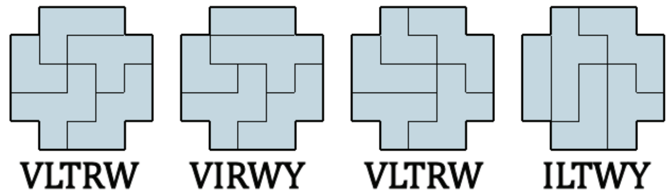
In this example there are three target shapes: "block", "gamma" and "cross". All have the same area, 21 units.



And there is a collection of pieces that is more than enough to cover that area. The pieces are called V, I, L, T, W, Y and R. They have a combined area of 29 units. So, any solution will use some of the puzzle pieces, but not all of them.

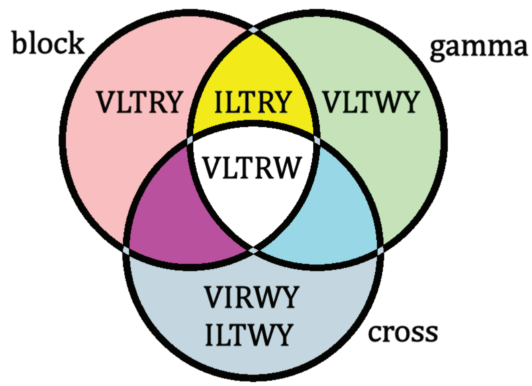


Here are some of the solutions for the "cross" target shape. The group of pieces for the first and third one is VLTRW, and the groups for the others are shown. But although there are many solutions, there are only three different groups.



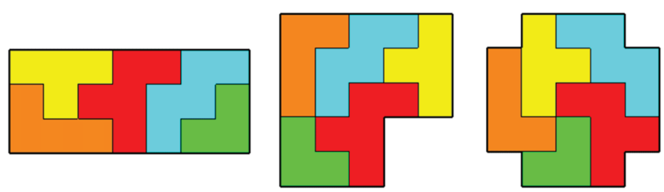
The shape “block” has several possible groups of pieces:
VLTRY, VLTRW, ILTRY.
And for “gamma”, the groups are:
VLTRW, ILTRY, VLTWY.

The task is to identify a single group of pieces that can make ALL THREE of the target shapes. The results can be drawn on a Venn diagram. Each of the circles is the set of groups of pieces that can make one particular shape.

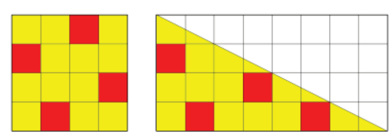


So the common group is VLTRW, shown in the centre:

And here it can be seen that, indeed, the group of pieces VLTRW can make each of the three target shapes.

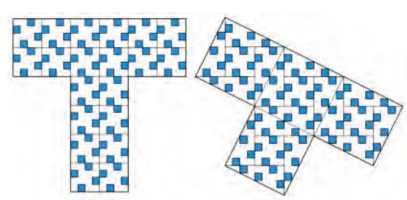


Adapting Burr Tools for the [1, 2, √5] triangle
Burr Tools usually deals with squares or equilateral triangles. So a modification is needed for the [1, 2, √5] triangle. These diagrams show how this was done. Everything was quadrupled in size and the unit square and triangle were represented like this:



It is a laborious and error-prone task to get the coding of all the puzzle pieces and polyomino shapes exactly right.

So a small shape definition program, written in Python, was used to help prepare the puzzle data. These diagrams show the T pentomino and the T tetromino.



The xmpuzzle file format
Burr Tools uses an XML format to describe the composition of puzzle pieces and target shapes. Everything is embedded in a file with an **`xmpuzzle`** extension. Usually this file is "zipped" before being written to disc. The **`xmpuzzle`** file also shows the details of the puzzle and, if some solutions have been found, these are embedded in the file as well before it is saved back to disc.

So it is possible to unzip these files, make some changes manually, and present them again to Burr Tools for further computation.

The structure of an **`xmpuzzle`** file is a bit complicated, but here is a condensed version of the file for the "block gamma cross" example puzzle above. Some of the XML has been removed for brevity, as well as parts of those sections which have a lot of repetition.

The main sections are these:

- A definition of all the shapes, both target shapes and puzzle pieces (in yellow).
- Indication of which shape is the target.
- Choice of the pieces to be used and how many of each (in blue). This might be a fixed number or a range of numbers.
- If the program has been run, the solutions are written back into the file (in pink).

```
<?xml version="1.0"?> <puzzle version="2"> <gridType type="0"/> <colors/>
 <shapes>
  <voxel x="3" y="7" z="1" type="0">#####################</voxel>
  <voxel x="5" y="5" z="1" type="0">###__###__###############</voxel>
          --(more lines like this)--
  <voxel x="4" y="2" z="1" type="0">####__#_</voxel>
  <voxel x="3" y="3" z="1" type="0">_#_##__##</voxel>
 </shapes>
 <problems>
  <problem state="2" assemblies="6" solutions="0" time="0">
   <shapes>
    <shape id="3" min="0" max="1"/>
    <shape id="4" min="0" max="1"/>
         --(more lines like this)--
    <shape id="9" min="0" max="1"/>
   </shapes>
   <result id="0"/>  <bitmap/>
   <solutions>   <solution>
     <assembly>0 0 0 0 x 2 6 0 10 1 0 0 18 x 0 3 0 18 2 4 0 10</assembly>
     <assembly>0 0 0 0 x 2 6 0 10 1 4 0 16 2 2 0 10 x 0 4 0 20</assembly>
        --(more lines like this)--
     <assembly>x 0 1 0 0 0 6 0 20 0 1 0 2 x 2 4 0 22 0 3 0 0</assembly>
    </solution>  </solutions>  </problem>  </problems>  <comment/>
</puzzle>
```

Without going into all the details, it can be seen that the **`<voxel>`** sections (in yellow) are describing the shapes of the three target shapes and the seven puzzle pieces. The symbols **`"#"`** and **`"_"`** (sharp and underscore) represent filled and empty cells in a matrix.

The **`<solutions>`** section (in pink) describes the solutions that have been found. The numbers come in sets of four (with a simple **`"x"`** if a piece is not being used). So it is a straightforward programming task to do some string processing and identify the group of pieces that has been used for any one solution. For our purposes the group is more relevant than the full solution.

Putting it all together

Having described the various logical components of the investigation, let's have a look at all the procedures involved. Two sorts of experiments were done:

- A search for a group of pieces, with no duplicates, which can make each of the 17 polyominoes.
- A search for the smallest number of pieces, this time allowing any number of duplicates.

Suppose we have a **BOX** of pieces, possibly containing some duplicates, and we present it to Burr Tools together with just one of the 17 polyomino shapes. We will then get a **COLLECTION** of **SOLUTIONS** for that particular shape. This process is then repeated 16 more times to cover all the target shapes.

The choice of the pieces in the box is a matter for the human investigator. If there are too few pieces, or if they are badly chosen, there may not be enough variety for a complete solution to emerge. But, if there are too many pieces, the computing complexity may be too great.

Within the collection of solutions for one target shape, there may be a group of pieces that is used for more than one solution. We are more interested in the **groups** than in the **solutions**. So the collection of groups needs to be reduced to a **SET** with no repetitions. The word **set** is being used in its mathematical sense of a collection of objects, no two of which are identical. Each object in the set is a group of puzzle pieces, usually between 8 and 11 pieces in any one group.

In this way, we will get 17 sets of groups of pieces and we need to find a group that is present in all 17 sets. So we make an "intersection" of the 17 sets, looking for the one element that is present in all of them. This way we hope to get a group of pieces that can be used to make all 17 polyominoes.

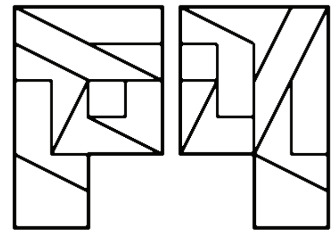
There may be several such groups, or there may be none at all. This would mean that, for that particular selection of pieces in the box, there is no group that can make all of the 17 target shapes. So it may be necessary to adjust the box of plausible pieces and try again. That might mean doubling up on a few pieces, including or inventing new ones and leaving out others.

Mirror Image Target Shapes

And now for a small complication. The puzzle pieces that we are working with are all non-symmetrical, except for the unit square. But the only angles in the polyomino target shapes are right angles.

So it is not possible to turn over just one puzzle piece and leave the others as they were. Therefore, unlike most put-together puzzles, it is NOT permitted to turn over any piece, unless all of them are turned over.

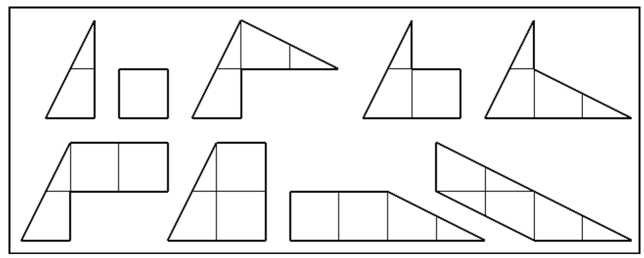
This means that if there is a solution to one target shape, the P pentomino for example, there will be quite a different solution for its mirror image, as seen in this nine-piece assembly.



Or there could be a polyomino which has a perfectly valid solution, but its mirror image has no solution at all.

So, although there are just 12 pentominoes and 5 tetrominoes, there are actually 8 more shapes to be considered if the mirror images of the non-symmetric polyominoes are included. This means that we can, for example, include the "skew tetromino" shape if there is a solution for its "Z" configuration, even though there is no solution for its "S" configuration.

A Nine-Piece Group with no Duplicate Pieces
Here is a group of 9 pieces that are all different. It can make all the 17 polyominoes.

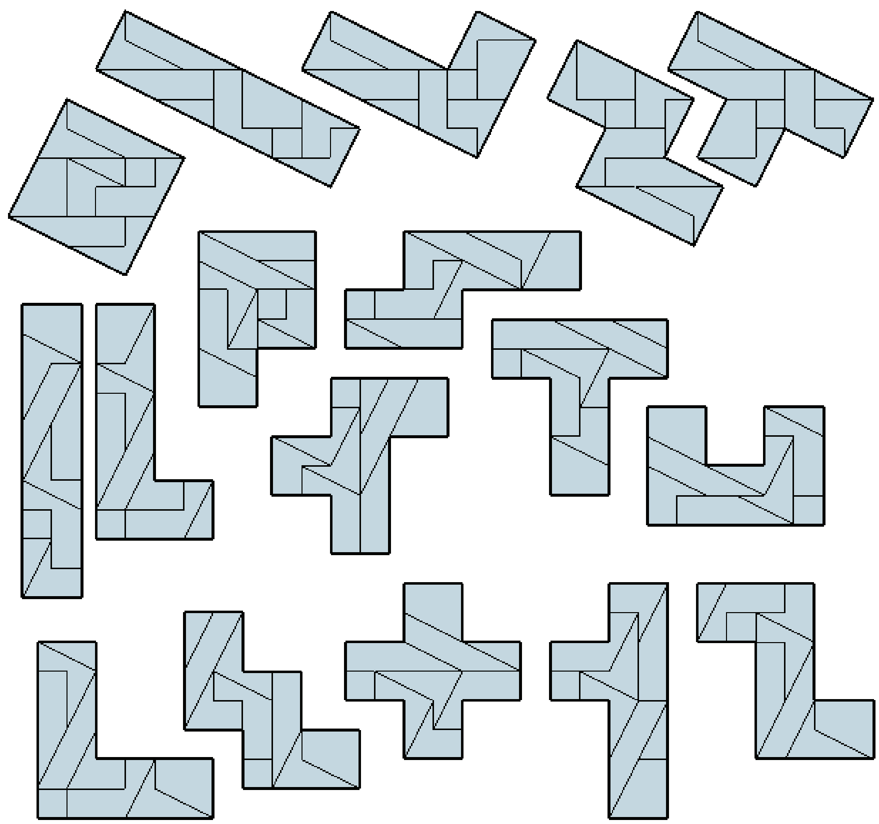


Using this nine-piece group, there are several solutions for all the polyominoes and some, but not all, of their mirror images.

This diagram gives an indication of the complexity of the solutions in this project.

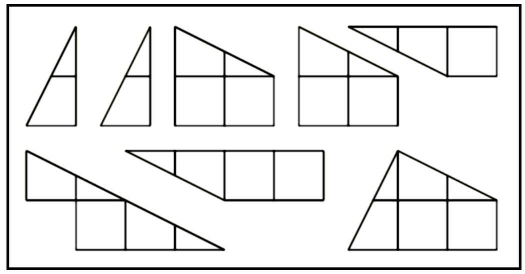
The skew tetromino has been drawn in its "Z" configuration. That is because there is not a solution in its "S" configuration.

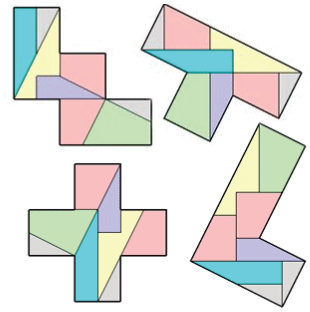
There are also no solutions for the mirror image shapes of the L tetromino and the R, S, and Z pentominoes.



Eight-Piece Groups
So far, three groups of eight pieces have been found, one of which is shown here.

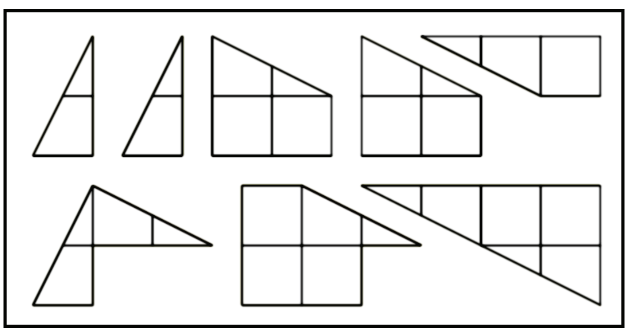
Some solutions using this group are shown at far right.

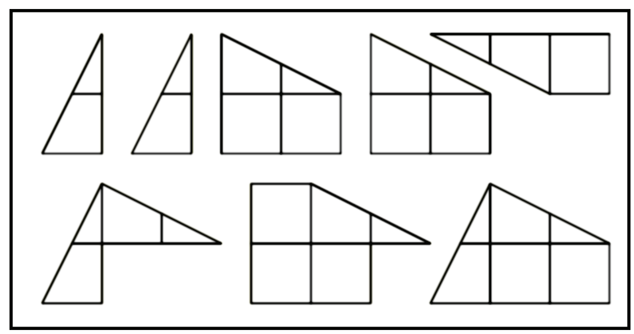




This was my Exchange Gift at the Gathering for Gardner in Atlanta in 2018. It has quite a small number of solutions for all of the tetrominoes and pentominoes and nearly all of the mirror image shapes, too. So it is a challenging collection of puzzles. It can't make the "Z" configuration of the skew tetromino, so the total number of puzzles is 24, not 25.

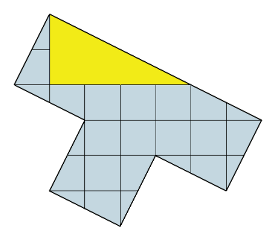
And here are two more groups of eight pieces which can make the 17 polyominoes. In each case the top row of five pieces is the same as in the group above.





Conclusions and Opportunities for Further Work
The procedure to find these groups of pieces was far from straightforward. The number of intermediate solutions found by Burr Tools was huge, and the computer frequently ran out of storage and just stopped.

Sometimes a consideration of the target shapes demonstrated that a particular puzzle piece could be used once but not twice. The long edge of the big triangle can't, for example, fit twice into the perimeter of the T tetromino.



So it has not been possible to do an exhaustive search for the very best groups of pieces that can make all 17 polyominoes. But the difficulty of finding a group of just eight pieces suggests strongly that no seven-piece group exists.

The solution in nine pieces may not be the only one, and it is possible that there is an eight-piece group with all the pieces different, which is still waiting to be found. Bigger computer needed!

Please email me with comments, discoveries and suggestions.

One Puzzle

Colin Beveridge

March 10, 2018

1. The Broken Calculator

A calculator is missing all of its keys but sin, cos, tan, SHIFT[1] and =. It initially starts with 0 on screen. Show that the calculator can produce any positive rational number.

[1] That is to say: the inverse trigonometric functions are also available.

Some functions

By applying one of the three inverse functions to a number (assuming it is in the relevant domain) and one of the direct functions to the result, we end up with a (generally different) number. It's worth exploring some of the things we can do with such compositions.

A useful composition would be one that took a number greater than one and returned its inverse, so that the output is in the domain of all three inverse functions.

This can be arranged by considering a right-angled triangle as pictured, with $q > p$. $\arctan\left(\frac{q}{p}\right)$ gives angle Q. The cosine of Q is $\frac{p}{\sqrt{p^2+q^2}}$, and the arcsine of this is angle P. Finally, $\tan(P) = \frac{p}{q}$, the reciprocal of the original argument.

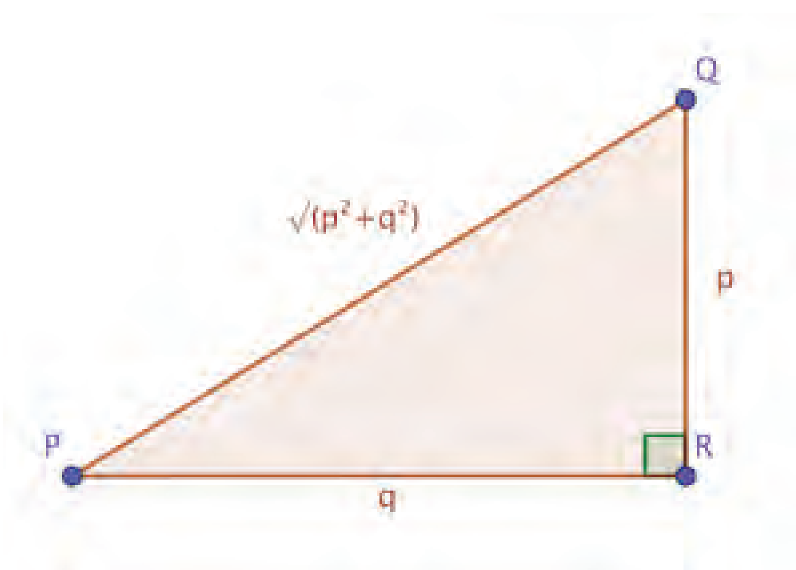


Figure 1: A triangle

Definition: Let $R(x) = \tan(\arcsin(\cos(\arctan(x)))) = \frac{1}{x}$.

Given a number smaller than one, where do the various compositions leave us? Ignoring the self-inverse compositions, and assuming $p < q$, we have:

- $\sin\left(\arccos\left(\frac{p}{q}\right)\right) = \frac{\sqrt{q^2-p^2}}{q}$
- $\cos\left(\arcsin\left(\frac{p}{q}\right)\right) = \frac{\sqrt{q^2-p^2}}{q}$
- $\tan\left(\arccos\left(\frac{p}{q}\right)\right) = \frac{\sqrt{q^2-p^2}}{p}$
- $\cos\left(\arctan\left(\frac{p}{q}\right)\right) = \frac{q}{\sqrt{p^2+q^2}}$
- $\tan\left(\arcsin\left(\frac{p}{q}\right)\right) = \frac{p}{\sqrt{q^2-p^2}}$
- $\sin\left(\arctan\left(\frac{p}{q}\right)\right) = \frac{p}{\sqrt{p^2+q^2}}$

I've arranged these in three pairs, such that each element of a pair is the other's inverse over a domain of at least $0 \le \frac{p}{q} \le 1$.

The first pair of functions aren't especially interesting, but either of the last two pairs can be used to great effect. I'll pick the last pair, and give them names.

Definition: Let $T_s(x) = \tan(\arcsin(x))$.

Definition: Let $S_t(x) = \sin(\arctan(x))$.

With these two functions, and $R(x)$ from before, we can solve the puzzle.

A solution

Proposition: Any positive rational number can be produced by applying a composition of the functions sin, cos, tan and their usual restricted inverses to 0.

Remark: $\cos(0) = 1$, so 1 can be produced.

Demonstration: Suppose we wish to produce a rational number, $r = \frac{p}{q}$, with p and q coprime positive integers.

If $p > q$, then r can be produced if $\frac{q}{p}$ can; therefore, we need only show that all positive rational numbers smaller than 1 can be reached.

Assuming $r < 1$, it can be reached (by way of S_t) if $T_s(r) = \frac{p}{\sqrt{q^2-p^2}}$ can.

This is not (generally) a rational number, but it is the square root of a rational number. Its numerator is smaller than q, by supposition; its denominator is also smaller than q because of geometry and/or algebra[2].

Remark: The key point here is that $T_s\left(\frac{p}{q}\right)$ is a fraction with a numerator and denominator both of which are square roots of integers, and both strictly smaller than q.

Applying R if needed, this means $\frac{p}{q}$ can be generated from some number of the form $\frac{\sqrt{a}}{\sqrt{b}}$ with $1 \leq a \leq b < q$, with a, b and q all integers[3].

Repeating the process leads to still smaller elements of the fraction; a decreasing sequence of integers bounded inclusively from below by 1 must eventually reach 1.

Since we know we can produce 1, all positive rational numbers can be produced ■.

[2] You can see this by considering a right-angled triangle with hypotenuse q and a leg of p. The second leg is $\sqrt{q^2 - p^2}$, which is smaller than q.

[3] Regarding $a \leq b$: equality holds here iff $q = \frac{1}{\sqrt{2}}$.

An example

Suppose we want to produce $r = \frac{4}{3}$, everyone's favourite triangle-related fraction.

- $\frac{4}{3}$ can be produced if $\frac{3}{4}$ can; $r = R\left(\frac{3}{4}\right)$.
- $\frac{3}{4}$ can be produced if $\frac{3}{\sqrt{7}}$ can: $r = R\left(S_t\left(\frac{3}{\sqrt{7}}\right)\right)$.
- $\frac{3}{\sqrt{7}}$ can be produced if $\frac{\sqrt{7}}{3}$ can: $r = R\left(S_t\left(R\left(\frac{\sqrt{7}}{3}\right)\right)\right)$.
- $\frac{\sqrt{7}}{3}$ can be produced if $\frac{\sqrt{7}}{\sqrt{2}}$ can: $r = R\left(S_t\left(R\left(S_t\left(\frac{\sqrt{7}}{\sqrt{2}}\right)\right)\right)\right)$.
- $\frac{\sqrt{7}}{\sqrt{2}}$ can be produced if $\frac{\sqrt{2}}{\sqrt{7}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(\frac{\sqrt{2}}{\sqrt{7}}\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{\sqrt{7}}$ can be produced if $\frac{\sqrt{2}}{\sqrt{5}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(\frac{\sqrt{2}}{\sqrt{5}}\right)\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{\sqrt{5}}$ can be produced if $\frac{\sqrt{2}}{\sqrt{3}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right)\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{\sqrt{3}}$ can be produced if $\frac{\sqrt{2}}{1}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(\frac{\sqrt{2}}{1}\right)\right)\right)\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{1}$ can be produced if $\frac{1}{\sqrt{2}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(R\left(\frac{1}{\sqrt{2}}\right)\right)\right)\right)\right)\right)\right)\right)\right)$.
- $\frac{1}{\sqrt{2}}$ can be produced if $\frac{1}{1}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(R\left(S_t\left(\frac{1}{1}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)$.
- $1 = \arccos(0)$, so $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(R\left(S_t\left(\arccos(0)\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)$.

Therefore $\frac{4}{3}$ can be produced.

A connection

"Why are you writing all this, Colin? It's a diverting enough puzzle, but… why?"

I'm writing about it because it gave me such a lovely revelation, I nearly jumped out of the bath.

Suppose we write our target fraction as $r = \frac{\sqrt{P}}{\sqrt{Q}}$, with $P = p^2$ and $Q = q^2$. Then our algorithm for working backwards to show 1 can be produced from r (and, hence, by way of inverses, r from 1) is:

While $Q \neq P$:

- If $Q < P$, swap P and Q (this is the effect of $R\left(\frac{q}{p}\right)$).
- Let $Q = Q - P$ (this is the effect of $T_s\left(\frac{p}{q}\right)$).

This is Euclid's algorithm for finding the greatest common factor of P and Q! Since, by supposition, P and Q are coprime, their GCF is 1. Therefore, 1 can be produced from r and hence r can be produced from 1 $\blacksquare$.

MARTIN GARDNER

Puzzle-Games

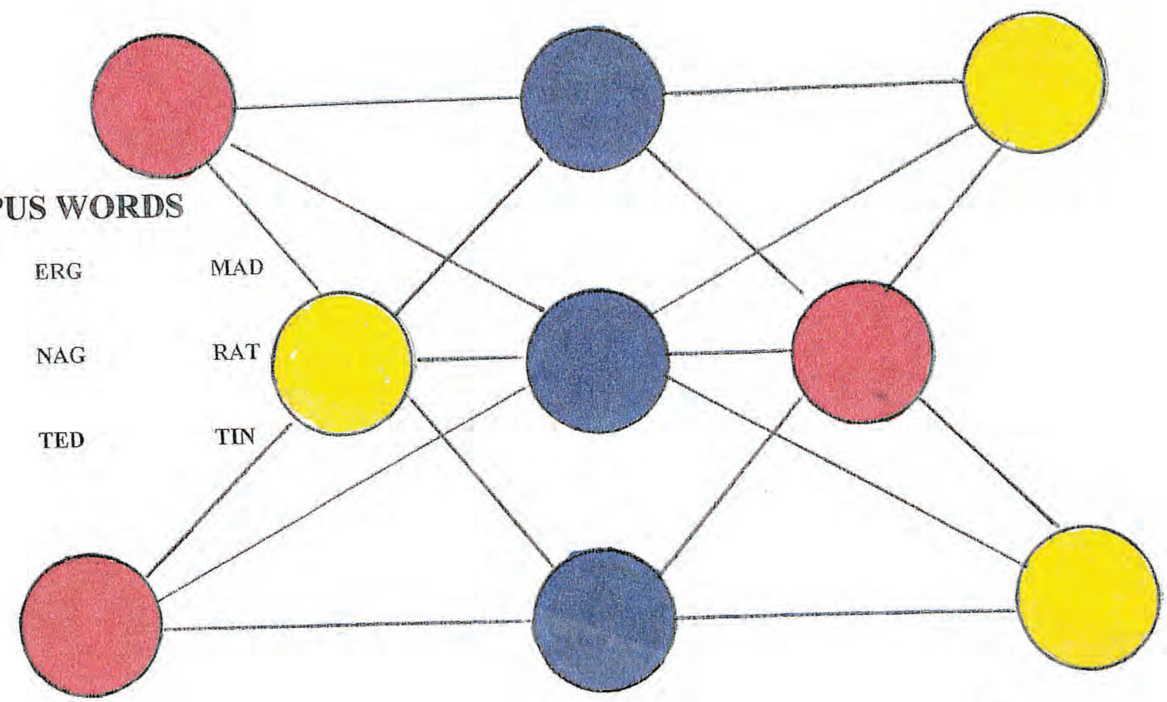
By Stephen Bloom, Lacey Echols, Jeremiah Farrell and Shannon Lieb

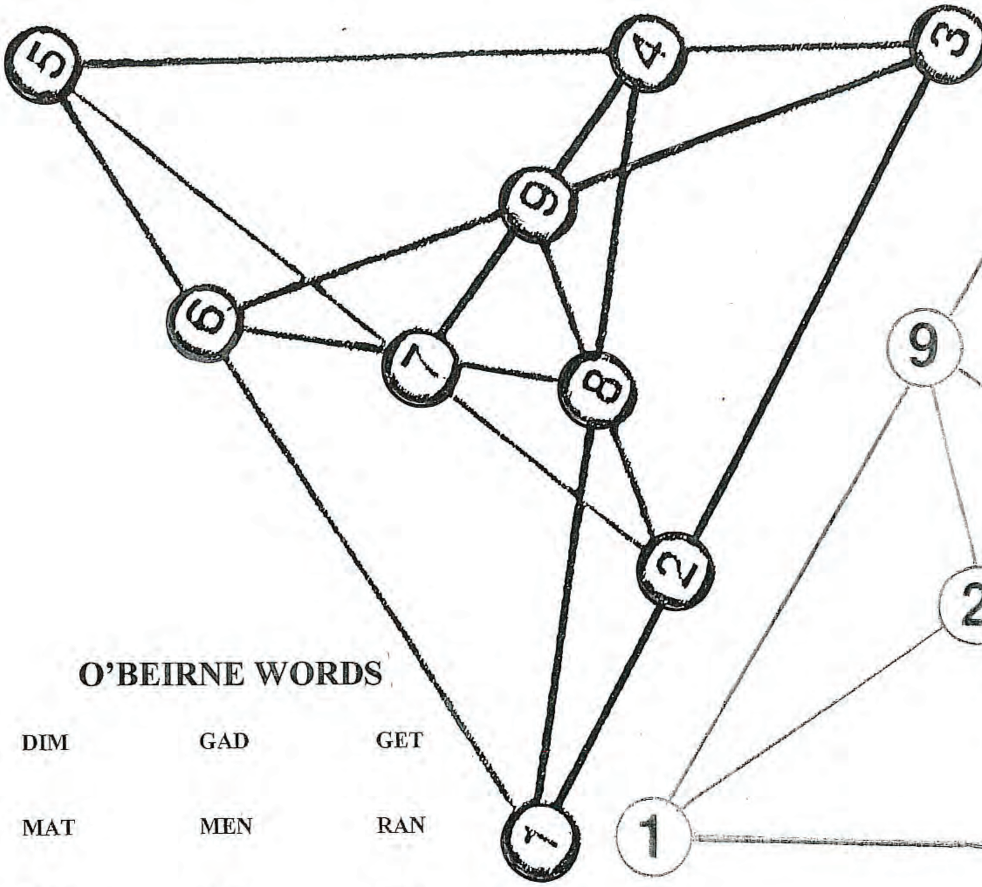
Butler University has offered for the last three years a fall semester class on the various exploits of Martin Gardner. The four of us have had the great pleasure of participating in the class and fully expect it to be a regular offering in the coming years.

We note that there are nine different letters in the name MARTIN GARDNER and we will use each letter exactly three times to form several sets of words that will turn out to be (9,3) symmetric configurations. The article "Configuration Games" by Jeremiah Farrell, Martin Gardner, and Thomas Rodgers in *Tribute to a Mathematician*, 2005 AK Peters, Wellsley, MA, edited by B. Cipra, E.D. Demaine, M. L. Demaine and T. Rodgers relates the mathematics of symmetric configurations, The article describes the three different (9,3) configurations, calling their line graphs Pappus's Mousetrap, O'Beirne's Mousetrap, and Mousetrap. For each of these we supply a set of nine Martin Gardner words which as a puzzle are to be placed on the respective graphs so that every line of three has a letter in common.

PAPPUS WORDS

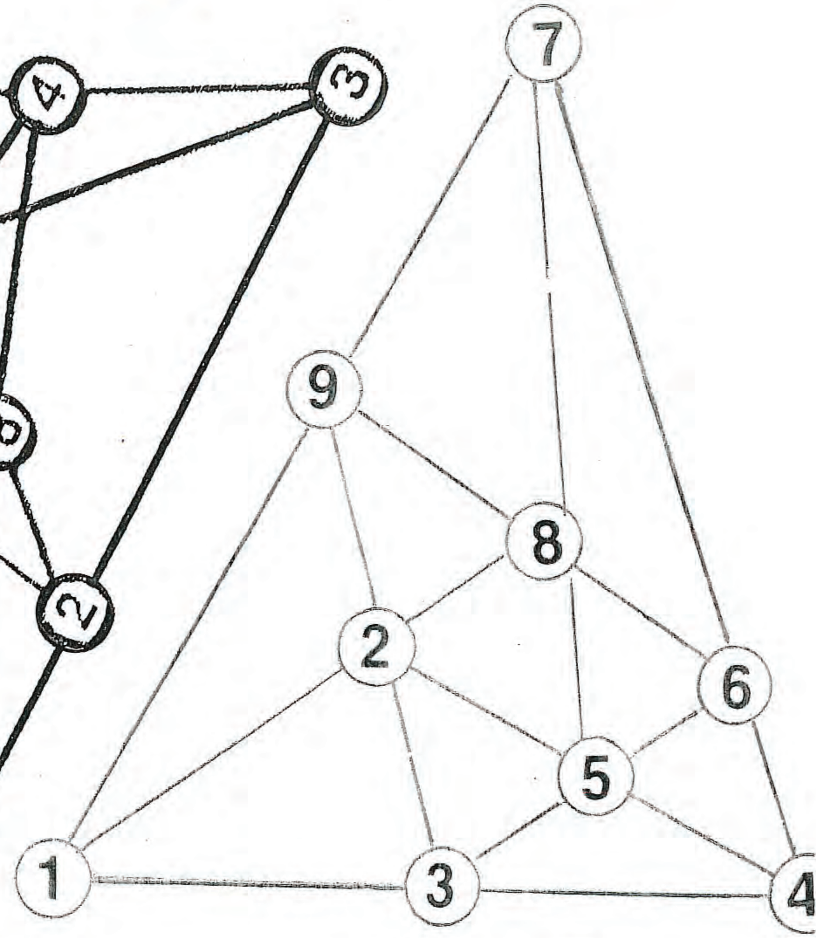
DIG	ERG	MAD
MEN	NAG	RAT
RIM	TED	TIN





O'BEIRNE WORDS

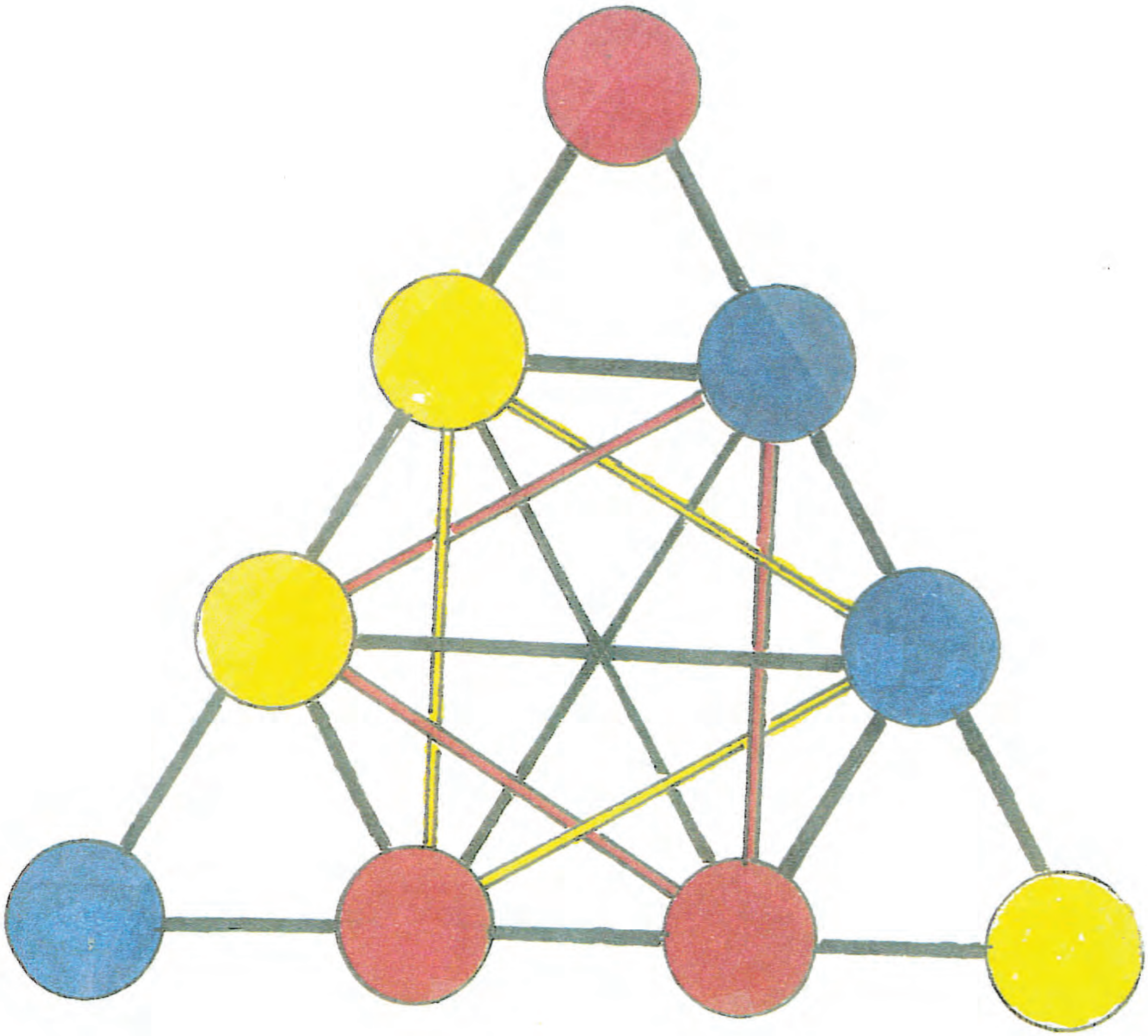
DIM	GAD	GET
MAT	MEN	RAN
RED	RIG	TIN



MOUSETRAP WORDS

DAM	DIG	END
ERG	MTN	NAG
RAT	RIM	TIE

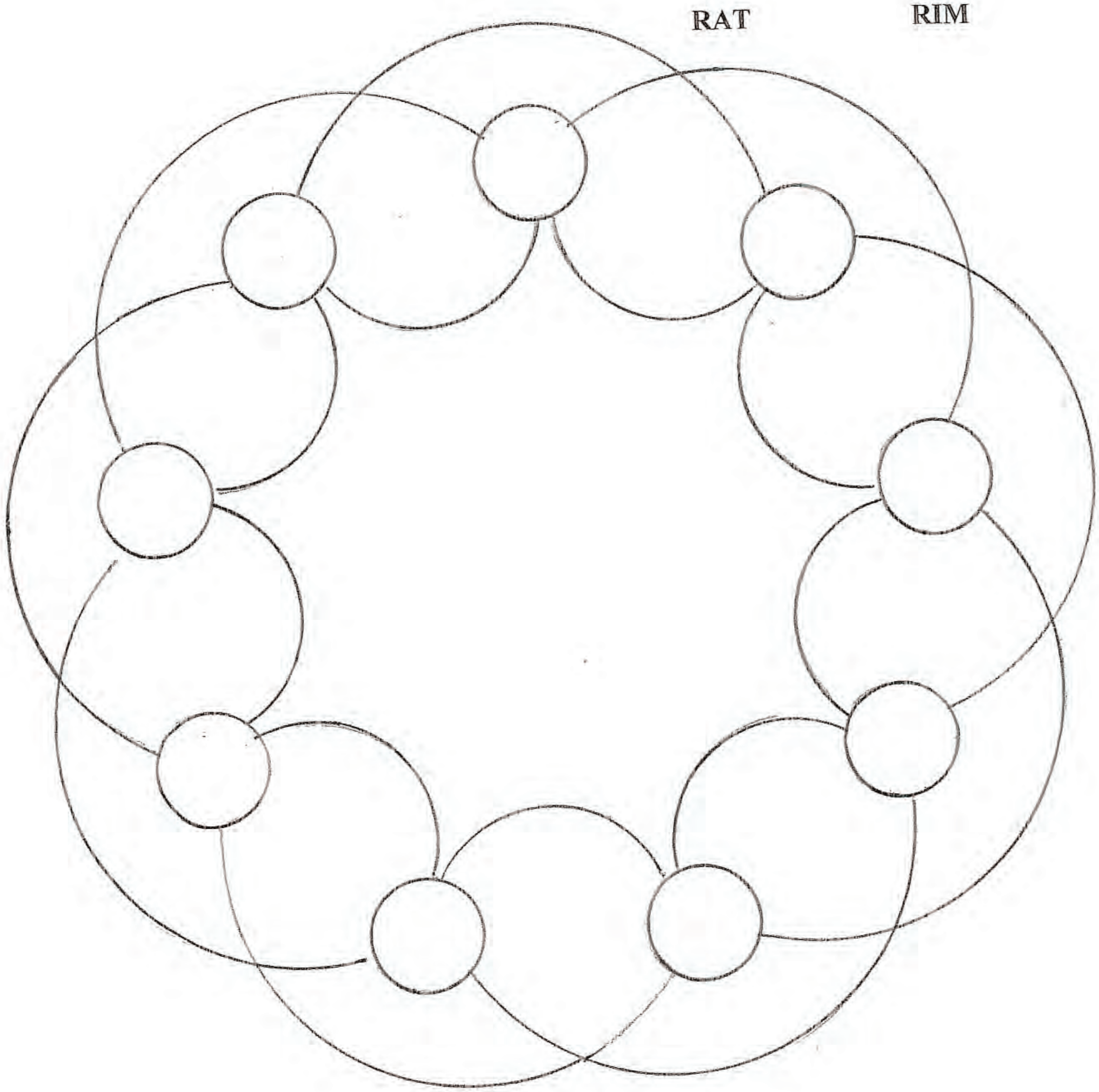
There are many interesting variations. Pappus can be refigured as equilateral triangles instead of straight lines. That is each of the nodes of the graph is to have a word so that each node of an equilateral triangle has a letter in common.



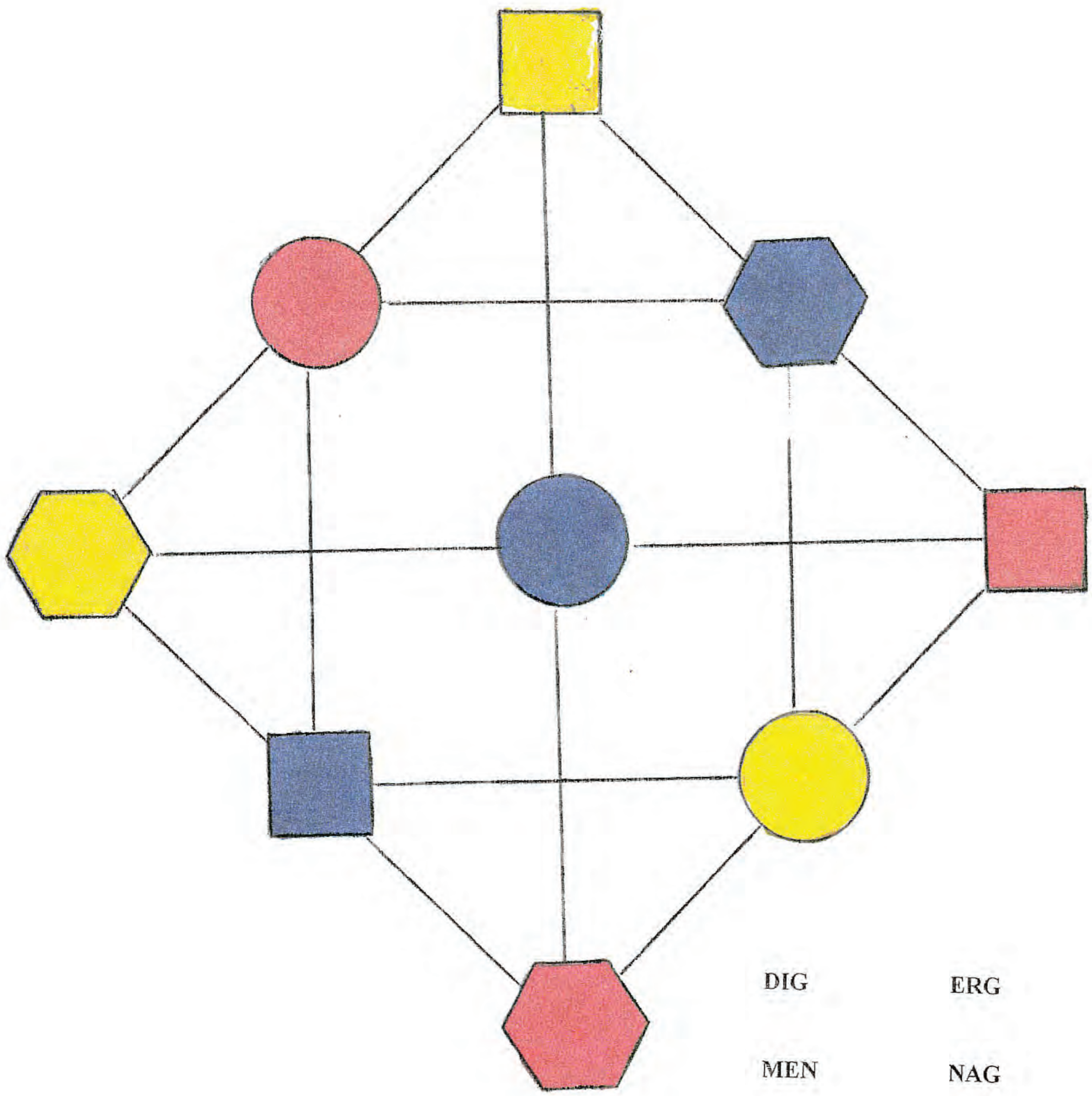
DIG	ERG	MAD
MEN	NAG	RAT
RIM	TED	TIN

Mousetrap words can be placed on this graph so that every connected node has a letter in common.

DAM	**DIG**	**END**
ERG	**MTN**	**NAG**
RAT	**RIM**	**TIE**



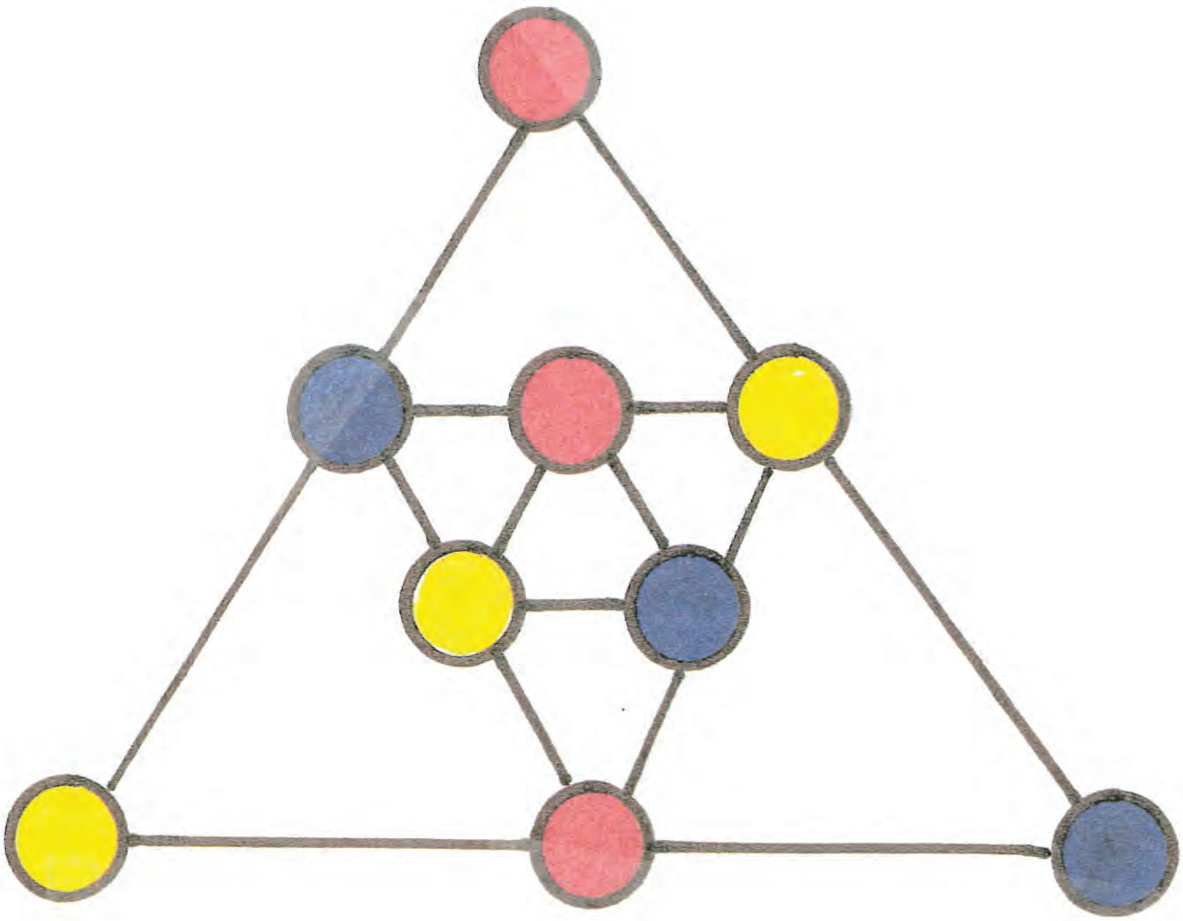
The following is another graph for Pappus. Here the same nine words are to be placed on the nodes so that any two connected nodes have a letter in common. In addition, each set of three shapes is to have a common letter.



DIG	ERG	MAD
MEN	NAG	RAT
RIM	TED	TIN

For O'Beirne the following diagram can be used with the name MARTIN. Each letter will be used exactly three times to form the nine chemical symbols which are then to be placed on the nodes so that any two connected nodes have a letter in common.

Am, Americum; Ir, Iridium; Ma, Manganese; Ni, Nickel; Ra, Radium; Rn, Radon; Ta, Tantalum; Ti, Titanium; and Tm, Thulium.



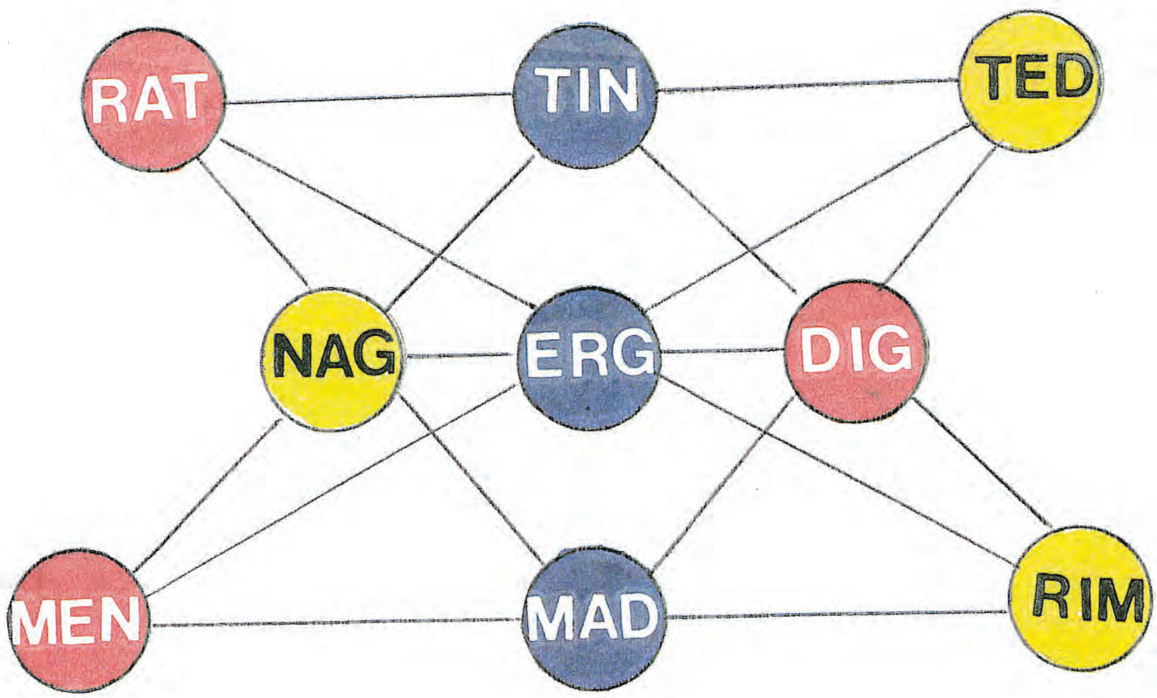
It is also possible to place the words from the former O'Beirne puzzle on the nodes of the above graph so that any two connected nodes have a letter in common.

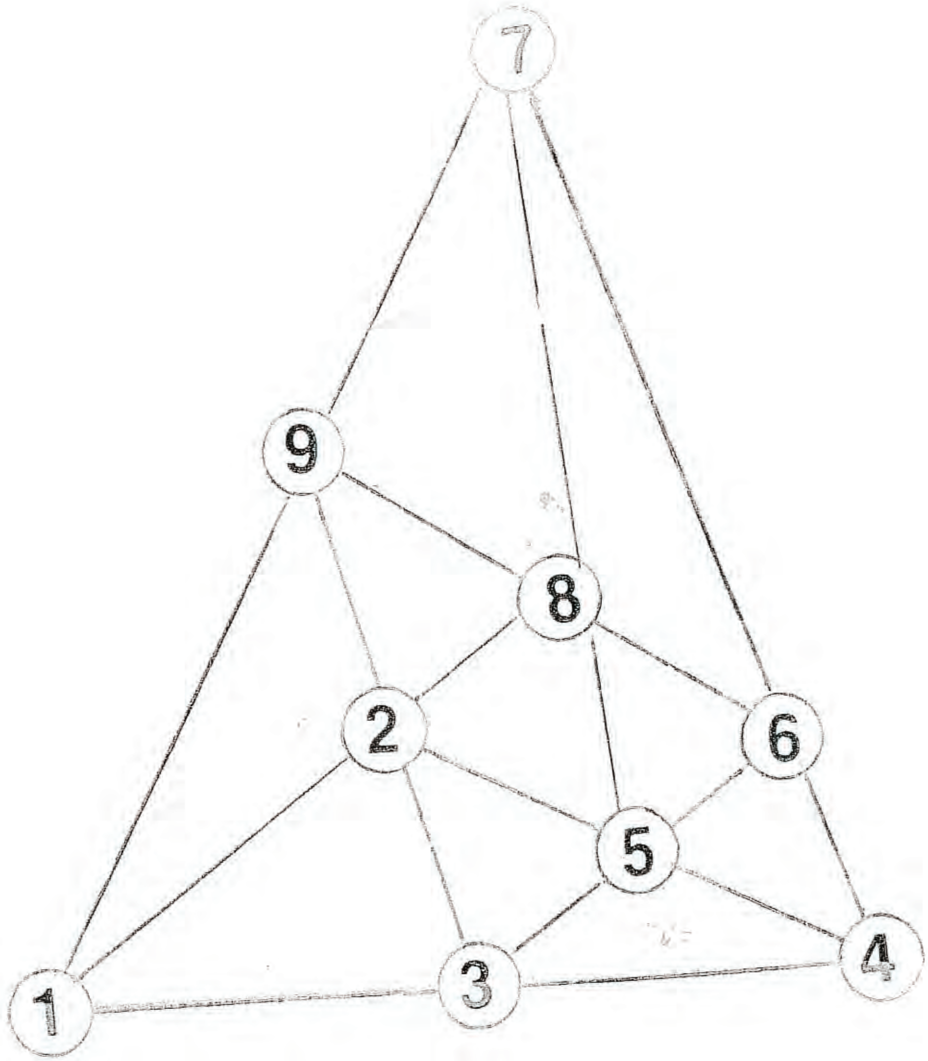
DIM	**GAD**	**GET**
MAT	**MEN**	**RAN**
RED	**RIG**	**TIN**

Possible answers to each puzzle follow.

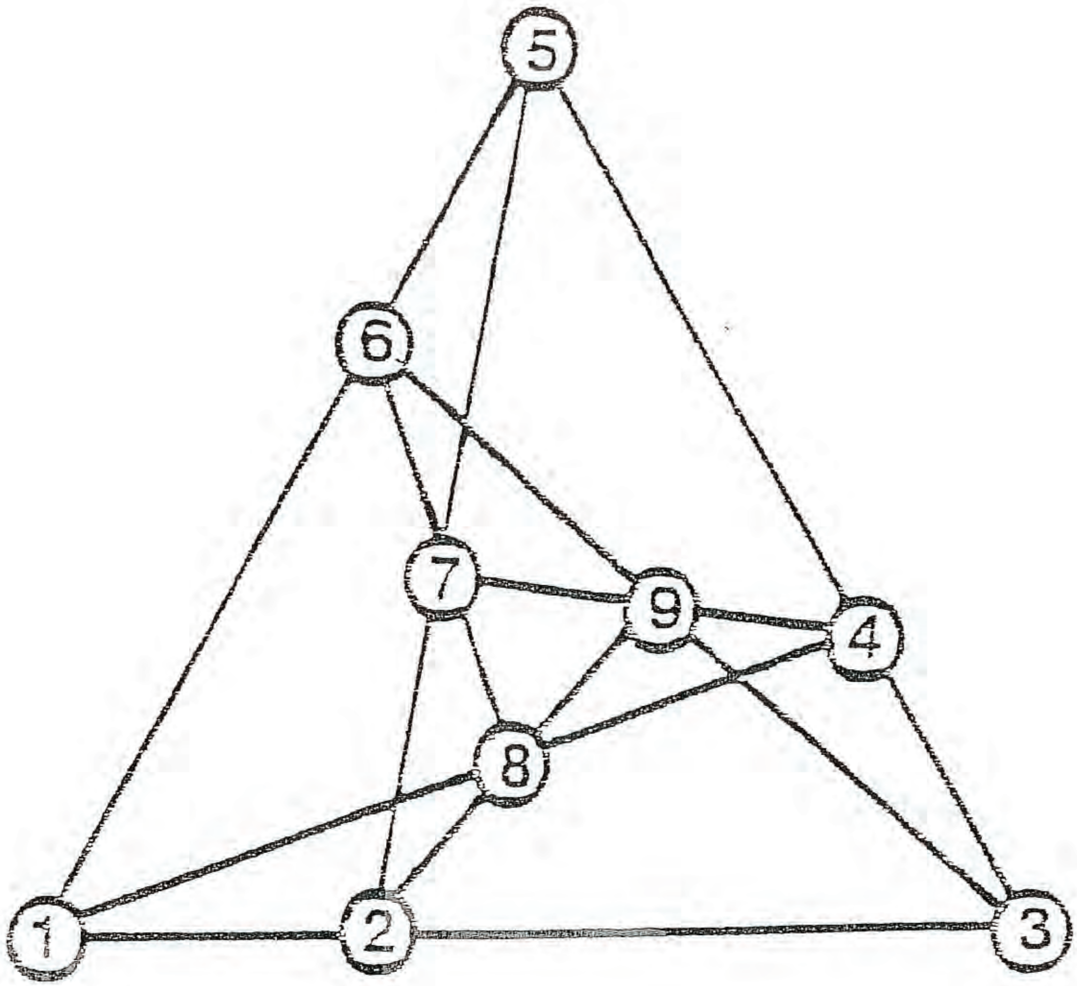
Now for the games. Two players have four distinctive tokens each and alternately play on the nodes of a complete puzzle. If First can select three nodes with a common letter, First wins. Otherwise, Second is awarded the win. It turns out that Second can only force a win on one of the graphs. We will explain later.

Be sure to play the games on completed puzzles so that the location of the letters can plainly be seen. Special note. For O'Beirne any player can win by selecting three of the same color.

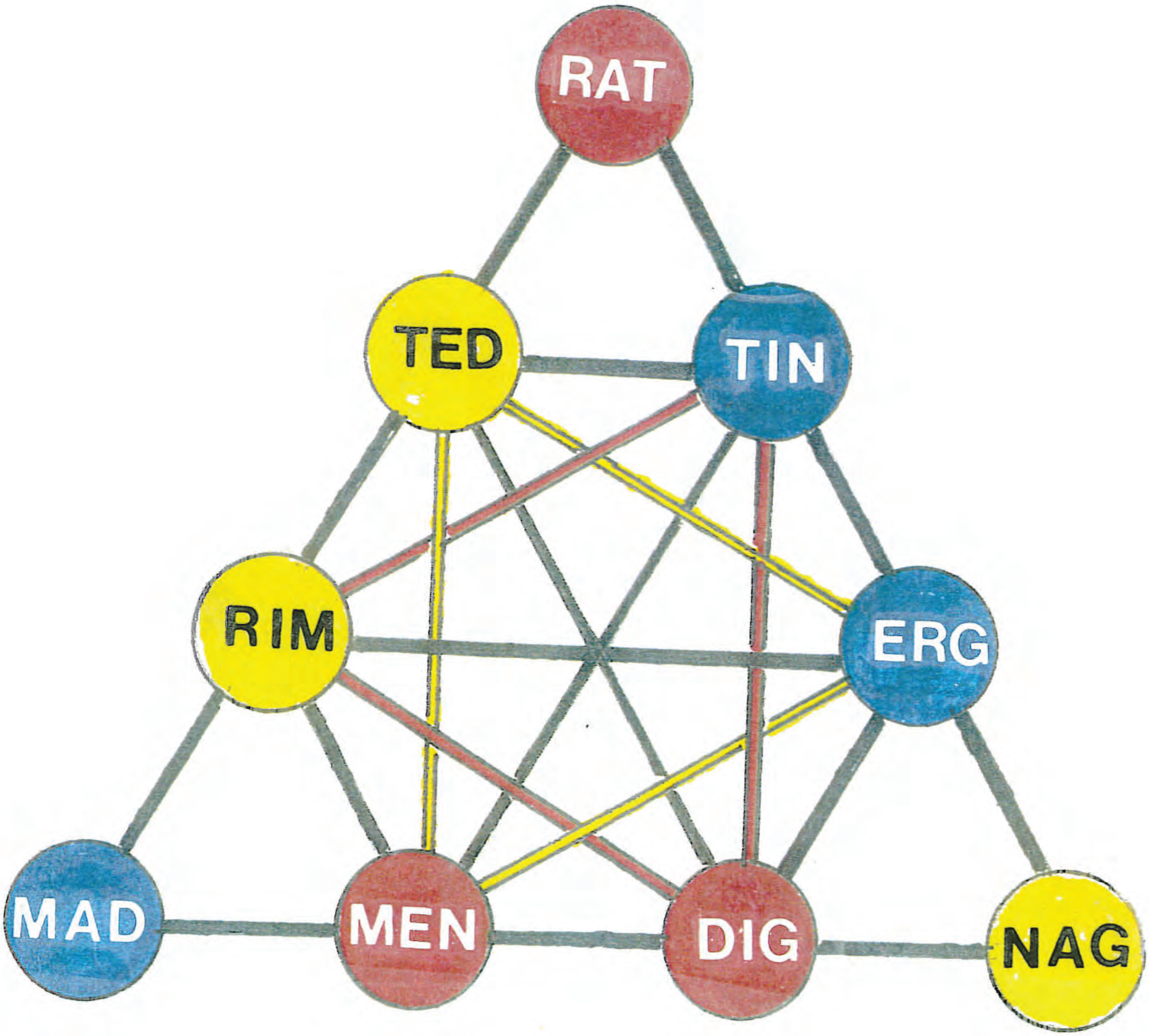


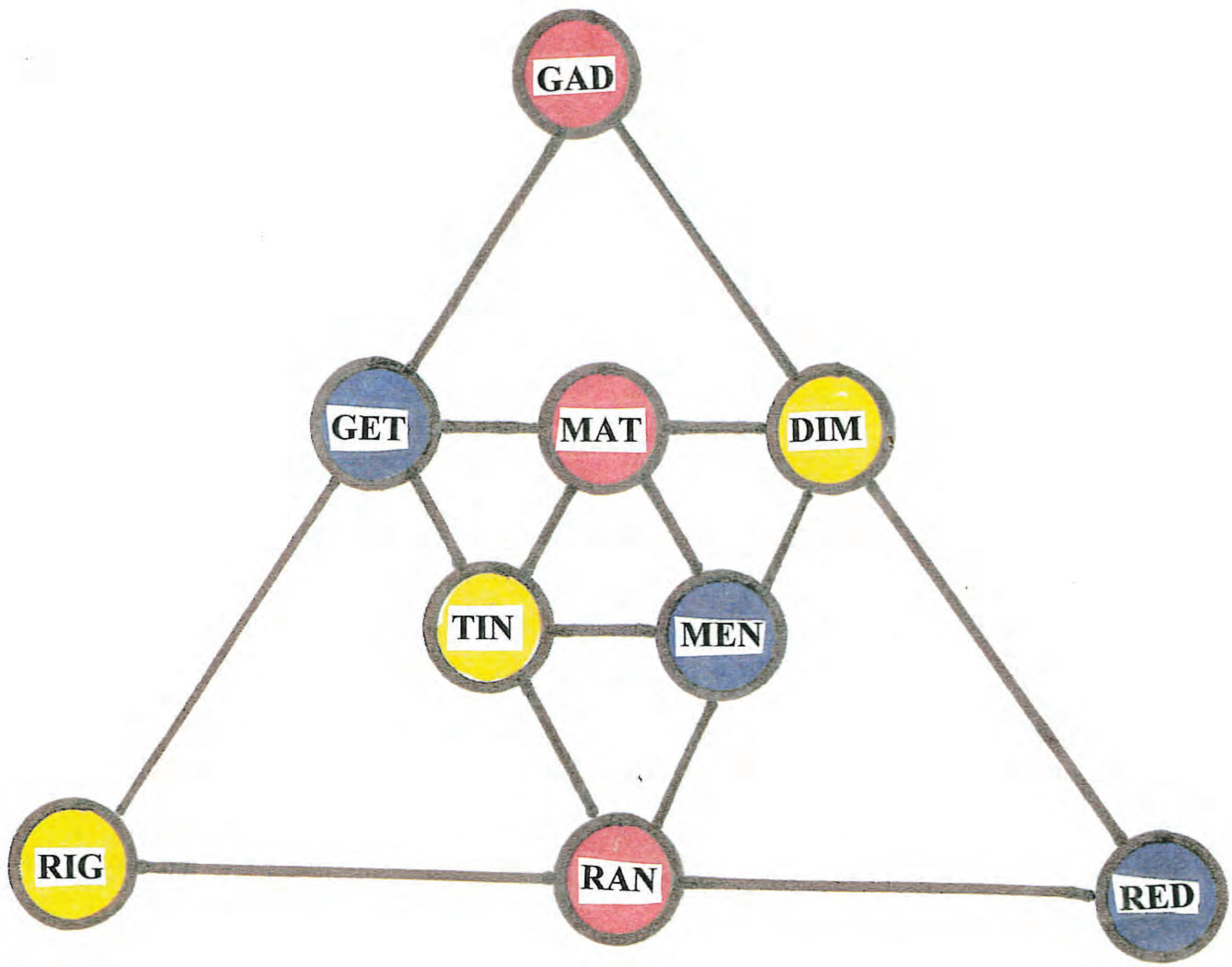


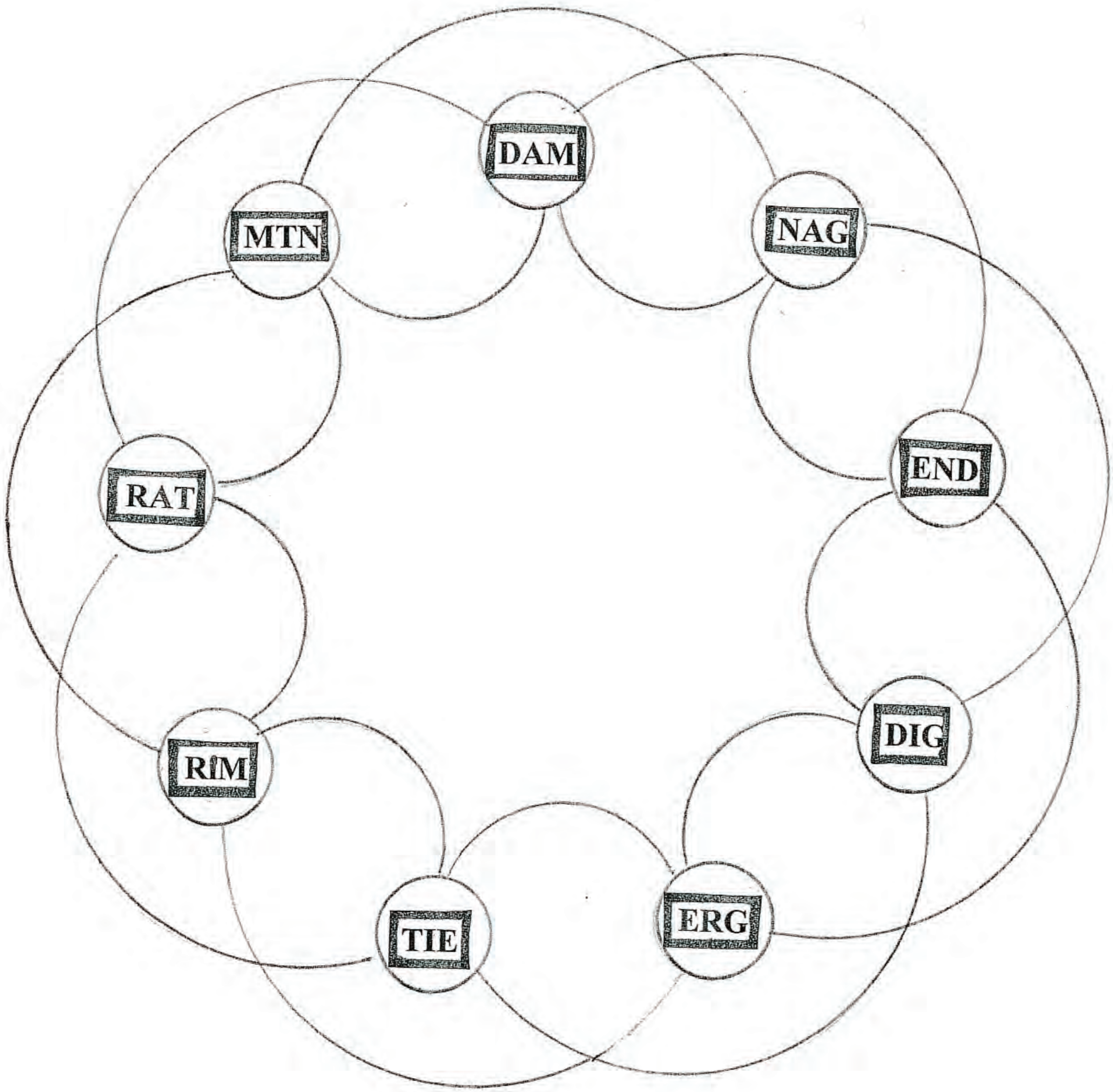
1 DAM	4 DIG	3 END
5 ERG	9 MTN	2 NAG
8 RAT	7 RIM	6 TIE

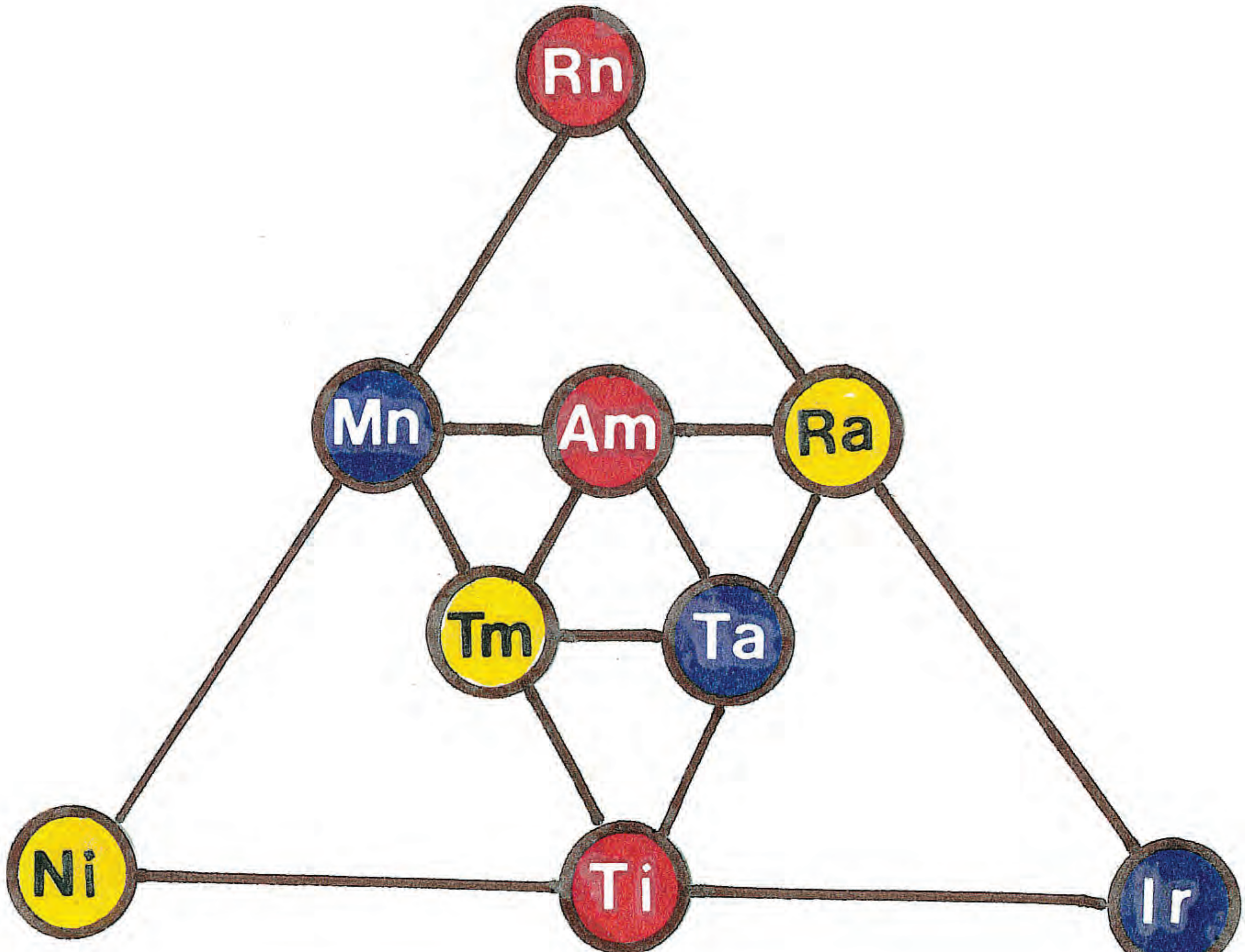


2 DIM	8 GAD	6 GET
1 MAT	3 MEN	4 RAN
9 RED	7 RIG	5 TIN


RAT
TED
TIN
RIM
ERG
MAD
MEN
DIG
NAG


GAD
GET
MAT
DIM
TIN
MEN
RIG
RAN
RED


DAM
NAG
END
DIG
ERG
TIE
RIM
RAT
MTN



How to win at the games.

Pappus types. First plays on any color. If Second plays on the same color, First wins by playing the last of that color. If instead Second plays on a different color, First wins by forcing Second to play on another of Second's colors.

O'Beirne's types. First wins by playing one of DIM , RAN, or GET. If Second plays another of these three, First takes the remaining one and wins with careful play. When Second plays any other node on the first turn, First must force Second to waste a move by forcing Second to play a node not connected to Second's initial node. For the MARTIB version the keys are Mn, Ra and Ti.

Mousetrap types. Second can win by playing the next higher number to First's choice (1 if the choice is 9). Careful play after this will force First to use up four moves with no win.

New Old School (NOS) Burrs

by Frans de Vreugd

Introduction

Gregory Benedetti is a puzzle designer from France who has designed some very nice and unusual puzzles. Several of his designs were entries in the Nob Yoshigahara puzzle design competition. One of Greg's fascinations is with puzzles that have a different internal mechanism than you might expect from the outside. His Blind Burr (entry in 2010) is a good example of that. Aspecial group of puzzles he has been working on is called NOS burrs (New Old School Burrs) On the outside the puzzles look like a standard six piece burr (a.k.a. Chinese Knot), but hidden in the inside is a completely different mechanism.

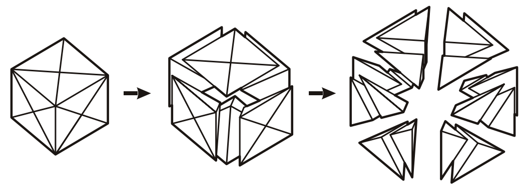
Interlocking puzzles can be classified in many different ways. One way to divide them into different classes is to look at the movement of the pieces. The vast majority of interlocking puzzles have rectilinear moves for the pieces. However, there is also a considerable group of puzzles that use coordinate motion (CM). For these puzzles, two or more pieces move at the same moment in different directions. The internal mechanism of such puzzles mostly use diagonal cuts in the pieces to allow this type of movement. Stewart Coffin (USA) has designed many of these in the past, and nowadays Vinco Obsivac (CZ) is the specialist in this type of puzzle.

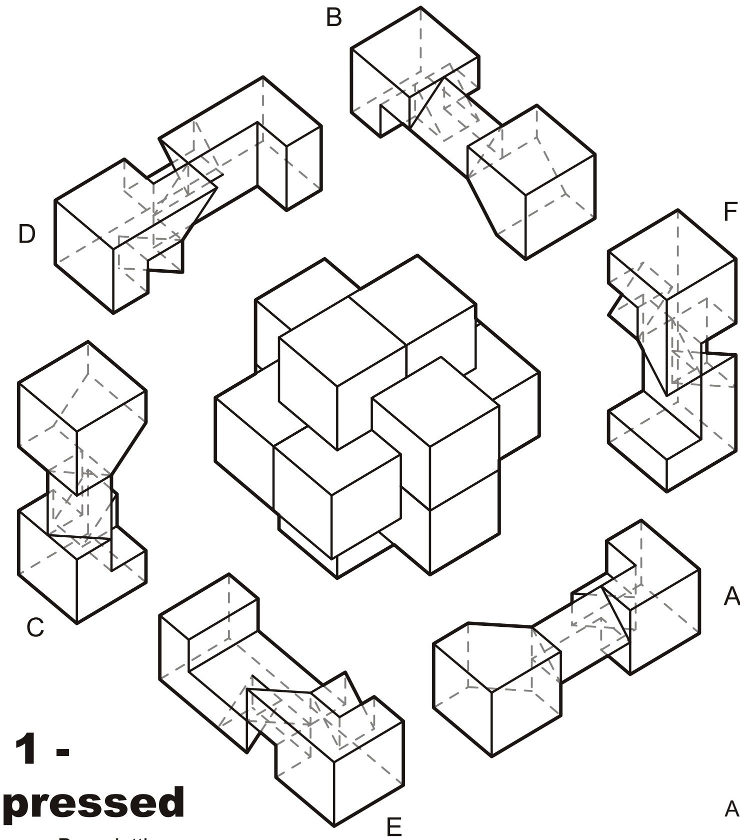
CM puzzles are quite different from 'standard' interlocking puzzles. For disassembling a CM puzzle finding the exact positions to put your fingers can be quite a challenge, and for assembling it often requires some dexterity to align the pieces exactly to their correct position. In the NOS burrs normal rectilinear moves are combined with coordinated motion moves. This is a wonderful surprise while playing with the puzzle.

Using non-orthogonal units

At IPP 32 in Washington in 2012, Greg brought prototypes of his NOS burrs. The puzzles looks like standard (and simple) six piece burrs, but looking at the pieces the average woodworker might get a heart attack! Apart from using cubical units many internal units are diagonal half-cubes. This may sound simple but it can result in really weird pieces. Greg made seven different NOS designs, six of them use these diagonal half-cubes, the seventh includes even more complicated notches. The basic building block is much smaller than a standard diagonal half-cube. Imagine that you subdivide a cube into six square pyramids, and then cutting each of these across both diagonals (see picture below).

In total Greg designed seven different NOS burrs, drawings of each of them can be found in the following pages.



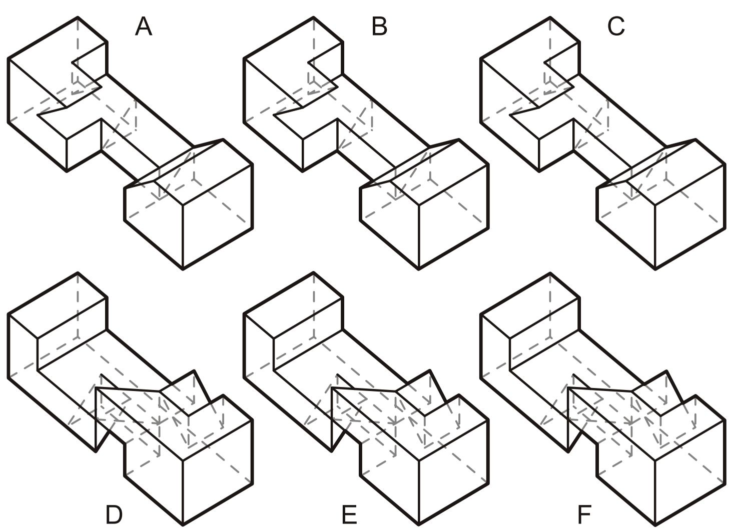


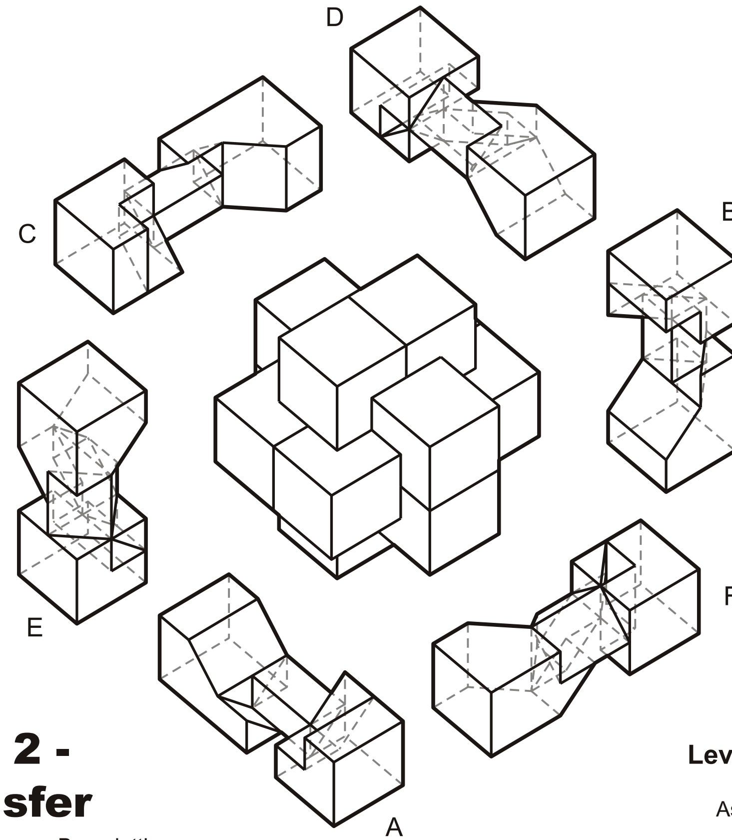
NOS 1 - Compressed

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 2

Assemblies: 1
Solutions: 1



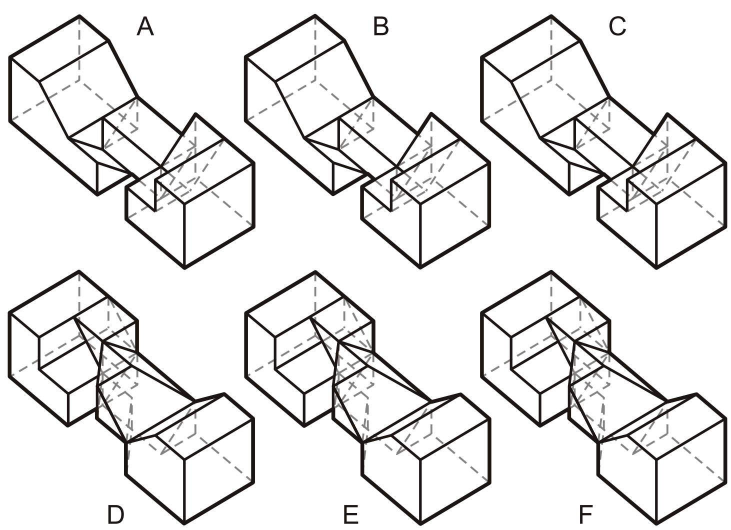


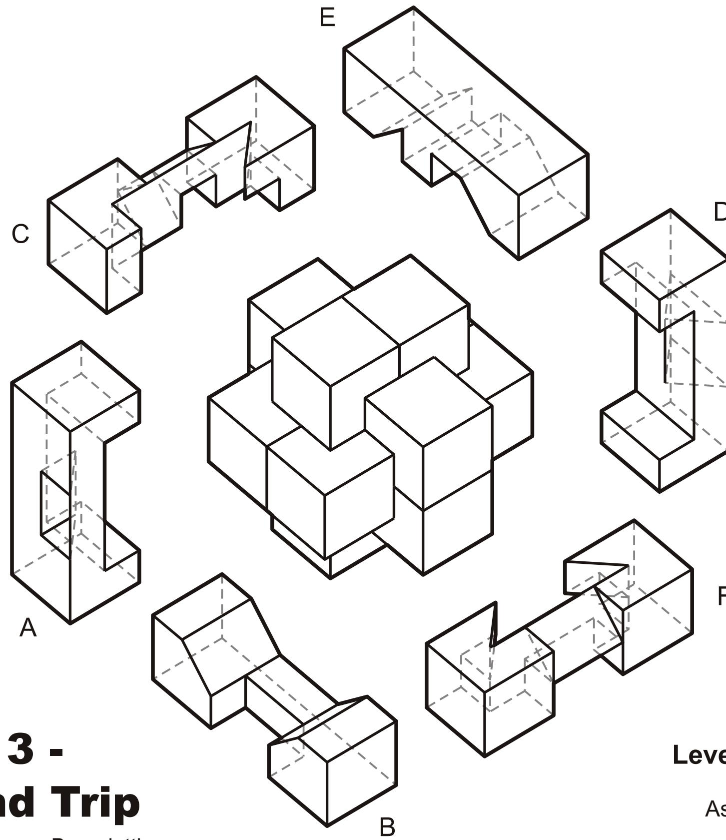
NOS 2 -
Transfer

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 2-2-2-2

Assemblies: 1
Solutions: 1



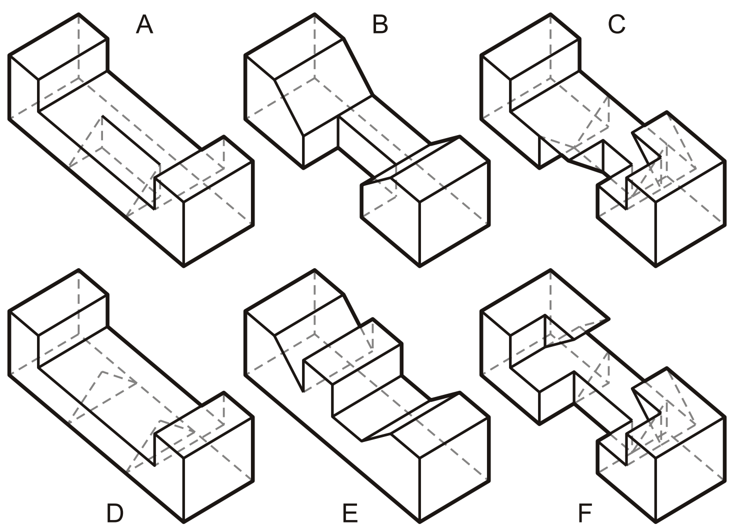


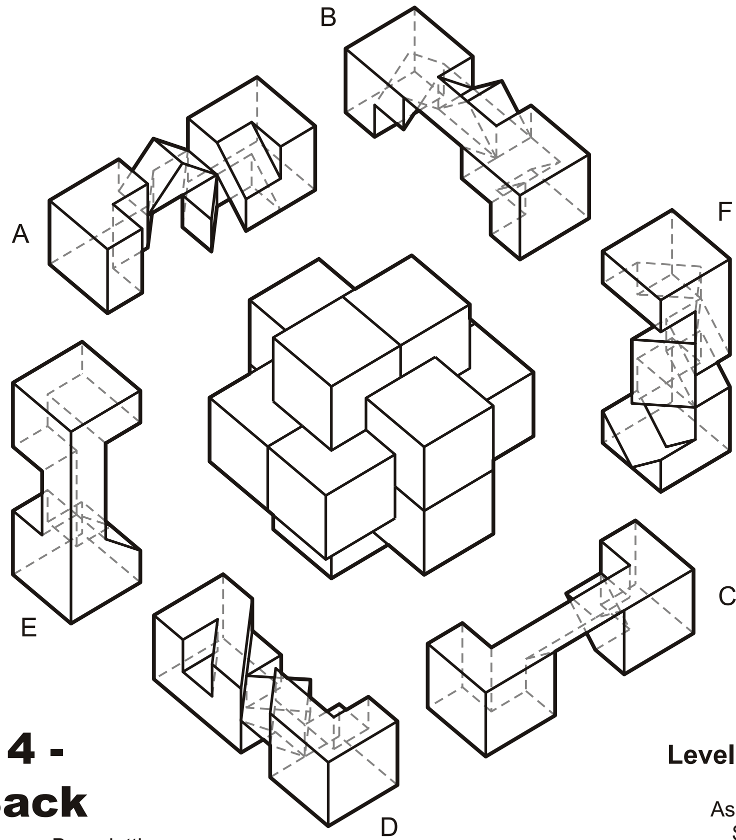
NOS 3 - Round Trip

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 4-8-2-2-1

Assemblies: 16
Solutions: 1



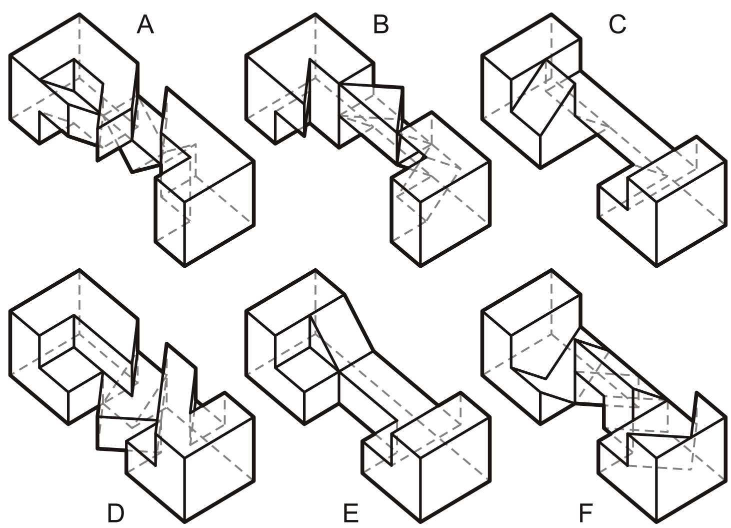


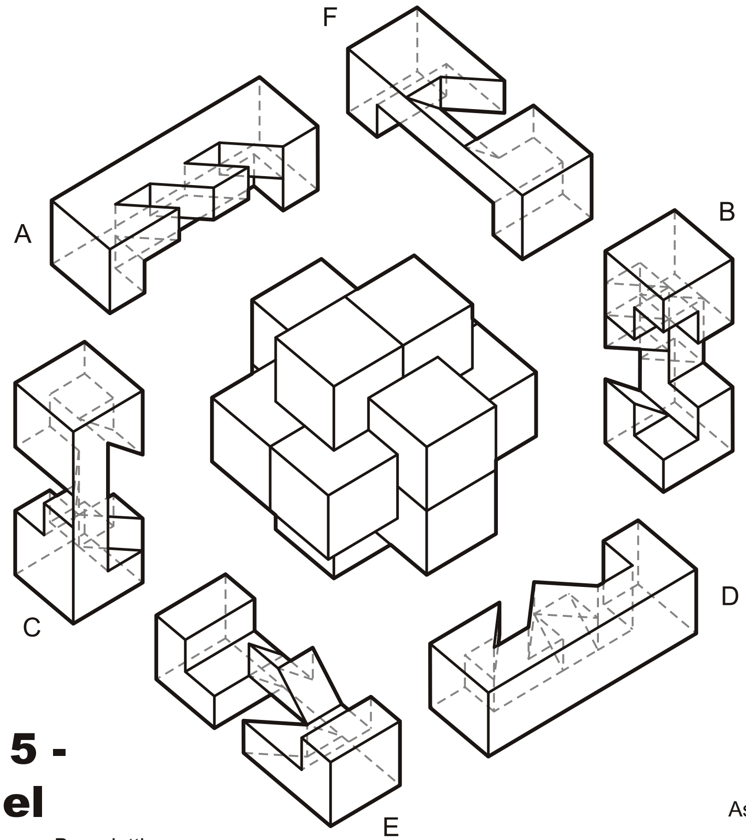
NOS 4 - Go Back

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 15-2-1-1

Assemblies: 2
Solutions: 1



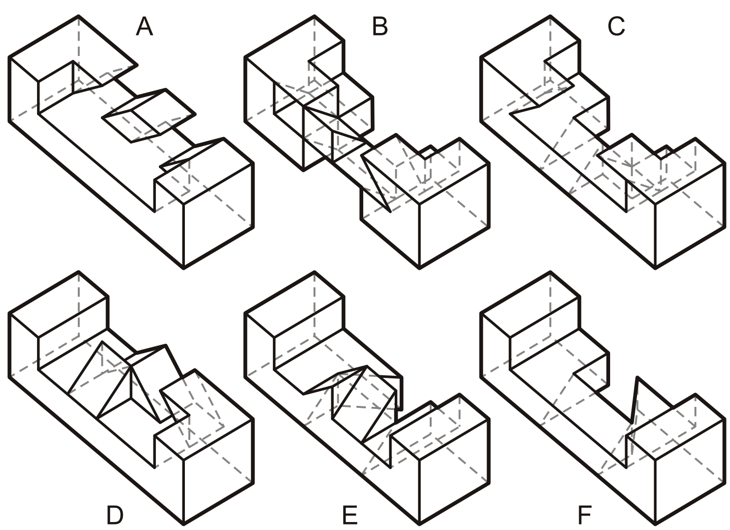


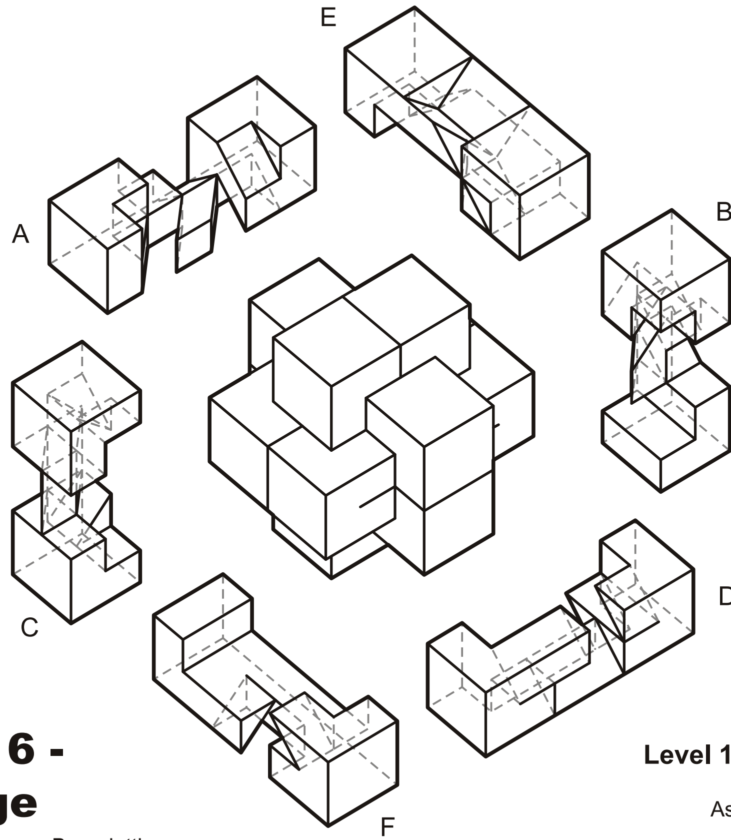
NOS 5 - Crenel

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 7-2

Assemblies: 1
Solutions: 1



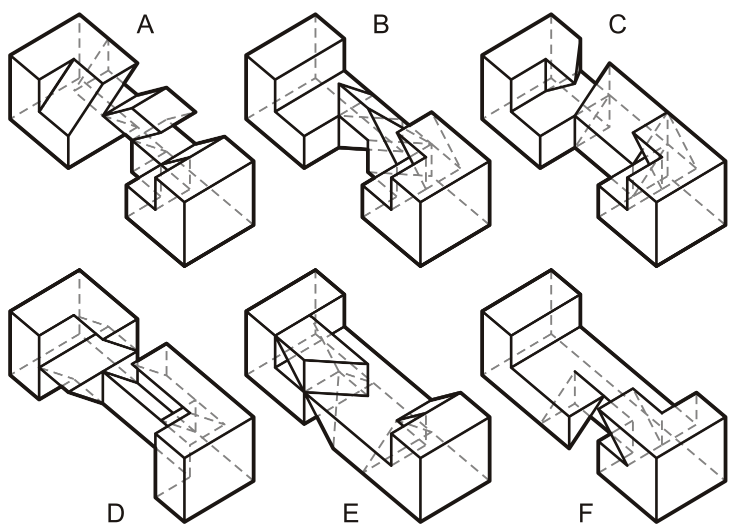


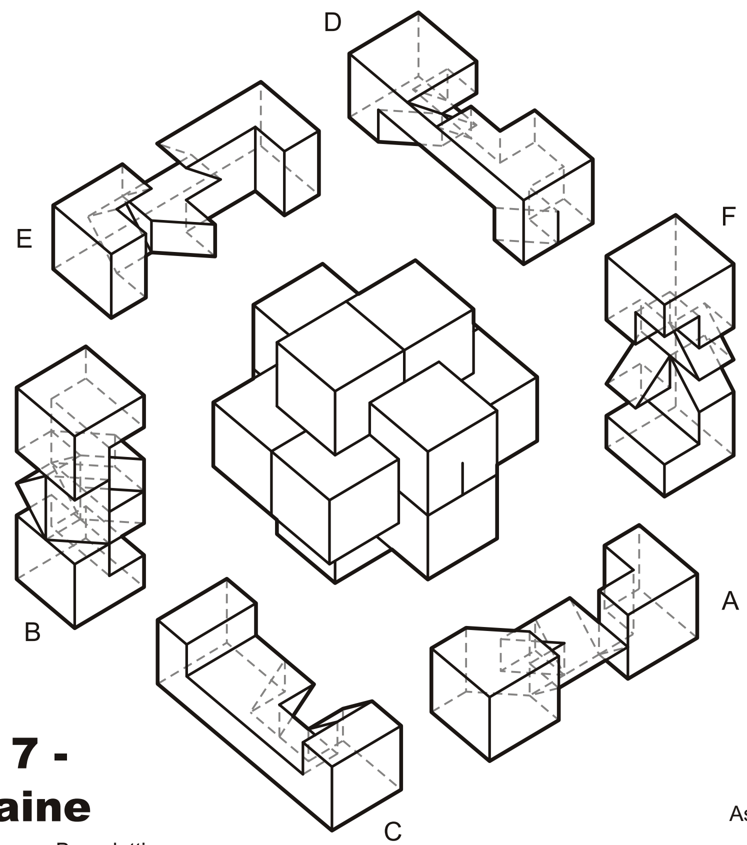
NOS 6 - Dodge

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 10-5-1-2-2

Assemblies: 3
Solutions: 1



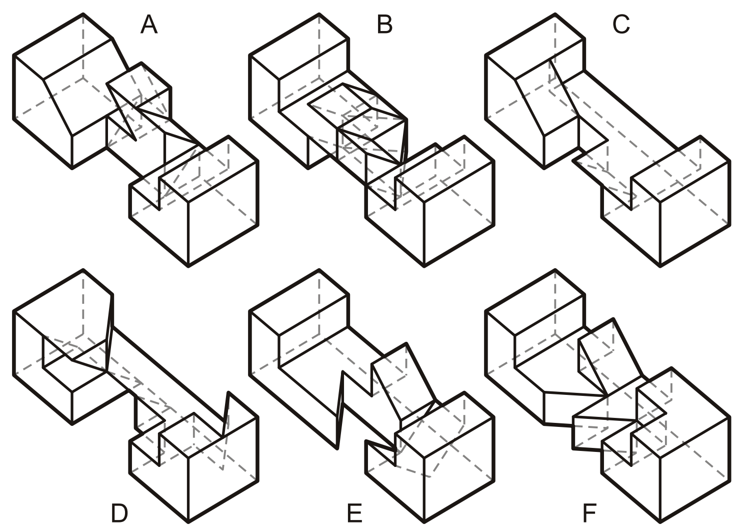


NOS 7 -
Seizaine

Design: Gregory Benedetti
Drawing by Frans de Vreugd

Level 16

Assemblies: 1
Solutions: 5



2,664 Coin-Sliding Font Puzzles

Erik D. Demaine* Martin L. Demaine*

Abstract

We present two font designs, each with 37 symbols (letters, digits, and slash), as grid configurations of the same number of coins. Each pair of symbols (say, A and B) forms a puzzle: re-arrange the first symbol (A) into the second (B) by a sequence of moves. Each move picks up one coin and places it in an empty grid cell that is adjacent to at least two other coins (the "2-adjacency" rule). We also present an online puzzle video game to play all 2,664 of these puzzles, where you can try to set the record on the minimum number of moves.

1 Coin-Sliding Puzzles

At our first G4G (G4G5 in 2002), we presented several new coin-sliding puzzles [DD04] based on our research with Helena Verrill [DDV02]. Figure 1 shows one example. In this type of puzzle, the goal is to transform the start configuration (drawn on the left) into the target configuration (drawn on the right) via a sequence of "moves". Each ***move*** picks up one coin and places it in an empty grid cell that is ***adjacent to at least two other coins*** (the ***2-adjacency*** rule).[1] A second goal is to minimize the number of moves that achieve the desired transformation.

Martin Gardner wrote about puzzles like this [Gar75], but on the triangular grid. Indeed, staying on the triangular grid is probably the original motivation for the 2-adjacency rule, as these

*MIT Computer Science and Artificial Intelligence Laboratory, 32 Vassar St., Cambridge, MA 02139, USA, {edemaine,mdemaine}@mit.edu

[1]We do not consider a "sliding" constraint (continuous planar motion of the coin without collision), which is present in only one puzzle in [Gar75]. The more precise name for these puzzles is "coin-moving puzzles", as in [DDV02], but we use the less formal term "coin-sliding" here.

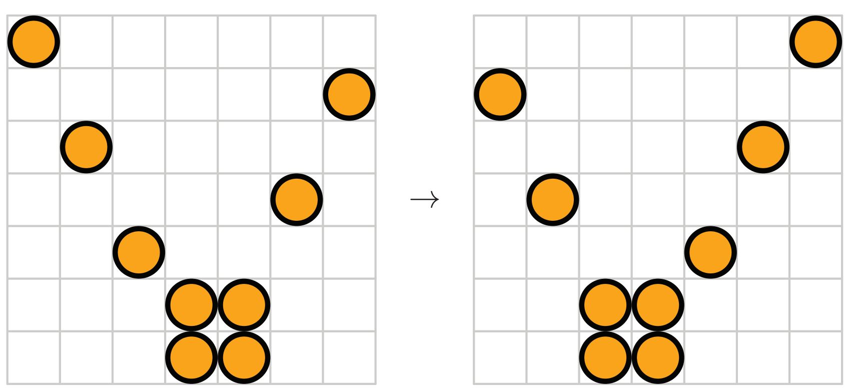


Figure 1: Puzzle 9 from [DD04]. The n-coin version of this puzzle is the asymptotically hardest puzzle: it requires $\Omega(n^3)$ moves, and all n-coin coin-sliding puzzles on the square grid can be solved in $O(n^3)$ moves [DDV02]. The exact constant factor is unknown, however.

moves force the coins to remain on a triangular grid. But triangular-grid coin-sliding puzzles turn out to be much simpler, both from a puzzle perspective and in terms of mathematics and algorithms [DDV02]. Thus we focus here on the square-grid coin-sliding puzzles, which originate with Harry Langman [Lan51].

Our main result with Verrill [DDV02] is a sufficient condition for a coin-sliding puzzle on the square grid to have a solution, and a corresponding algorithm to solve these puzzles. To state the result, we need to define the notion of "span" of a configuration of coins. Imagine you have a bag full of extra coins, and you place as many as you can onto the board while still respecting the 2-adjacency rule for each placement. The ***span*** is the resulting configuration, which is a rectangle or disjoint union of rectangles (with at least two blank rows in between the rectangles). The span represents the maximum set of reachable positions that the coins could reach (even without the bag of extra coins). Making moves can therefore never increase the span, only decrease it accidentally.

Our sufficient condition is that "two extra coins suffice" in the following technical sense:

Theorem 1 [DDV02, Theorem 2] *Configuration A of coins can be re-arranged into configuration B via 2-adjacency moves on the square grid if there are two "extra" coins e_1 and e_2, each adjacent to two other coins (not each other), such that the span of $A - e_1 - e_2$ contains the span of $B - e_1 - e_2$. The number of moves is $O(n^3)$ where n is the number of coins, and the moves can be found algorithmically in $O(n^3)$ time.*

This theorem tells us an easy way to design puzzles that are guaranteed solvable: just make sure the spans of the two configurations match (or configuration A's span is more than configuration B's span), and make sure there are two extra coins. However, it remains an open problem how to find the fewest moves to solve such a puzzle.

2 Coin-Sliding Fonts

Over the past dozen years, we have developed several different typefaces/fonts that express text through mathematical theorems or open problems in broadly accessible forms, often through the use of puzzles. The fonts are all free to play with on the web.[2]

In this paper, we revisit sliding-coin puzzles from the perspective of mathematical/puzzle fonts. Figures 2 and 3 show our two font designs, one with 12 coins on a 5×7 rectangle and one with 13 coins on a 5×9 rectangle. Each font consists of 37 symbols (26 letters, 10 digits, and slash[3]), where each symbol is made from the same number of coins arranged on the square grid within the same size of rectangle (which is also the span of the configuration). You can write messages in these fonts using our online web application.[4]

Every pair of symbols within the same font defines a coin-sliding puzzle. Thus we obtain $37 \cdot 36 = 1{,}332$ puzzles within each font, for a total of 2,664 puzzles.

2.1 Puzzle Video Game

We implemented a puzzle video game for playing all of these puzzles. You can play on any device with a web browser[5] or using an Android app[6]. Figure 4 shows what the user interface looks like.

[2]http://erikdemaine.org/fonts/
[3]We included slash because it plays a significant role in many of our early coin-sliding puzzles [DDV02, DD04].
[4]http://erikdemaine.org/fonts/coinsliding/
[5]https://coinsliding.erikdemaine.org/
[6]https://play.google.com/store/apps/details?id=org.erikdemaine.coinsliding

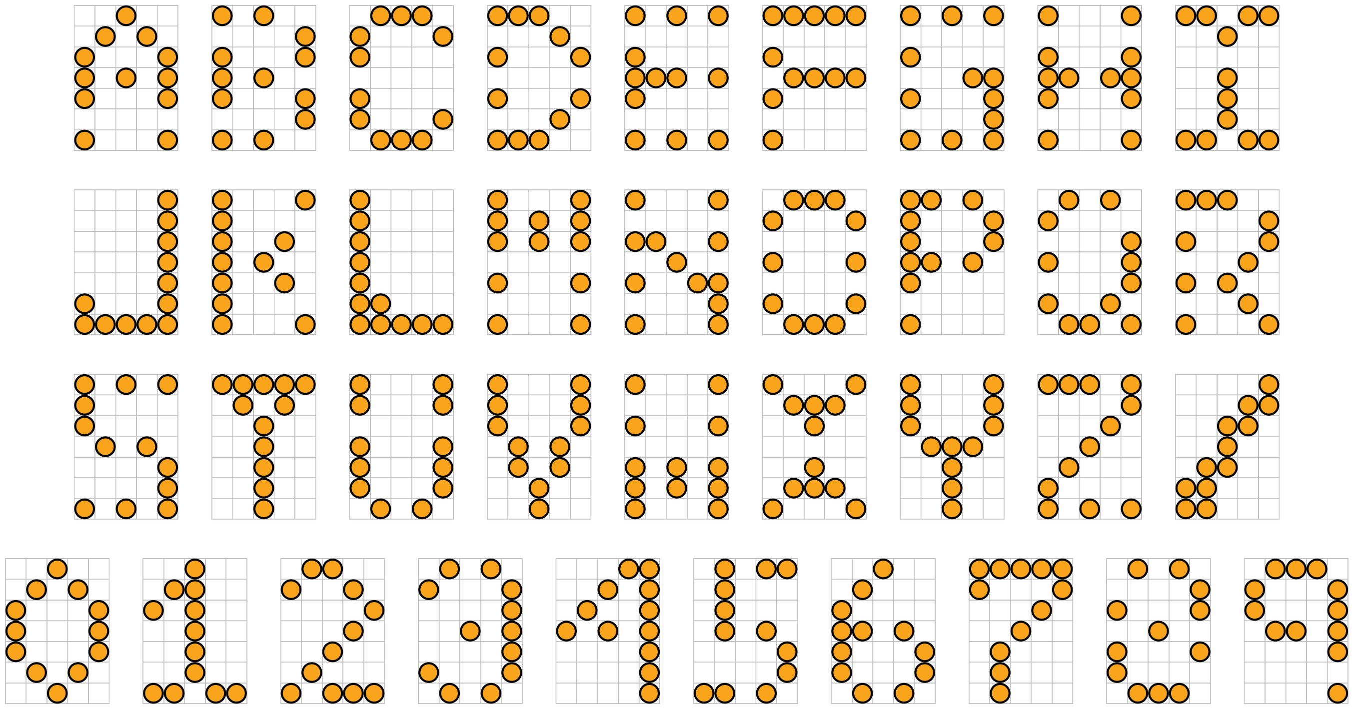


Figure 2: 5 × 7 coin-sliding font. Each symbol consists of 12 coins.

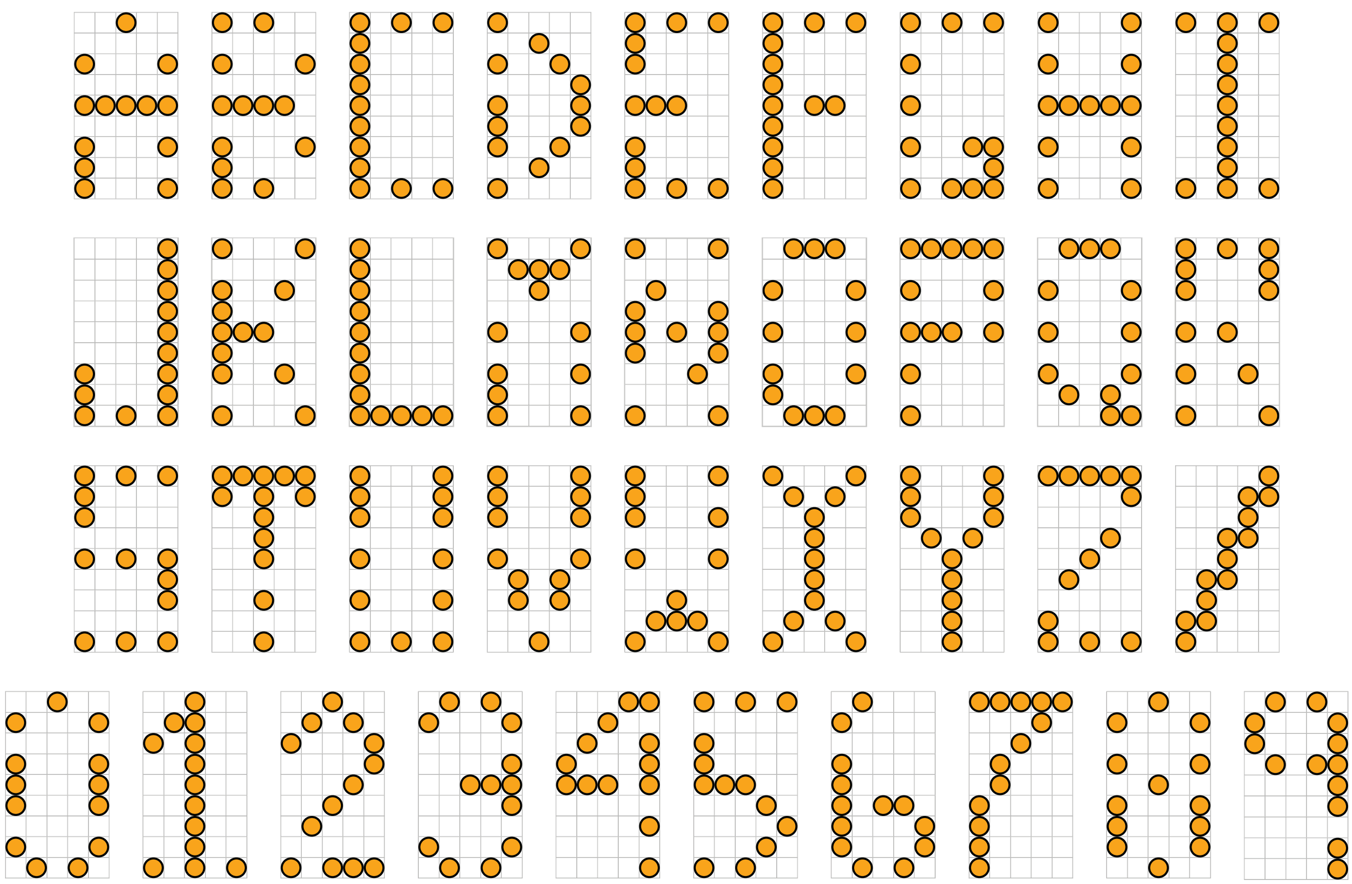


Figure 3: 5 × 9 coin-sliding font. Each symbol consists of 13 coins.

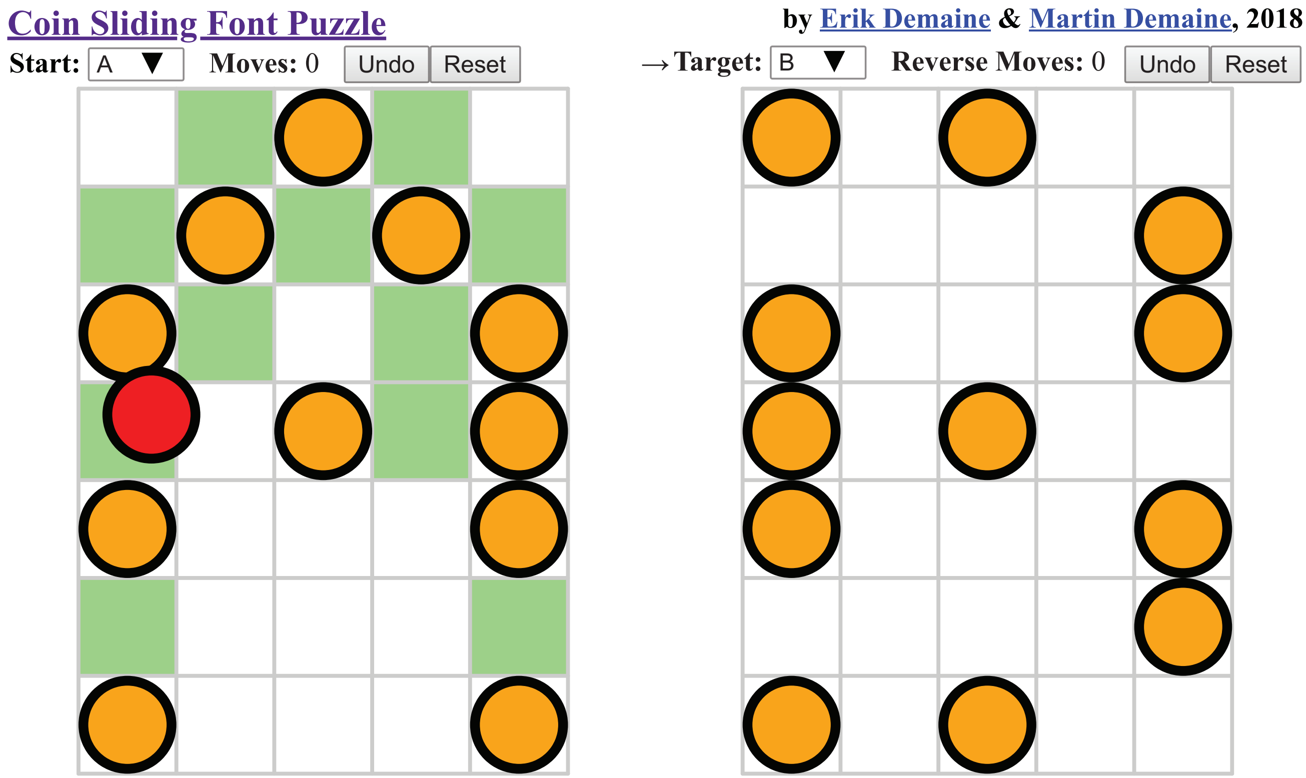


Figure 4: Coin-sliding puzzle video game. Play online at https://coinsliding.erikdemaine.org/

To play a puzzle, you select a font (5×7 or 5×9), then choose a puzzle from "All puzzles in family" or using the "Start" and "Target" selections. Dragging coins makes moves. If you get stuck, you can "Undo" move by move, or "Reset" to the beginning.

When you solve a puzzle, you can post your score (number of moves) along with your name. Help us find good solutions to all the puzzles! This will give us a better understanding of the number of moves required to solve coin sliding puzzles, which remains a mathematical mystery.

An example solution animation can be found on a special website.[7]

The source code is also available.[8]

2.2 Proof of Solvability

We prove that all of the puzzles are solvable. Theorem 1 covers most of the puzzles, as they all have span equal to the full rectangle (either 5×7 or 5×9), even after removing two well-chosen coins. However, not all of the configurations have extra coins neighboring two other coins, so they are not valid choices for the target configuration B in Theorem 1. Nonetheless, we can show that all symbol configurations are ***reachable*** from valid B configurations in Theorem 1 (and thus from all valid A configurations, including all other symbols). Figures 5 and 6 prove each case, either highlighting two suitable extra coins, or showing a sequence of ***reverse moves*** (with arrows) that free up two suitable extra coins. A reverse move is the exact opposite of a 2-adjacency move, i.e., it moves a coin from a position adjacent to at least two other coins to any other position. Each sequence of reverse moves can be verified by dragging coins on the right side of the puzzle video game's user interface.[4]

[7]http://erikdemaine.org/fonts/coinsliding/g4g.html
[8]https://github.com/edemaine/coinsliding

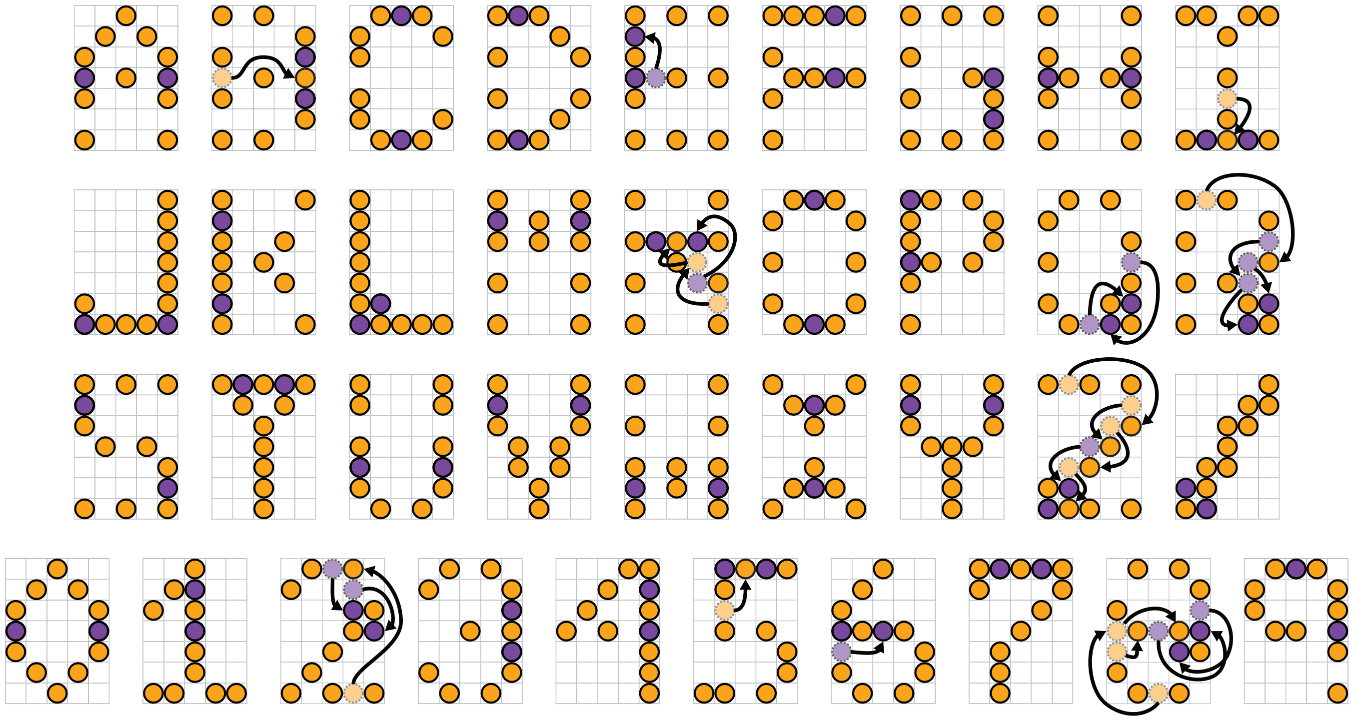


Figure 5: Reachability proof for 5 × 7 coin-sliding font.

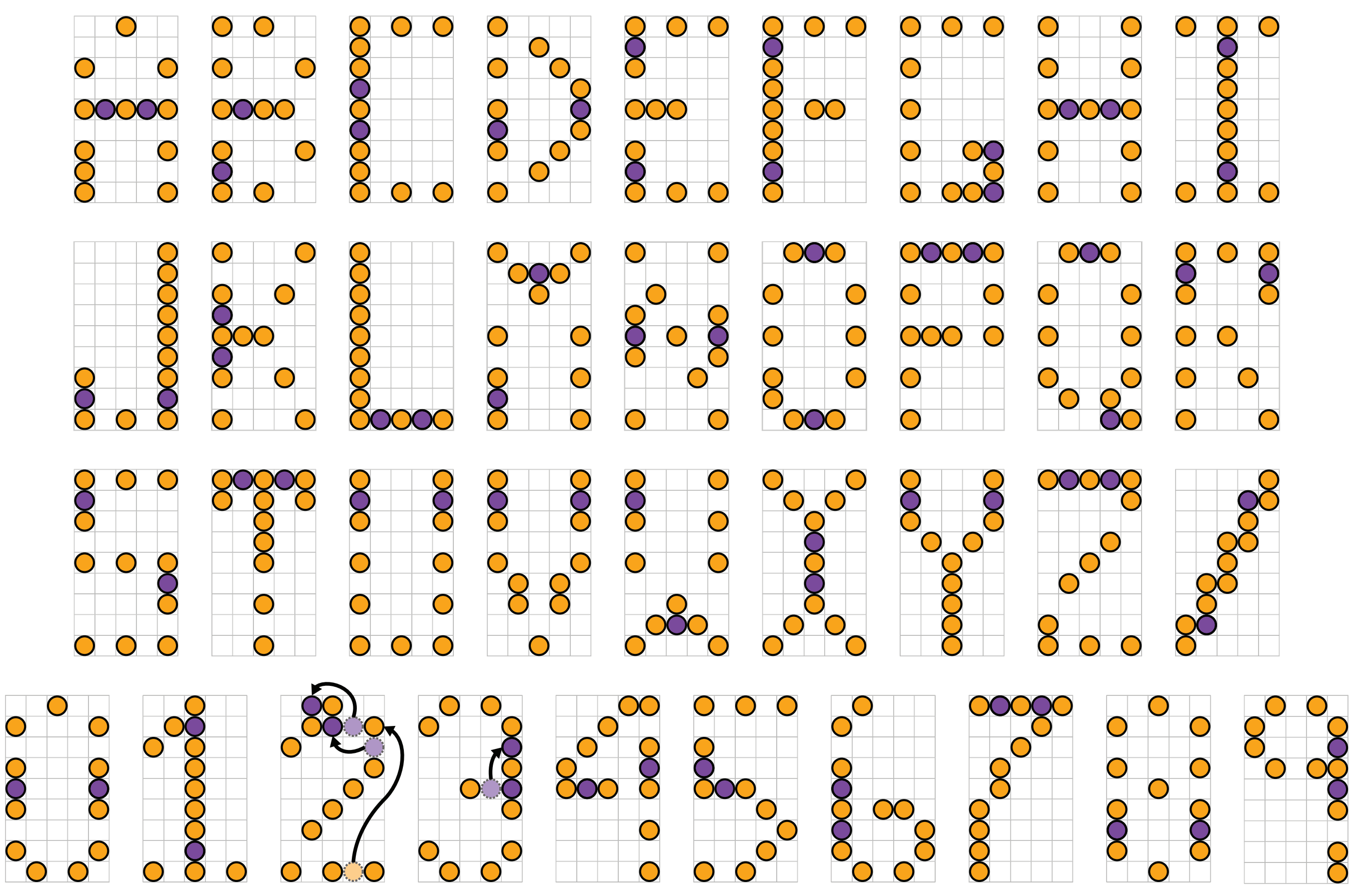


Figure 6: Reachability proof for 5 × 9 coin-sliding font.

References

[DD04] Erik D. Demaine and Martin L. Demaine. Sliding-coin puzzles. In *Tribute to a Mathemagician*, pages 63–72. A K Peters, 2004.

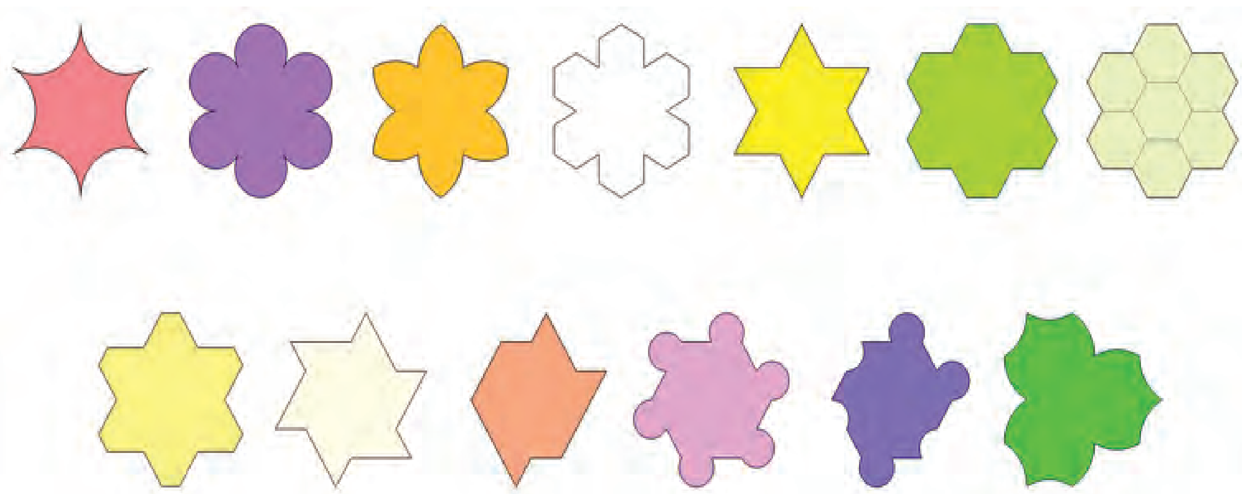
[DDV02] Erik D. Demaine, Martin L. Demaine, and Helena Verrill. Coin-moving puzzles. In R. J. Nowakowski, editor, *More Games of No Chance*, pages 405–431. Cambridge University Press, 2002. Collection of papers from the MSRI Combinatorial Game Theory Research Workshop, Berkeley, California, July 2000. http://www.arXiv.org/abs/cs.DM/0204002.

[Gar75] Martin Gardner. Penny puzzles. In *Mathematical Carnival*, chapter 2, pages 12–26. Alfred A. Knopf, New York, 1975. Appeared in *Scientific American*, 214(2):112–118, February 1966, with solutions in 214(3):116–117, March 1966.

[Lan51] Harry Langman. Curiosa 261: A disc puzzle. *Scripta Mathematica*, 17(1–2):144, March–June 1951.

LUCKY 13

A baker's dozen of combinatorial puzzles
by Michael Dowle



presented by Kate Jones
to the 13th Gathering for Gardner
April 11-15, 2018
Atlanta, Georgia

Combinatorial Puzzle Designs

by Michael D. Dowle

(2017)

Preface/Background

The vintage (late 1960s/early 1970s) "Beat the Computer" Pla-Puzzle No. 0 was the only puzzle published by Tenyo, Japan, with rounded puzzle pieces. This puzzle triggered my interest and an idea for a new puzzle design, and subsequently groups of puzzles. I purchased the Pla-Puzzle No. 0 in the early 1970s. I was, however, disappointed and frustrated by the puzzle design, since two of the thirteen pieces were identical. Each puzzle piece was a circle with up to six protuberances around the circumference, but there was no circular piece without any protuberances. Instead there were two circular pieces with one protuberance.

Replacing one of the duplicate pieces with a circle created a complete geometric set comprising thirteen different puzzle pieces. This set satisfyingly filled a template with three-fold circular symmetry.

Combinatorial Puzzle Designs

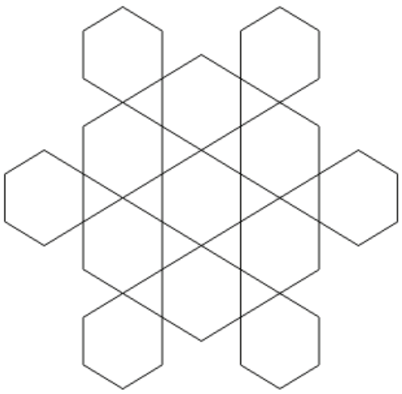
The Combinatorial Puzzles presented here require a set of 13 puzzle pieces to be fitted inside a template. The objective is to find 13 solutions. Each solution must have a different puzzle piece covering the center of the template (except for Puzzle 8). There may be alternate solutions for each puzzle piece.

The pieces for each puzzle are generated using the same principle and constitute a geometric set. A set of puzzle pieces is produced by arranging up to six shapes in every possible configuration around a differently shaped central piece that exhibits six-fold rotational symmetry (except for Puzzle 5). The templates possess three-fold rotational symmetry. These properties can be seen in the following illustrations.

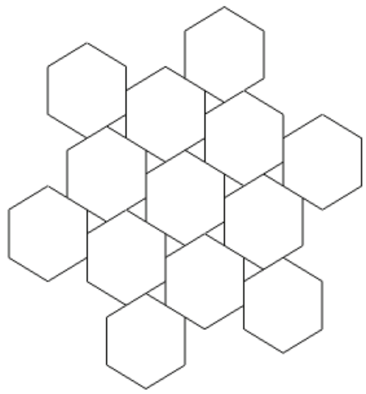
Three groups of Combinatorial Puzzles are described—each group has its own design characteristics, but all groups share the same common objective.

Each Combinatorial Puzzle is presented on a page in a common format, *viz.* puzzle pieces (on left); design grid structure and puzzle template (on right); puzzle solutions (bottom).

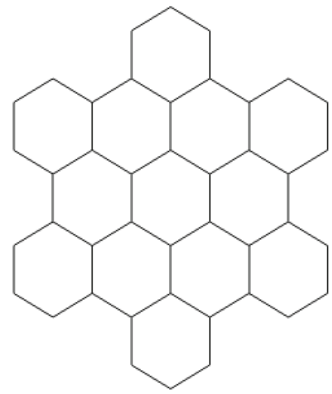
The designs of the puzzle pieces and corresponding templates are different for the three groups presented. For the first group (Puzzles 1 through 8), 12 of the 13 pieces have mirror symmetry (five of which also have rotational symmetry) while the 13th piece is chiral. The templates possess both three-fold rotational symmetry and mirror symmetry. The chiral piece may be used with either face upward. The pieces are vertex-connected.



Group 1—vertex-connected



Group 2—vertex to edge



Group 3—edge-connected

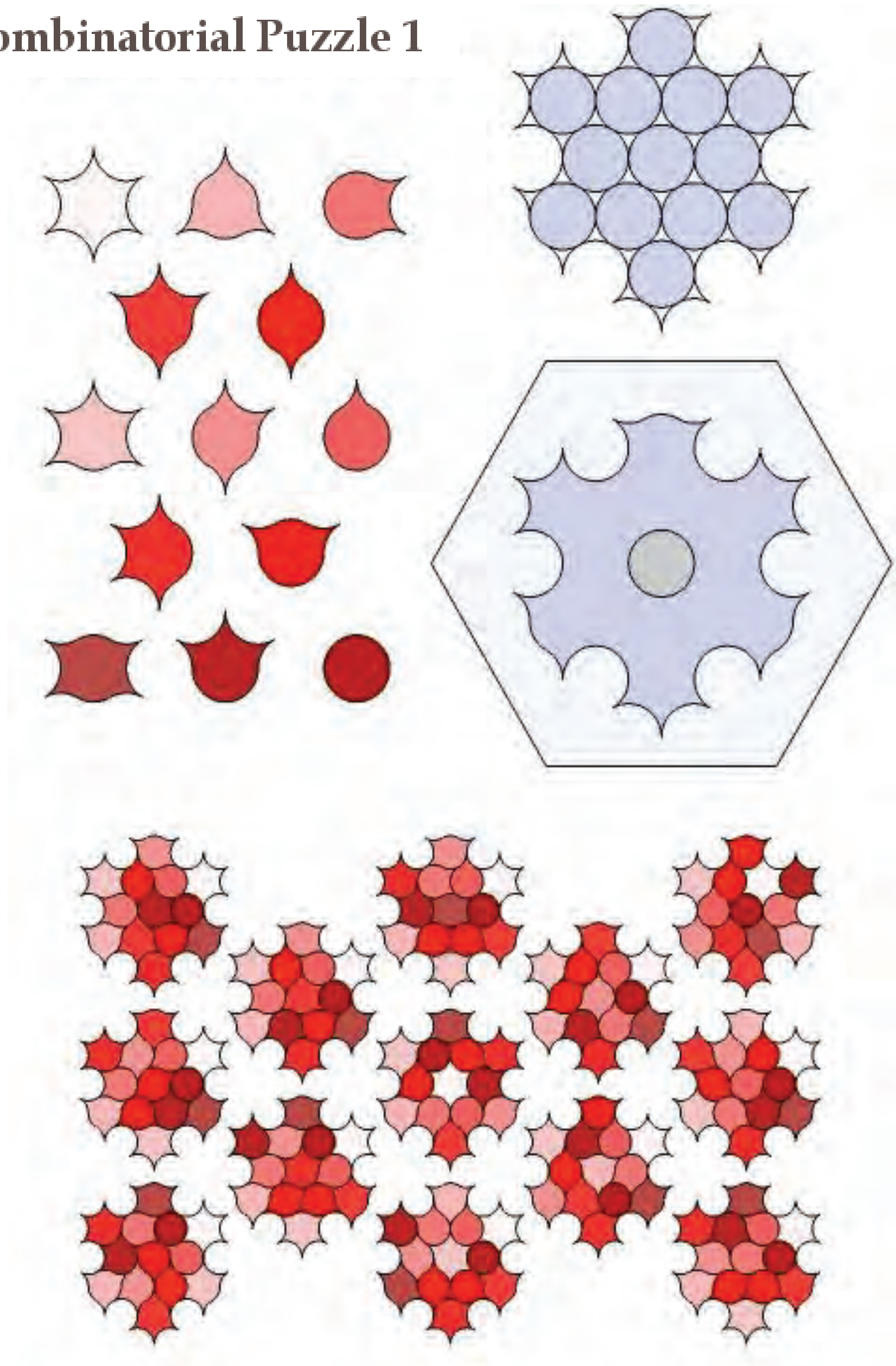
For the second group (Puzzles 9 through 12), 12 of the 13 pieces are chiral (4 of which have rotational symmetry) while the 13th piece has both rotational and mirror symmetry. The templates are chiral with three-fold rotational symmetry. The chiral pieces may be used with only one face upward, the face consistent with the chirality of the template. The chiral pieces can have two different shapes. The pieces are connected vertex-to-edge.

The third type (Puzzle 13), created by Jacques Griffioen and developed by Kate Jones, has 12 of the 13 pieces with mirror symmetry (5 of them also have rotational symmetry). The 13th piece is chiral and may be used with either face upward. The template has three-fold rotational symmetry and is chiral. The pieces are edge-connected.

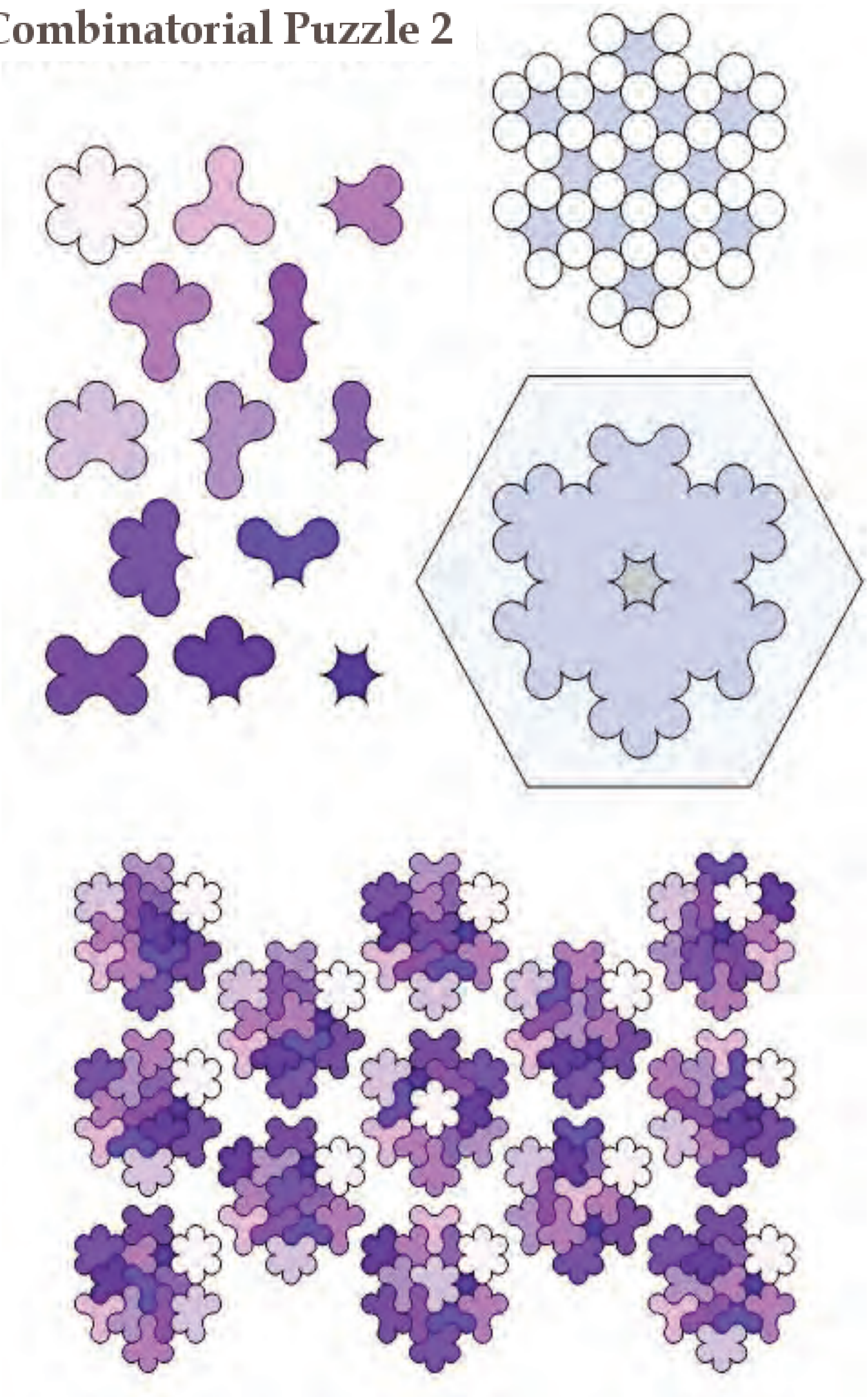
NOTES

- *Combinatorial Puzzle 1*—Twelve of its thirteen puzzle pieces appeared in the "Beat the Computer" Pia-Puzzle No. 0 published by Tenyo, Japan, in the 1960s-1970s. The "Beat the Computer" puzzle used a different template design and duplicated one of the puzzle pieces to obtain a thirteenth puzzle piece.
- *Combinatorial Puzzle 5*—Some of its complete geometric set of pieces are used in the STAR HEX™ puzzle published by Kadon Enterprises, Inc. The STAR HEX™ puzzle uses more pieces than Combinatorial Puzzle 5 and has different objectives.
- *Combinatorial Puzzle 7*—Some of this complete geometric set of pieces are used in the HEXNUT™ puzzles published by Kadon Enterprises, Inc. The HEXNUT™ puzzles use more pieces than Combinatorial Puzzle 7 and have different objectives.

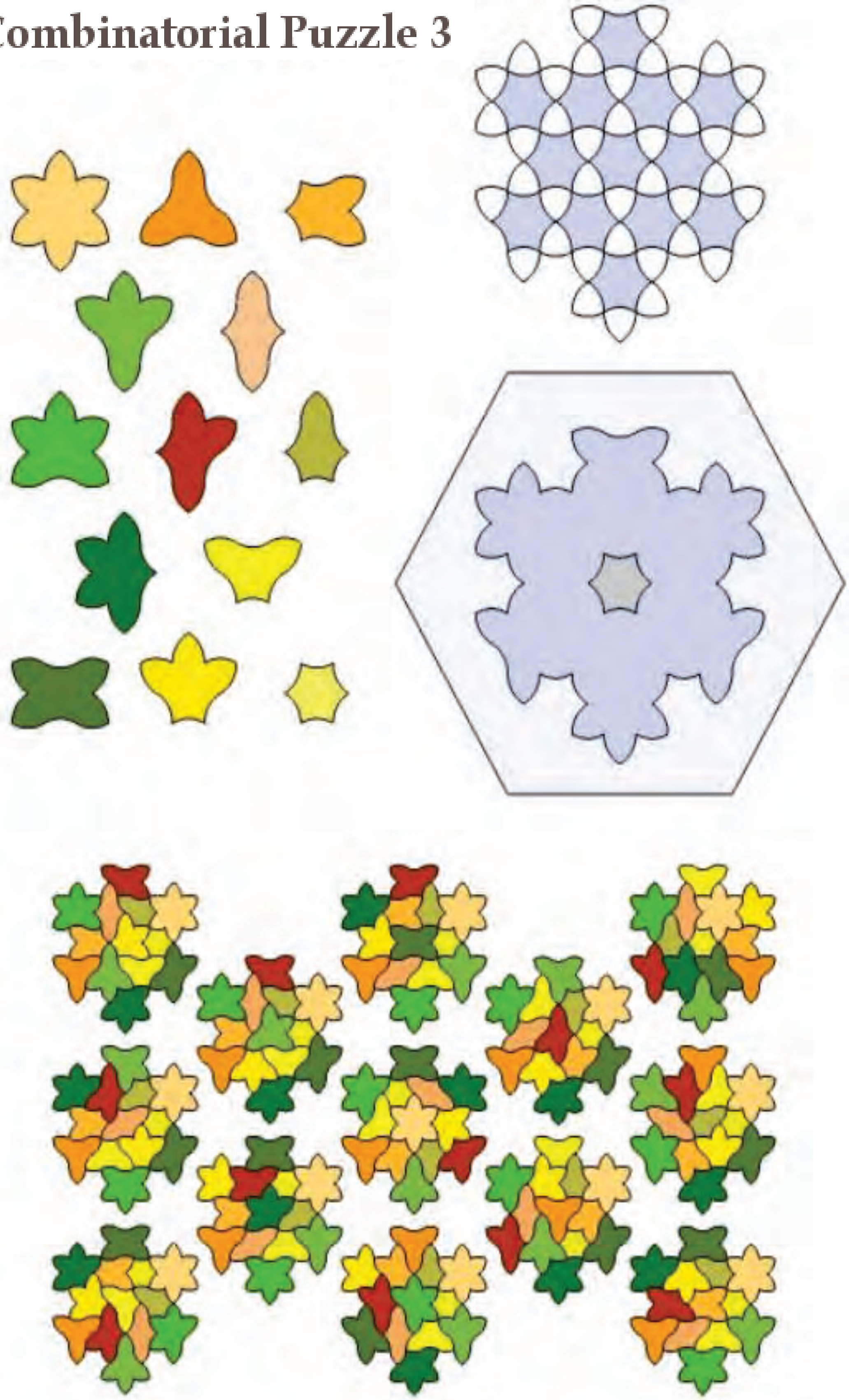
Combinatorial Puzzle 1



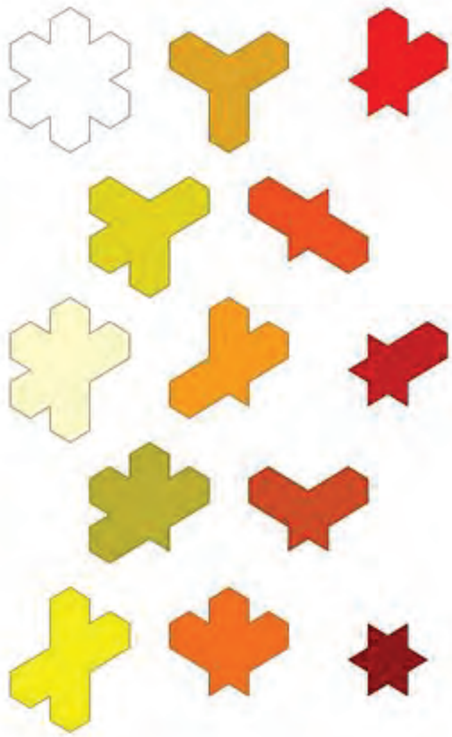
Combinatorial Puzzle 2

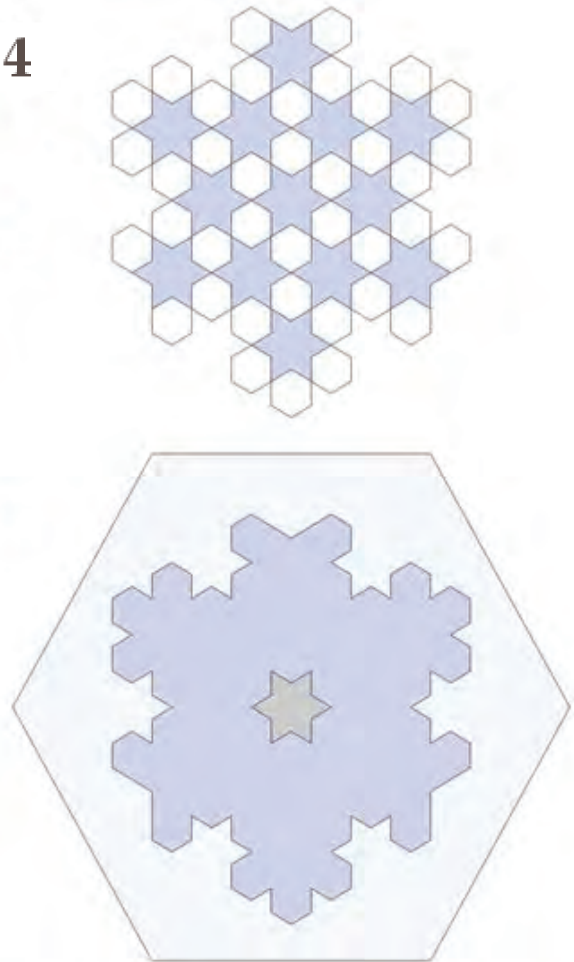


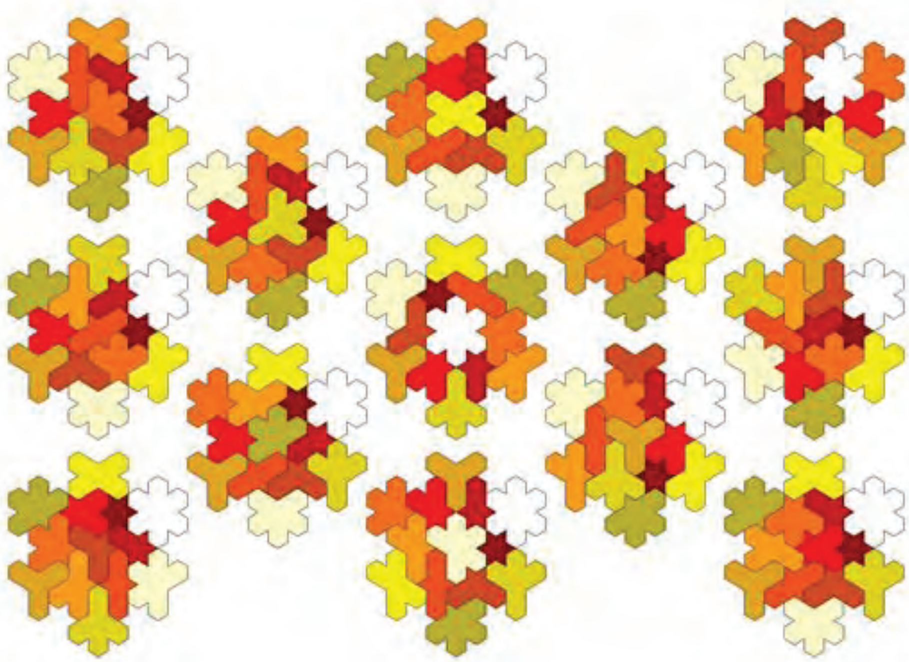
Combinatorial Puzzle 3



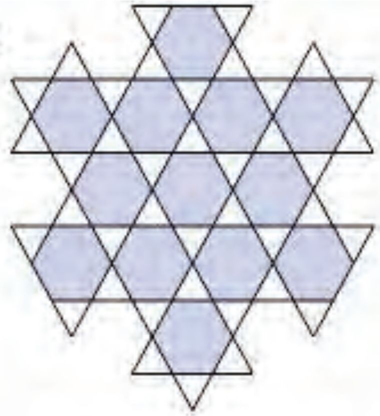
Combinatorial Puzzle 4

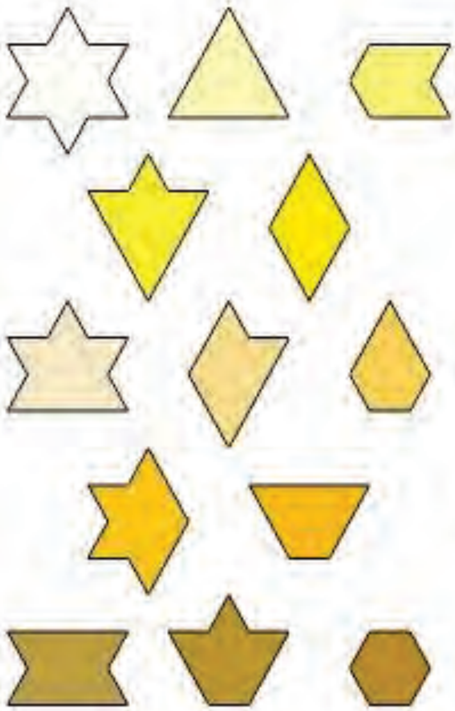


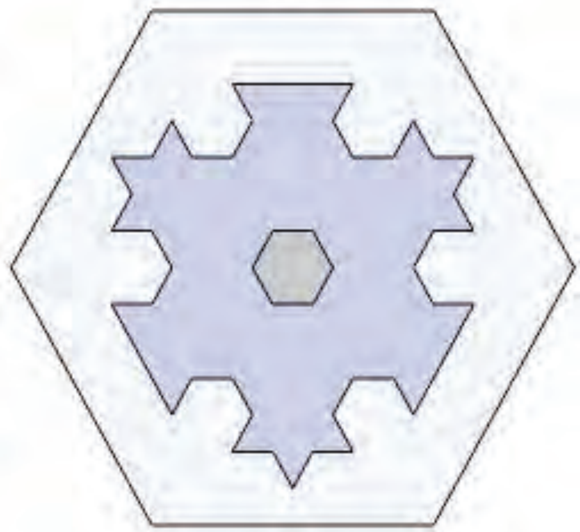


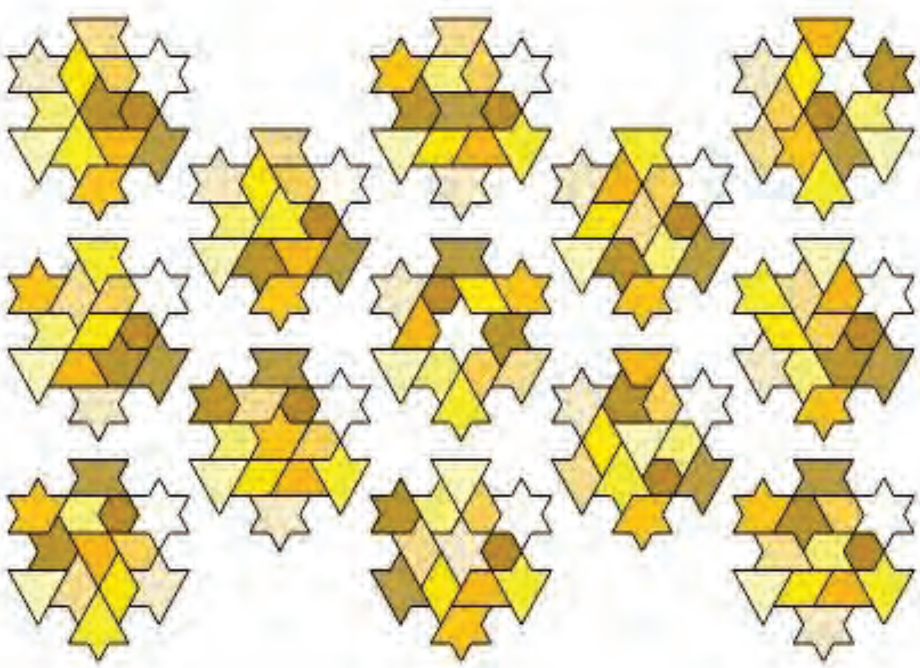


Combinatorial Puzzle 5

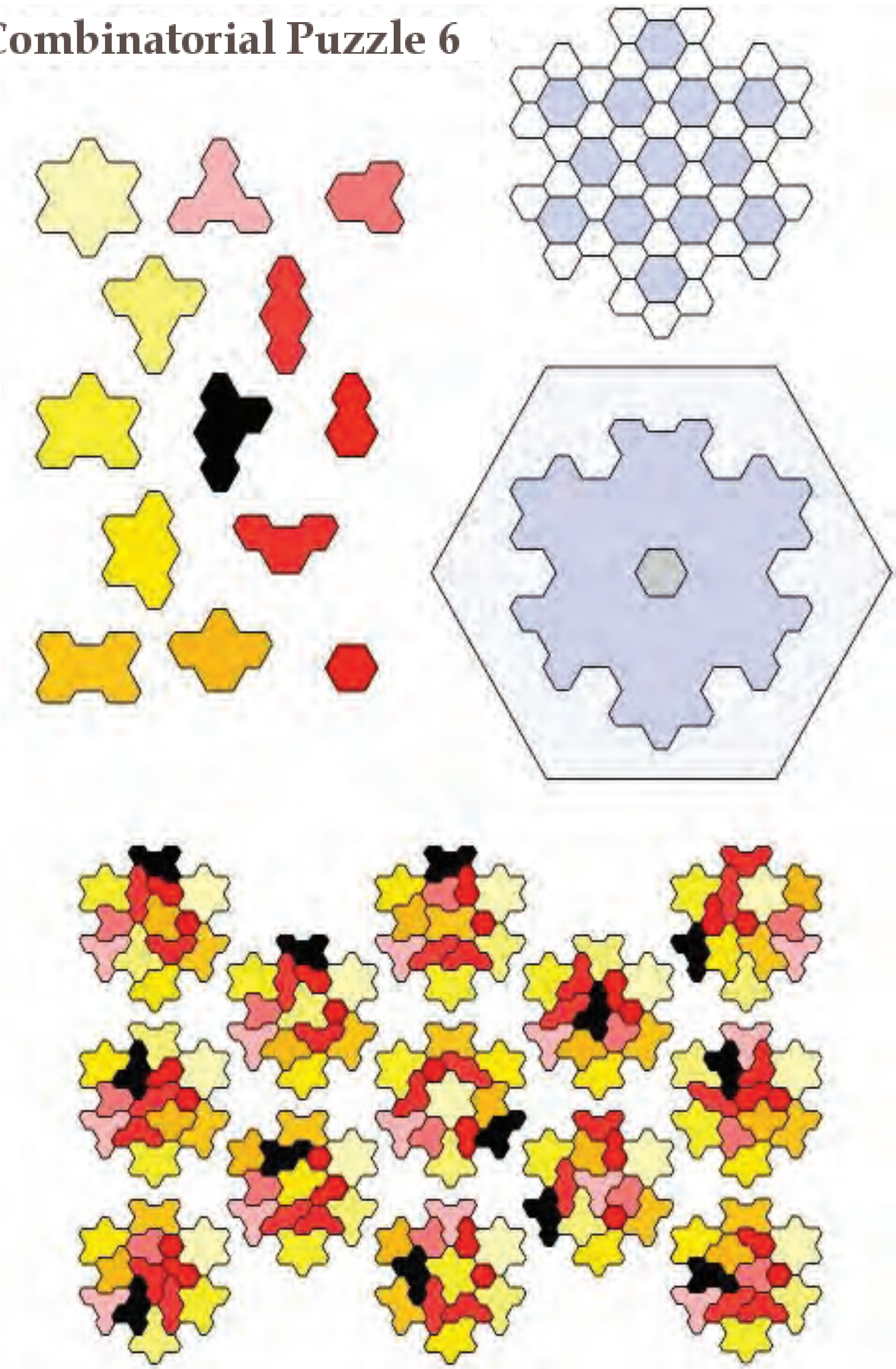




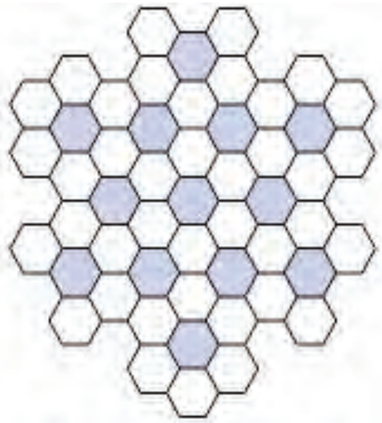


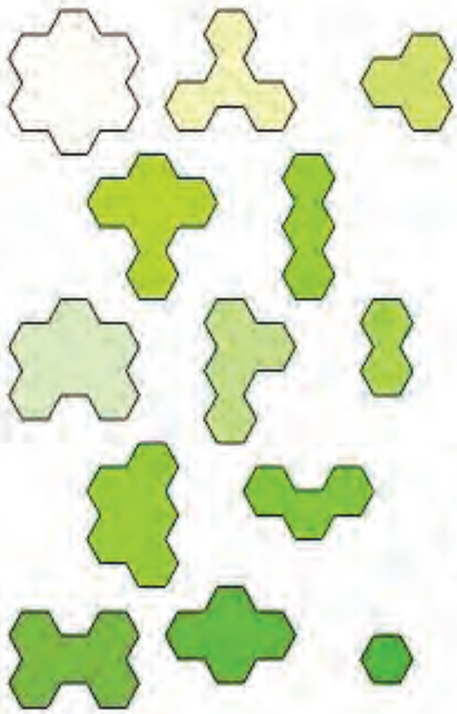


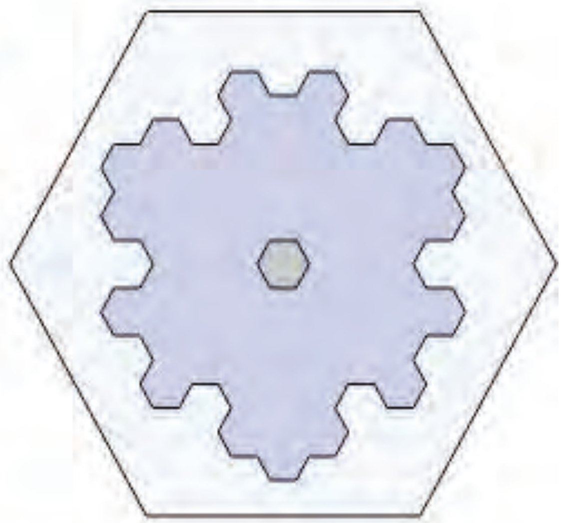
Combinatorial Puzzle 6

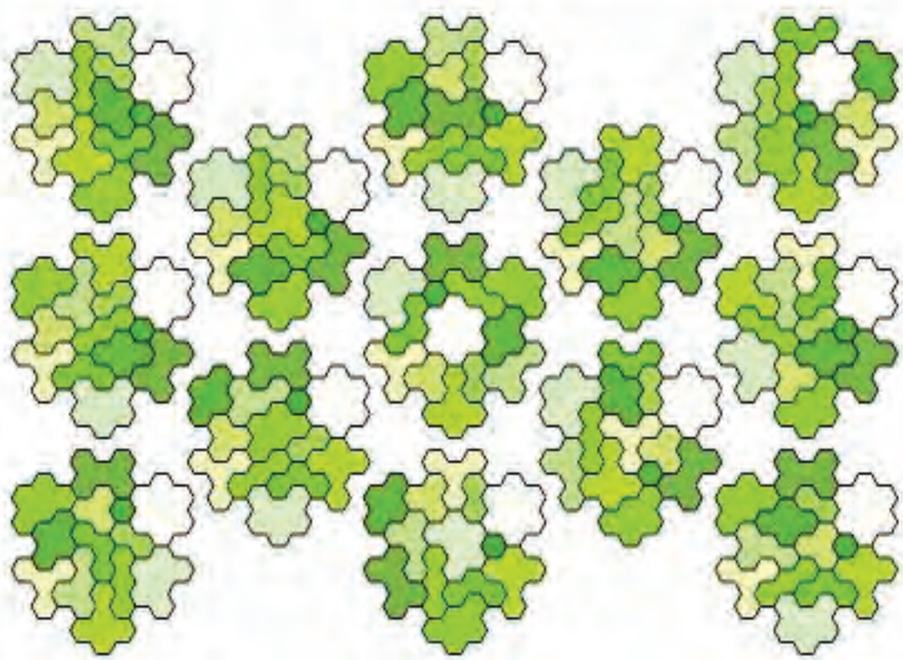


Combinatorial Puzzle 7

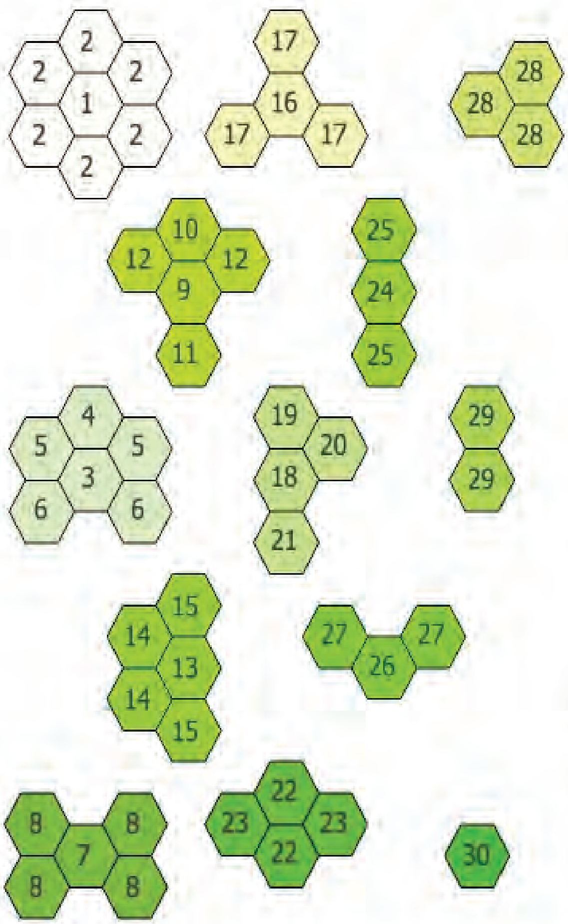


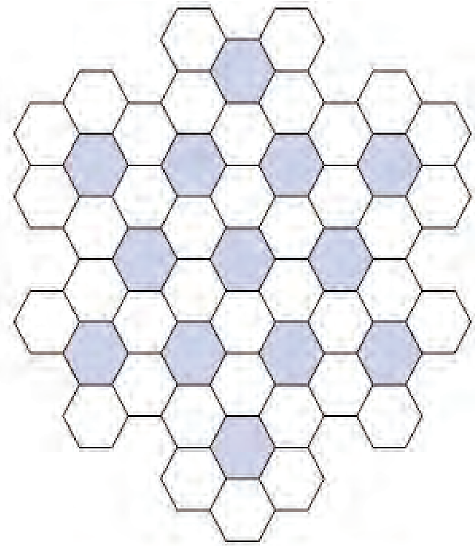


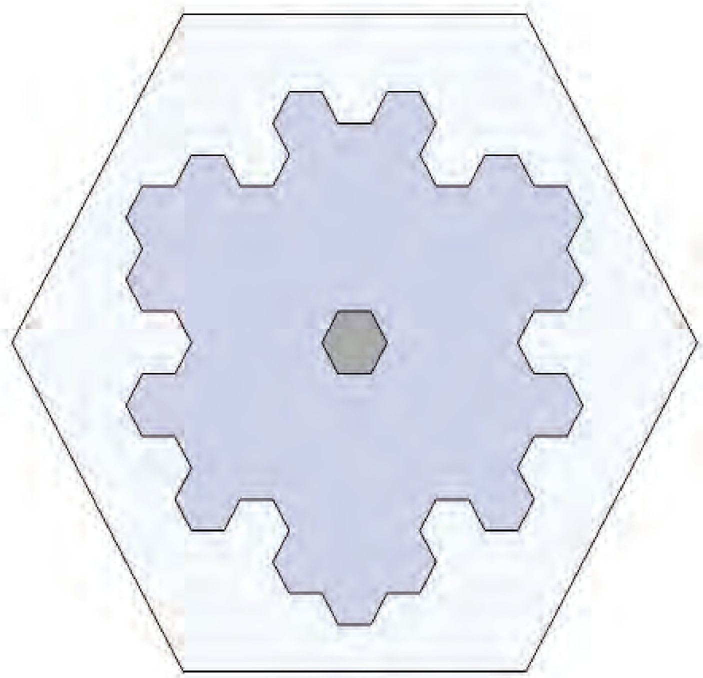




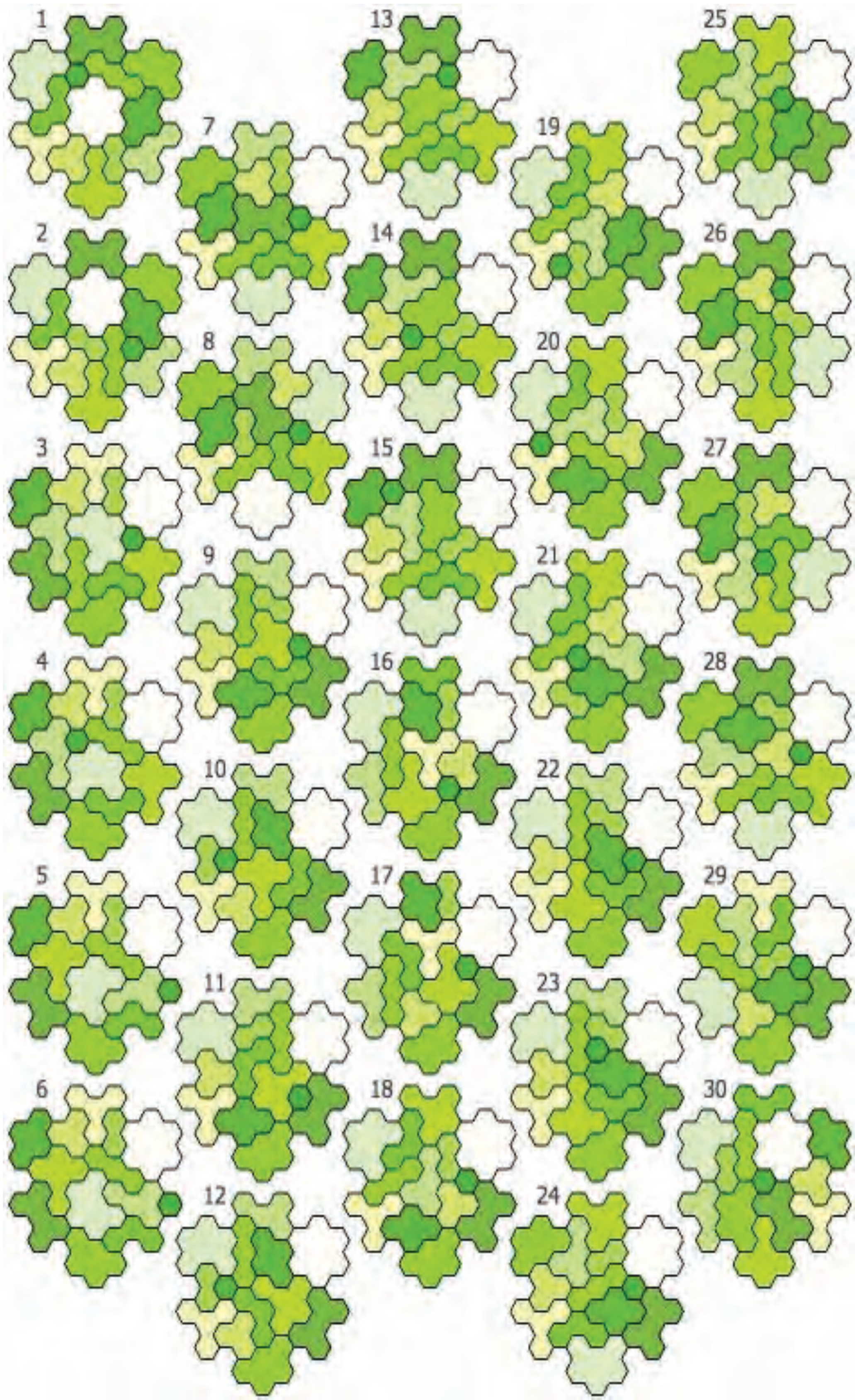
Combinatorial Puzzle 8



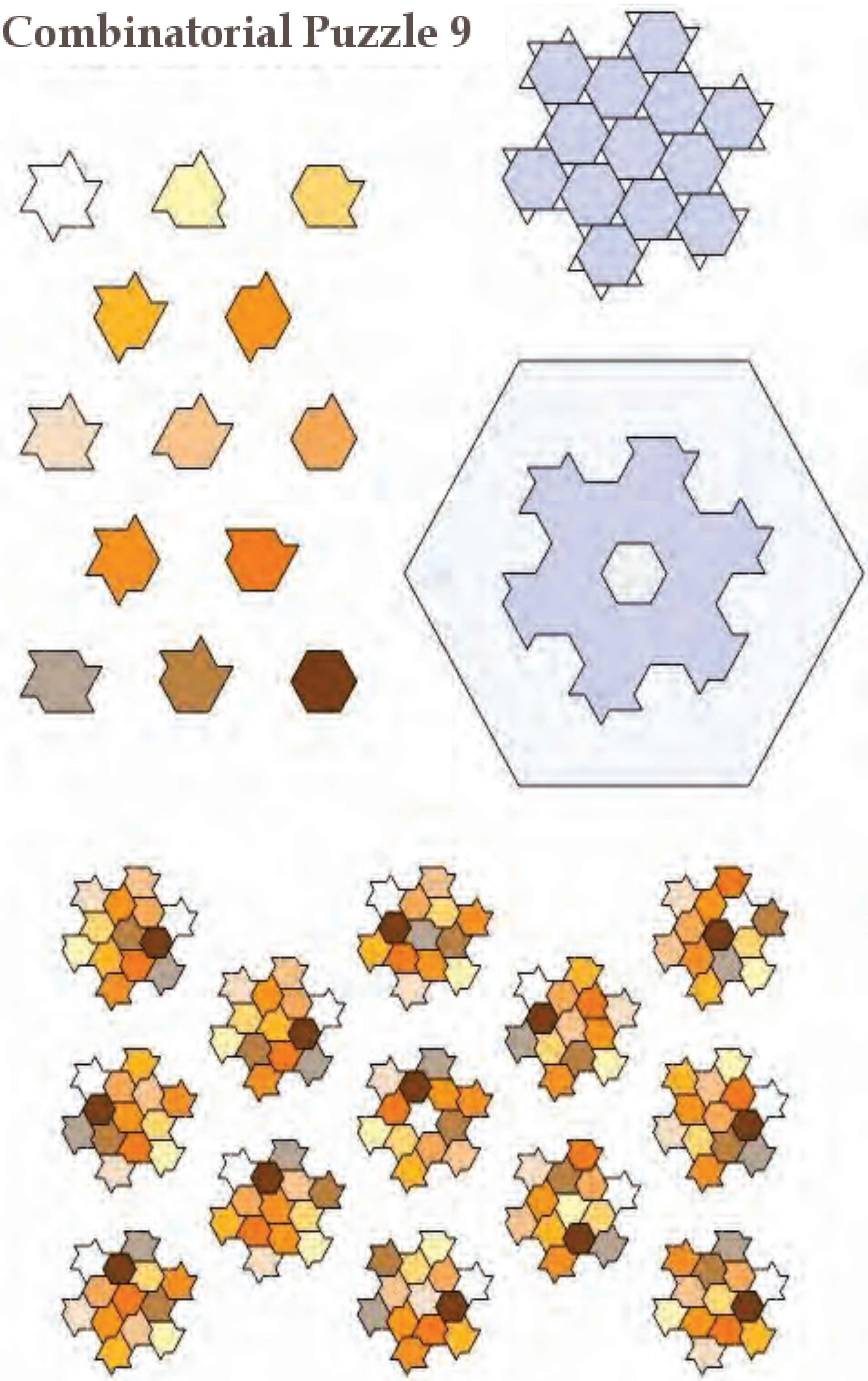




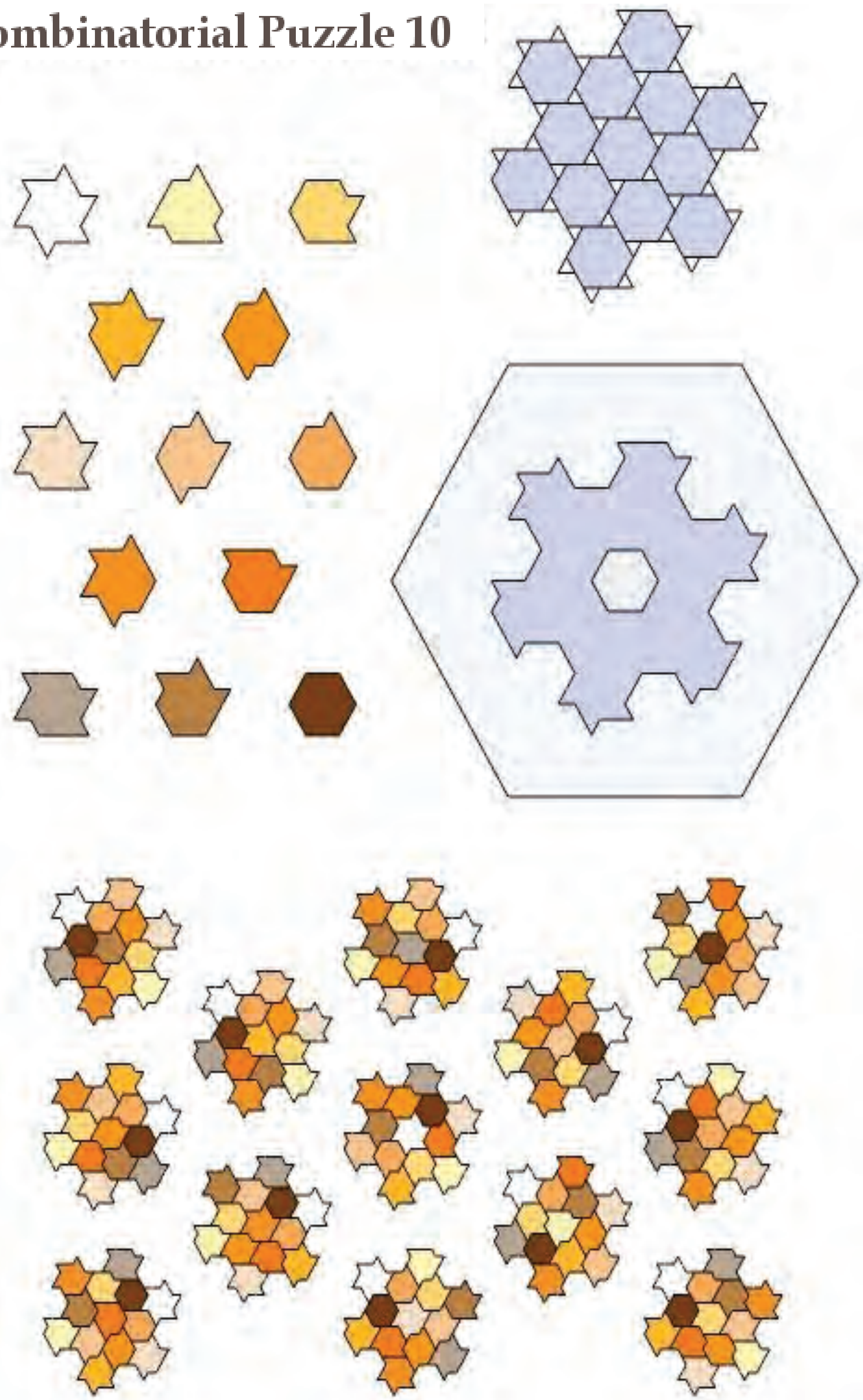
This puzzle differs from the previous versions inasmuch as the central shape and the surrounding shapes are congruent hexagons. An alternate objective for this puzzle is to find 30 solutions where every solution has a different hexagon at the center. See illustration below showing equivalent hexagons in the pieces.


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30

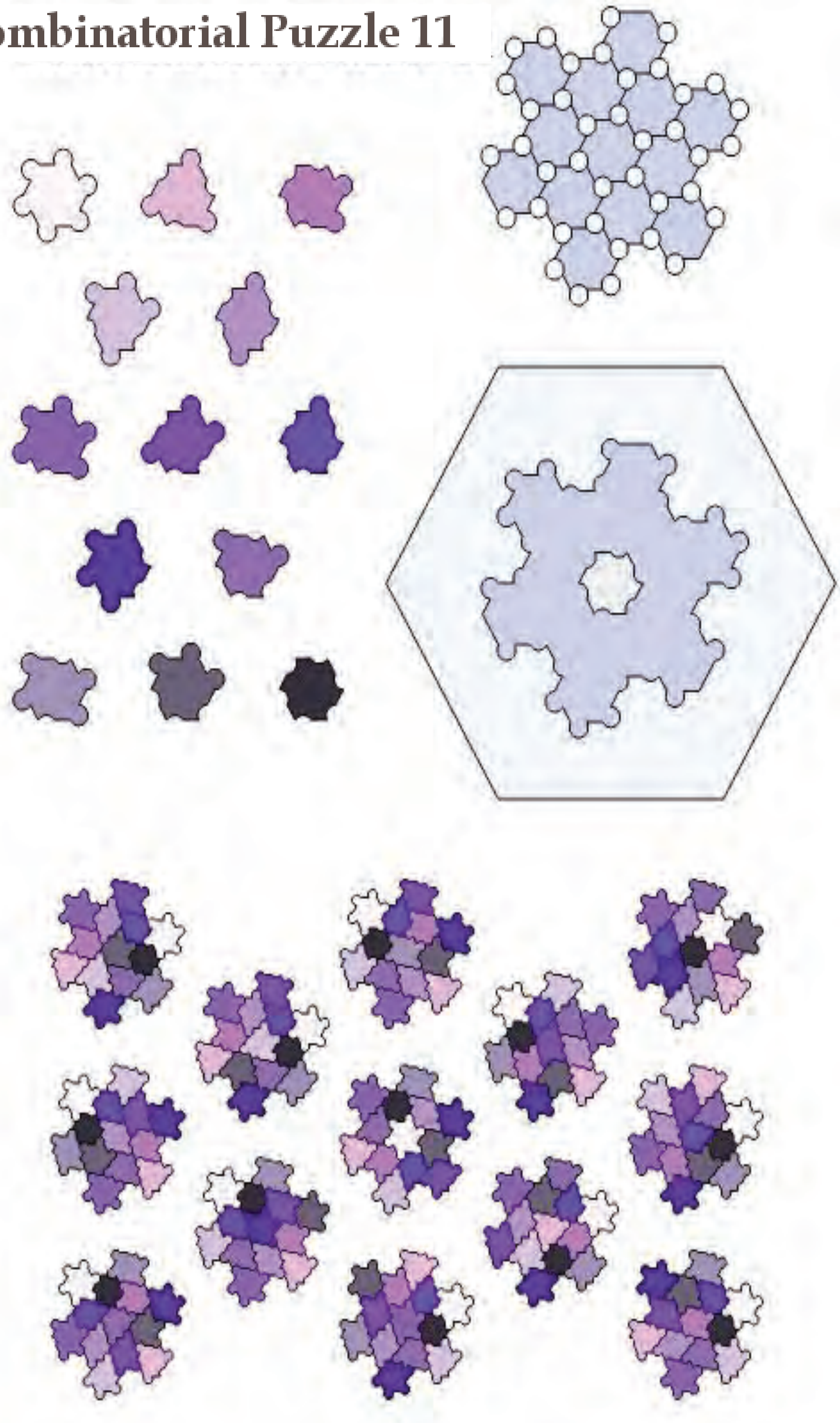
Combinatorial Puzzle 9



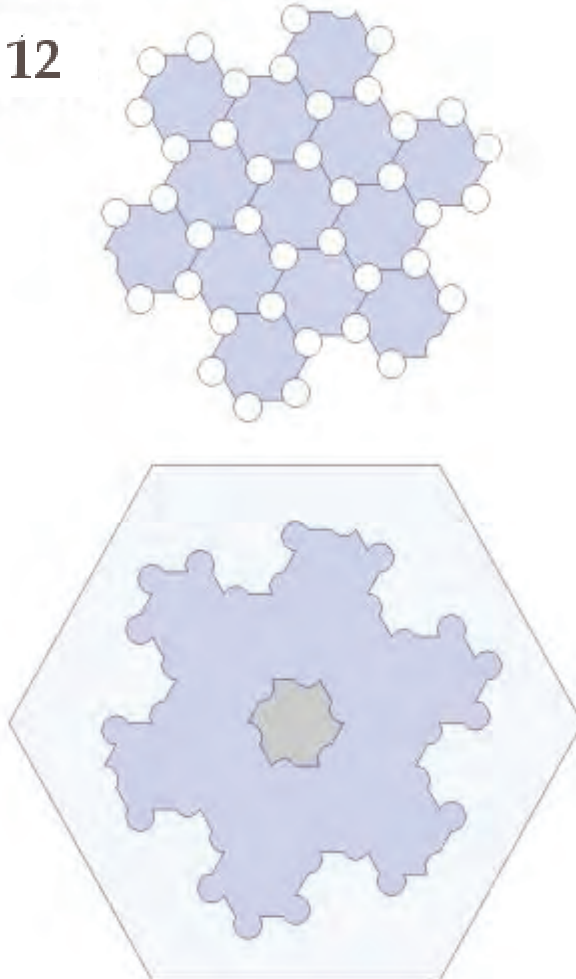
Combinatorial Puzzle 10

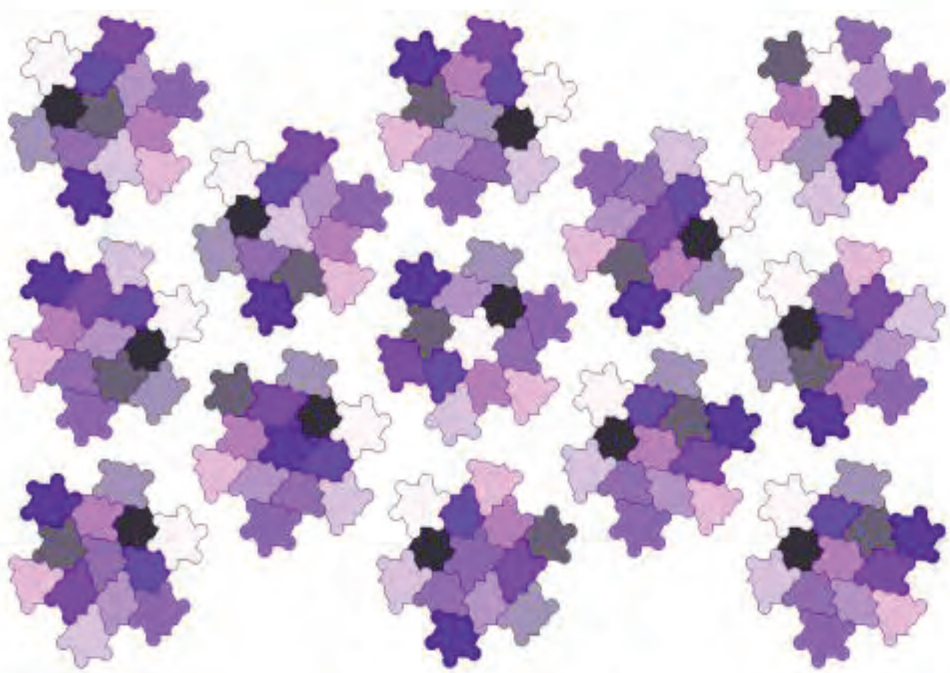


Combinatorial Puzzle 11

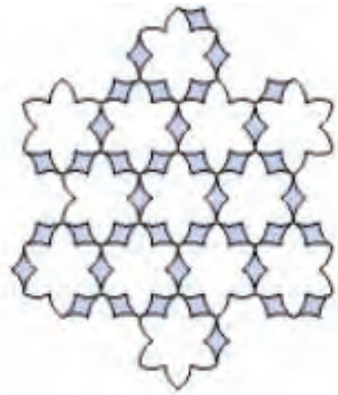


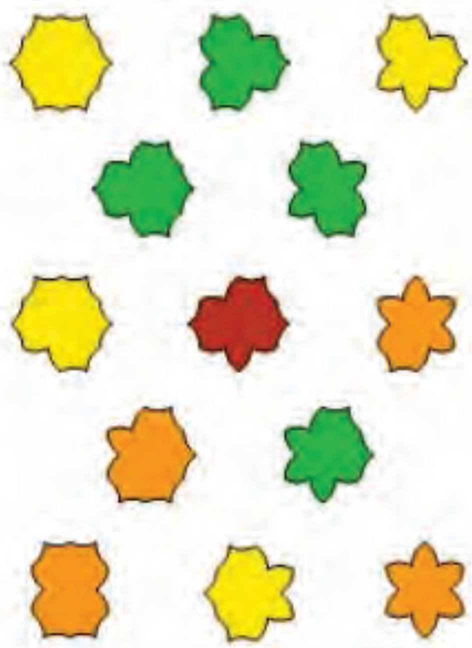
Combinatorial Puzzle 12

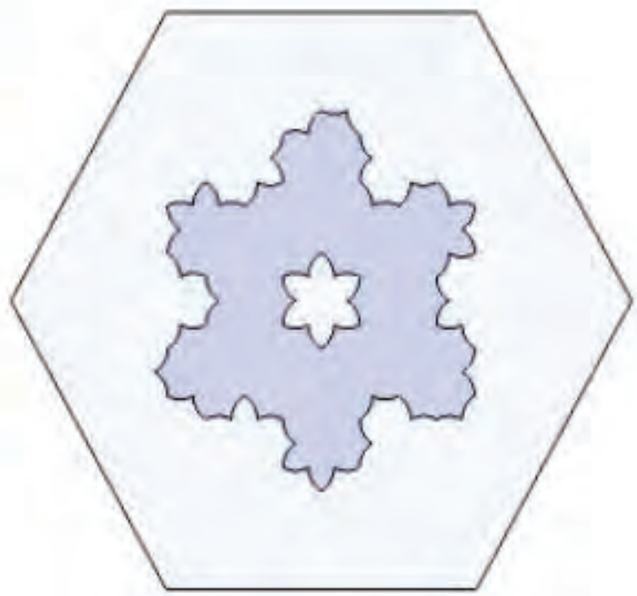


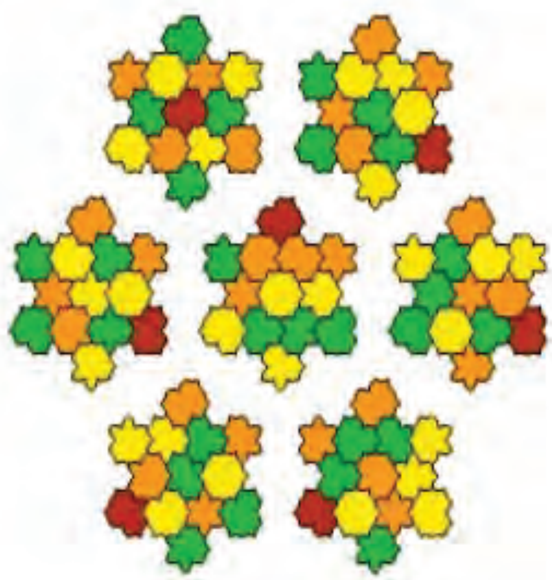


Combinatorial Puzzle 13
(Kadon's LEAVES Puzzle)

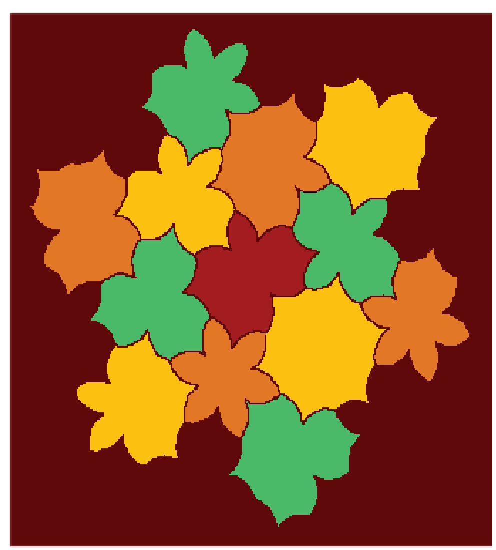




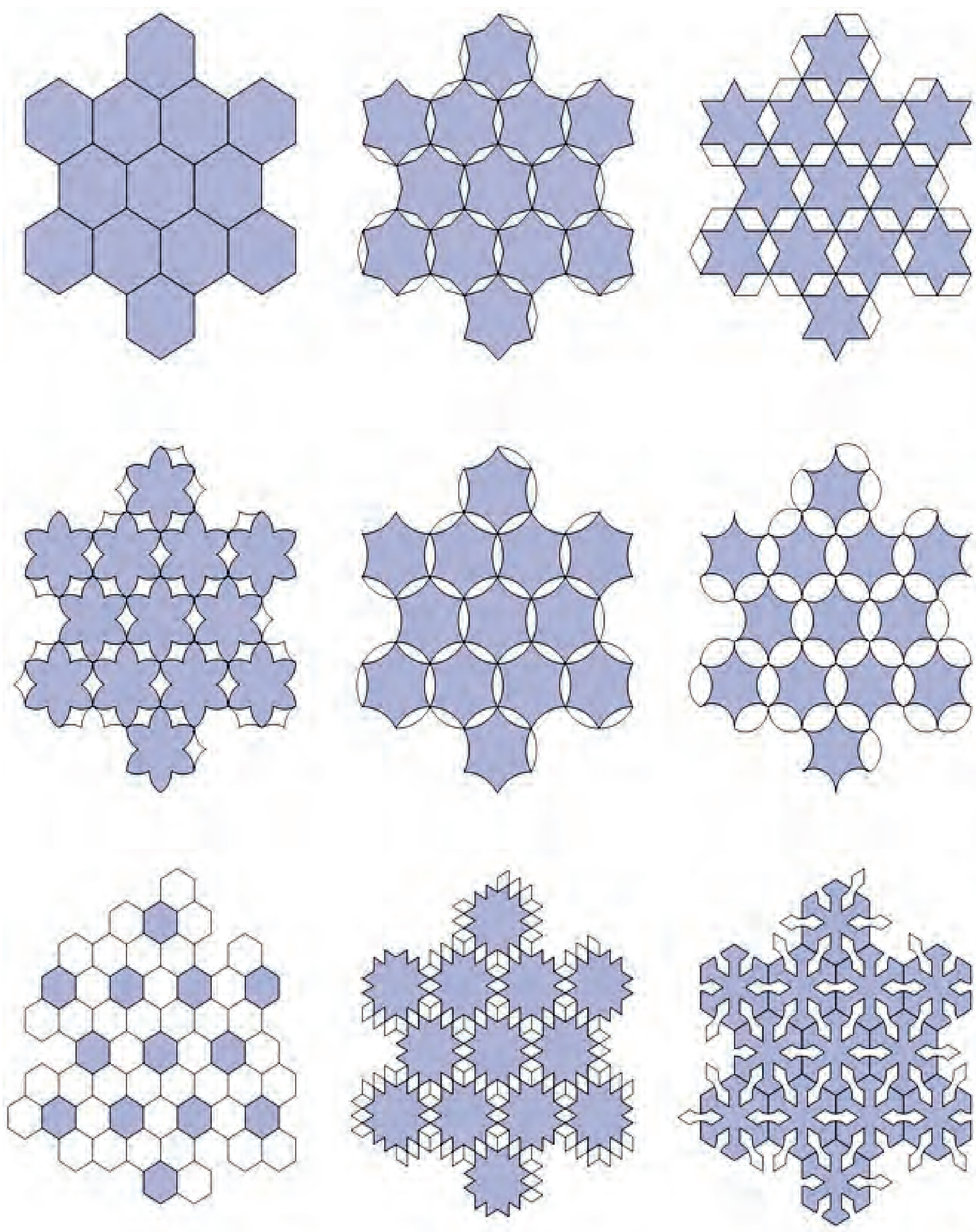




"LEAVES" is a trademark of Kadon Enterprises, Inc.



There are many alternate possibilities for the design of the LEAVES pieces and templates. Some samples of design grids by Michael Dowle are illustrated below.



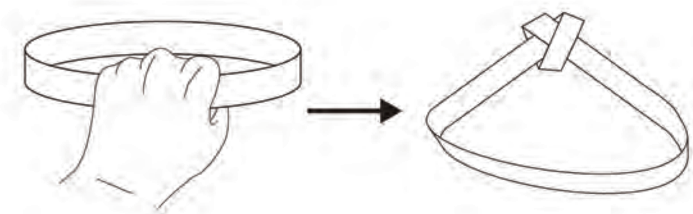
Some Paper Puzzles[1]

Yossi Elran

Paper Knot Puzzle

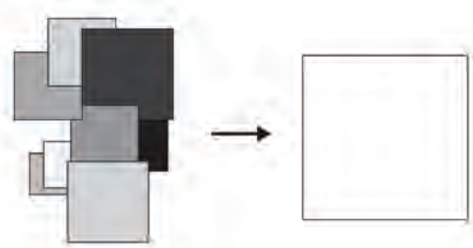
Make a band out of a strip of paper. Tie a knot in the band without cutting the band open (that is, without cutting the band along its width)!

Hint: What kind of a band is needed to begin with?



Overlapping Papers Puzzle

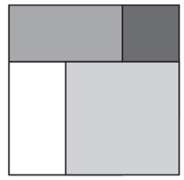
Arrange square sheets of paper one on top of the other to form a square. What is the smallest number of sheets needed to ensure that no sheet is fully visible? There are no other limits to this puzzle.



The following figure shows a counter-example using four different colored square sheets of paper. You can see that there is one sheet which is totally visible, which does not fulfil the requirement of the solution.

[1] Adapted with permission from:

Ilan Garibi, David Goodman and Yossi Elran, *"The Paper Puzzle Book: All You Need is Paper"*, World Scientific Press, New Jersey, 2017



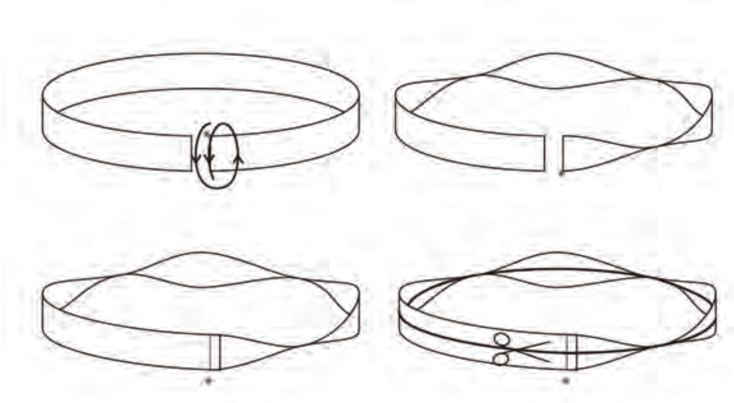
Follow up challenge: Arrange square sheets of paper, all the same size, one on top of the others to form a square. What is the smallest number of sheets needed if it is required that no sheet is fully visible?

Quadrisecting Rectangles into Triangles Puzzle

Find at least six <u>different</u> triangles that you can fold from a sheet of printer paper, where each triangles area is a quarter of the area of the whole sheet. You are allowed to use only two fold lines. A 'pinch' made to mark a certain point on the paper is not considered a fold line.

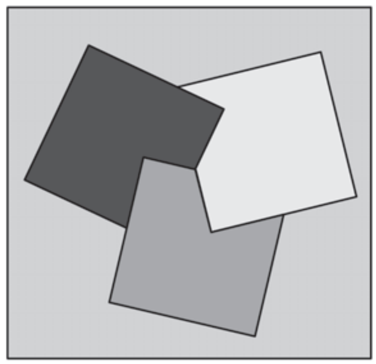
Paper Knot Puzzle Solution

The trick lies in the preparation of the paper band before you start cutting. The paper band has to be half-twisted three times. When cutting along the center line and opening up, a band with a knot in it is created. This is a less-known property of Möbius bands. Generally speaking, making *n* half-twists in a strip of paper and taping its ends will form either a one-sided or two-sided Möbius band, depending on the parity (odd *n* generate single-sided bands while even *n* generate two-sided bands). When cutting along the center line of these bands, either one (for odd *n*) or two (for even *n*) bands are created, with $\frac{1}{2}(n-1)$, for odd n, or $\frac{1}{2}(n-2)$, for even *n*, knots in them.

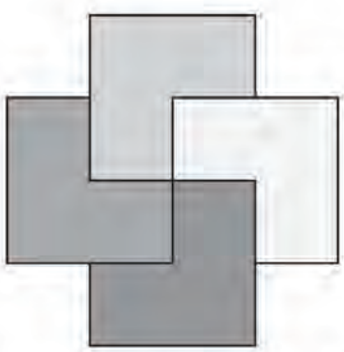


Overlapping Papers Puzzle Solution

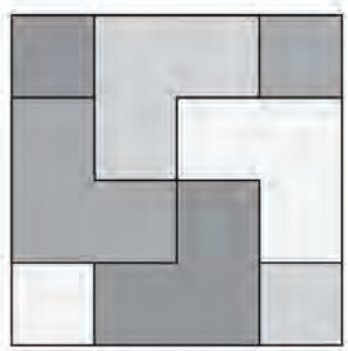
Four sheets is the minimal amount. Three that cover each other and the last sheet, large enough to encompass this assemble and placed behind them, solve the puzzle.



When the sheets have to be the same size, the minimum number of identical sheets you need is eight. The building block is the mutually overlapping ‘plus sign’ shape, shown below, made out of four sheets of paper. Add four more sheets for the corners.



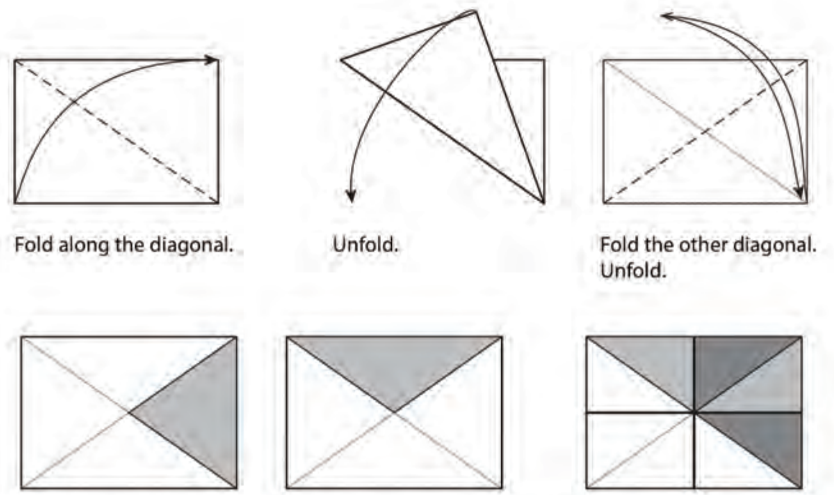
Four sheets to cover each other.



Four more underneath assemble the background square.

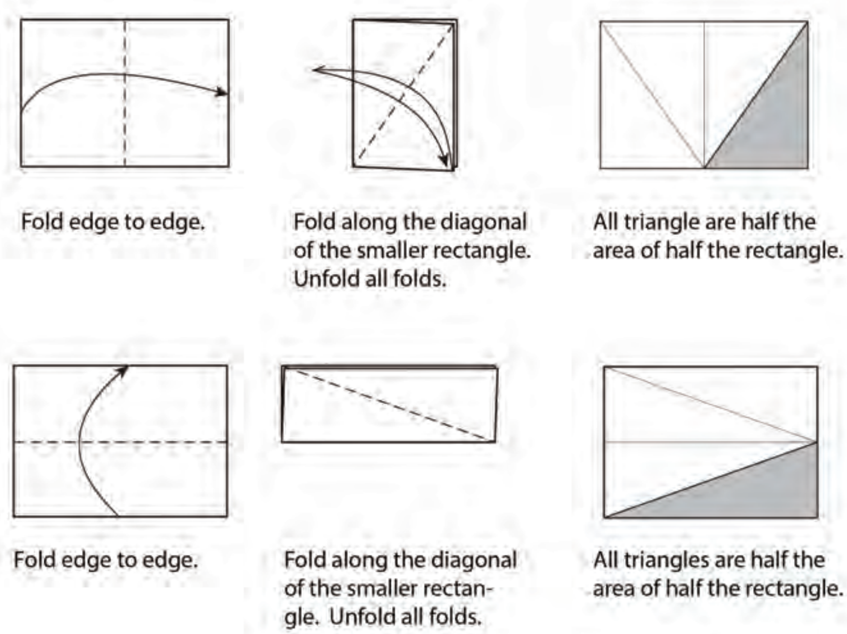
Quadrisecting Rectangles into Triangles Puzzle Solution

There are six different triangles. The first two emerge when you fold the two diagonals:

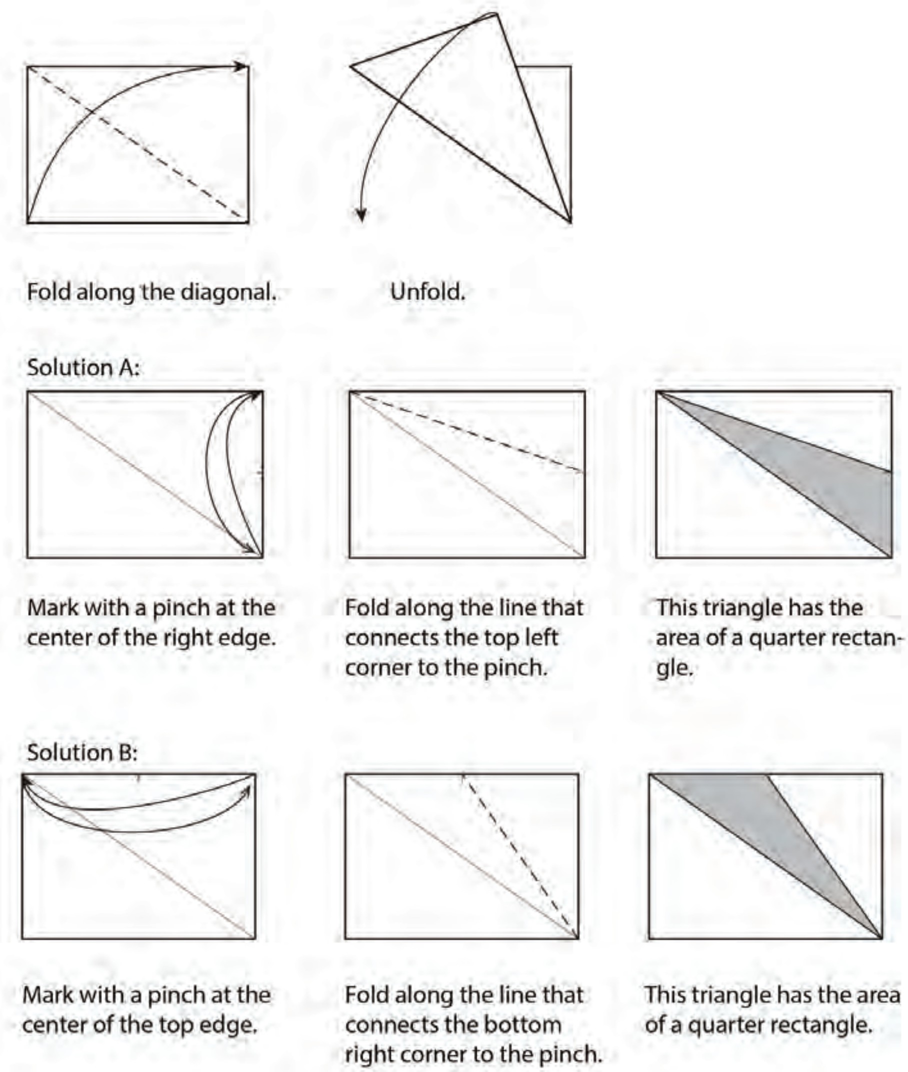


These two triangles are equal in area, and together they are half the area of the rectangle. A simple way to show this is to divide the rectangle into four smaller ones. You can see that each triangle is divided into two triangles and all are half of the quarter rectangle.

By folding the paper in half, you get two more triangles:



The last two triangles are shown below:



The explanation is based on the way you calculate the area of an triangle. You can see that both triangles in the half-rectangle triangle have the same base length and the same height, hence the same area.

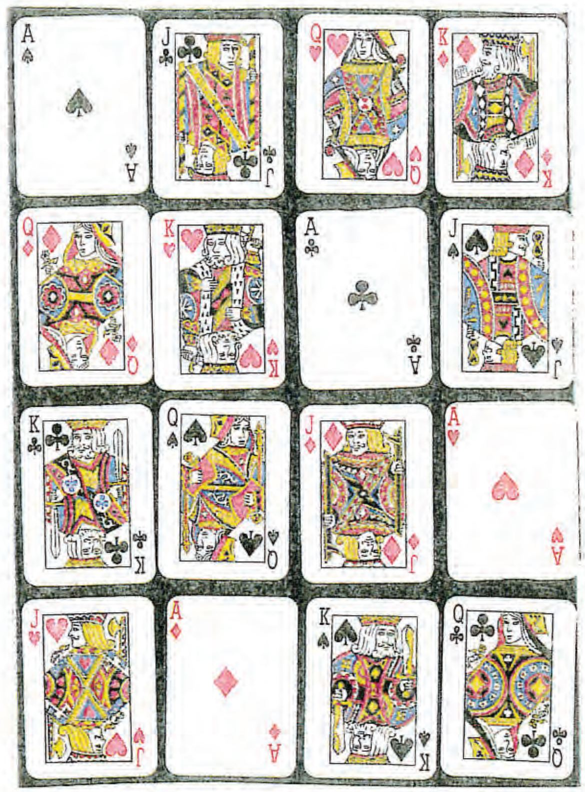
EULER ENTERTAINMENTS

By Jeremiah and Karen Farrell

Martin Gardner's article "Euler's Spoilers: The Discovery of an Order-10 Graeco-Latin Square" Chapter 14 in his book *New Mathematical Diversions from Scientific American*, Simon & Schuster, 1966 reports on a certain puzzle traced back to 1624 in a book by Claude Gaspar Bachet:

> Arrange the sixteen highest playing cards so that no value or suit appears twice in any row, column or the two main diagonals. To Euler the values are Graeco and the suits Latin. Now days usually abbreviated to simply Latin squares.

One possible solution (of 144) follows.

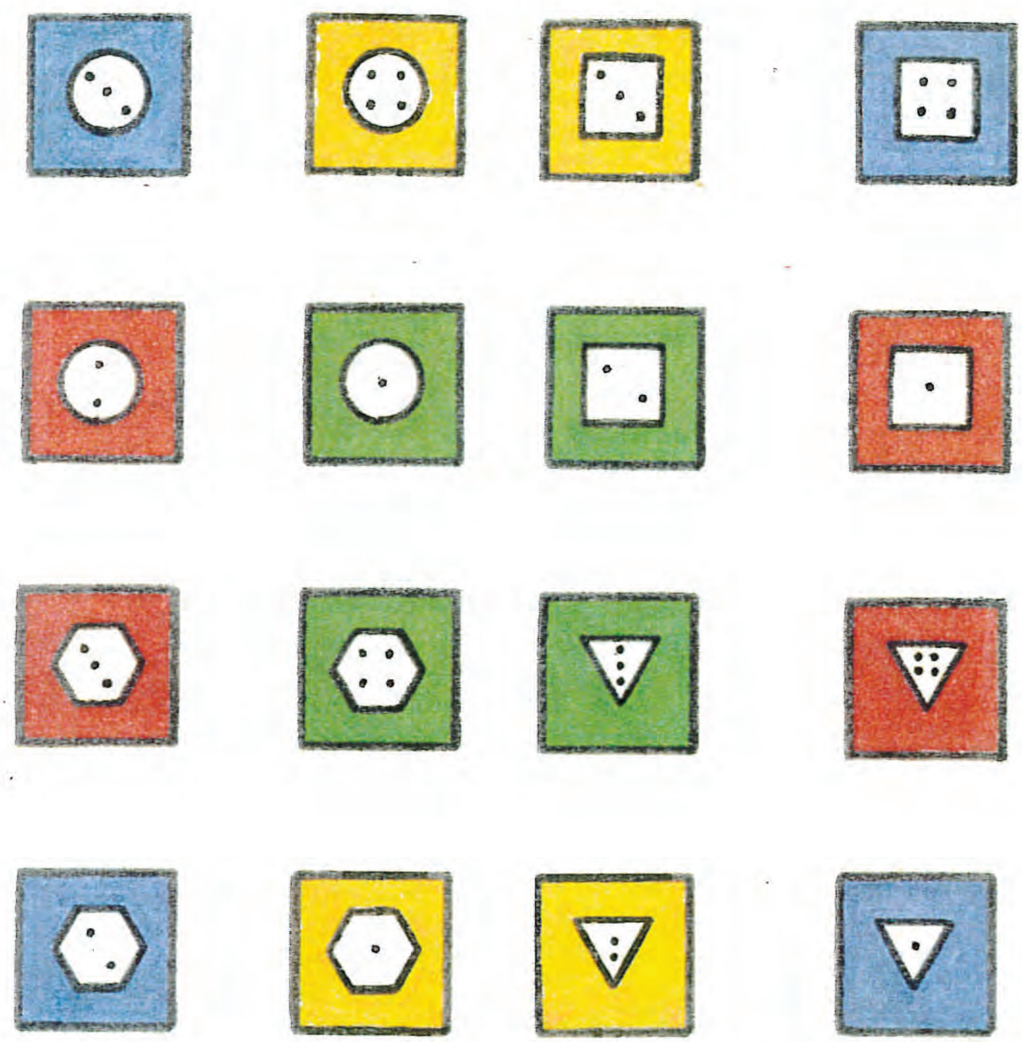


This square can be made magic with constant 34 by, say, labeling the values 1, 2, 3 and 4 and the suits 0, 4, 8 and 12 and then adding the pairs in each entry.

The two sets of four are what Leonard Euler (1707-1783) in the last years of his life called mutually orthogonal squares or 4x4 squares that could be superimposed so no two entries were duplicated. Euler tried to find such n x n Latin squares for all n. Euler knew that such squares exist if n is odd or if n = 4k but conjectured that no solution exists for n

= 4k+2. In 1901 Gaston Tarry published a proof no solution exists for n = 6, but E. T. Parker, R. C. Bose, and S. S. Shrikhand proved that Euler was wrong and n = 6 is the only exception. *Scientific American's* cover for November 1959 had staff artist Emi Kasdi's depiction of two order n = 10 Graeco-Latin squares.

Much more can be said about mutually orthogonal squares. Here is a list of sixteen entries combining three 4x4s using number, shape and color.



The reader is asked as a puzzle to arrange the pieces in a 4x4 square so that every row and column has no duplicated symbols. Euler called such squares semimagic and for order 4, three is the maximum number of mutually orthogonal squares possible.

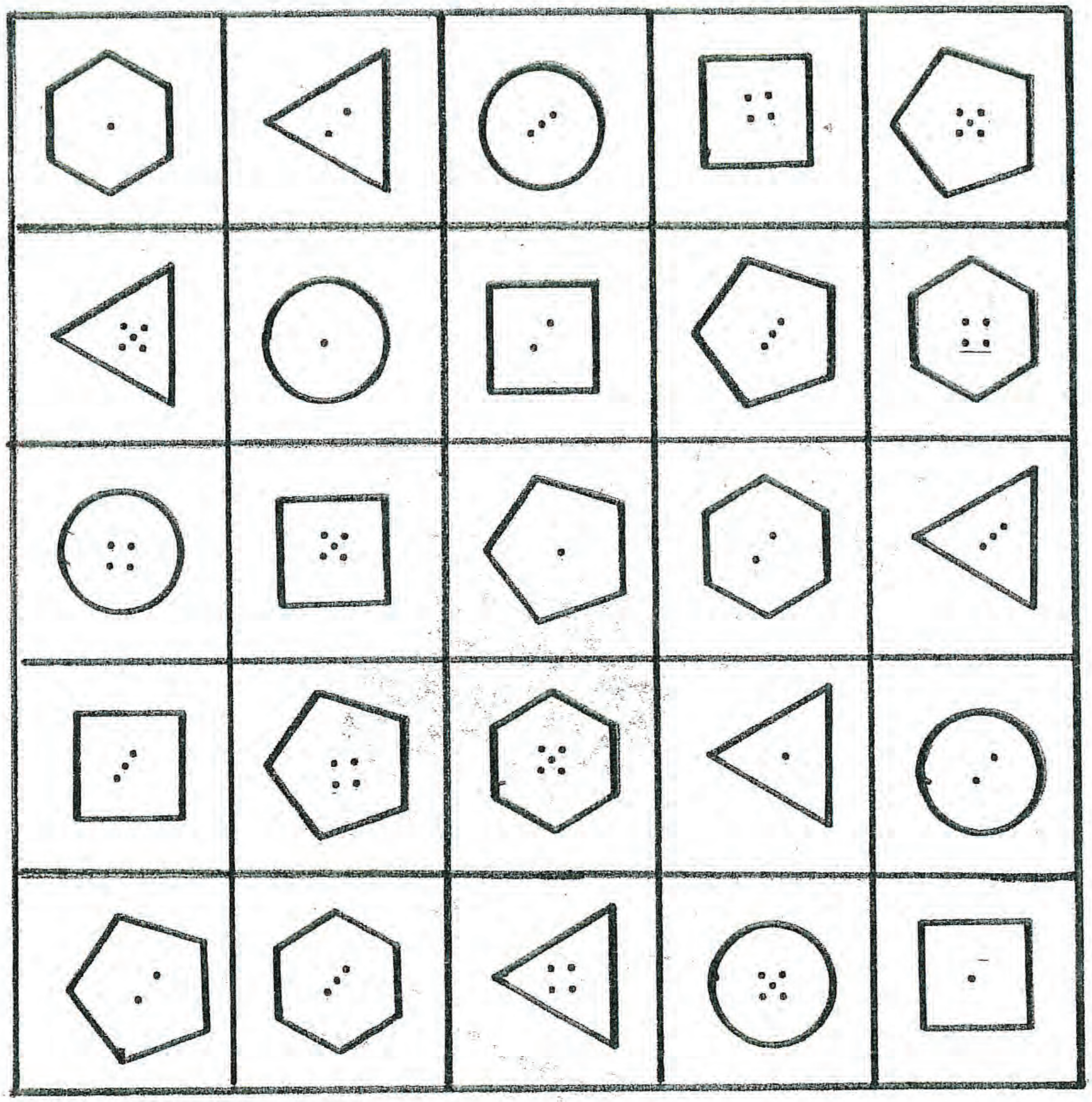
In fact for order n the maximum number of squares can only be n-1. This is not always obtained and to date it is not known for which orders this maximum is obtained.

For order 5 there can be the maximum 4 and it is possible to construct two puzzle-games using some of the possibilities. With the two such orthogonal pairs of number and shape shown, mix the 25 tokens face-down and each of two players draws 10 tokens. They will alternately place a token on a 5x5 board under one of the following two rules.
(1) No two tokens have a symbol in common in any row or column.
(2) (Cut-throat) Same as (1) but in addition no two tokens in any diagonal, broken or not, can have a symbol in common.

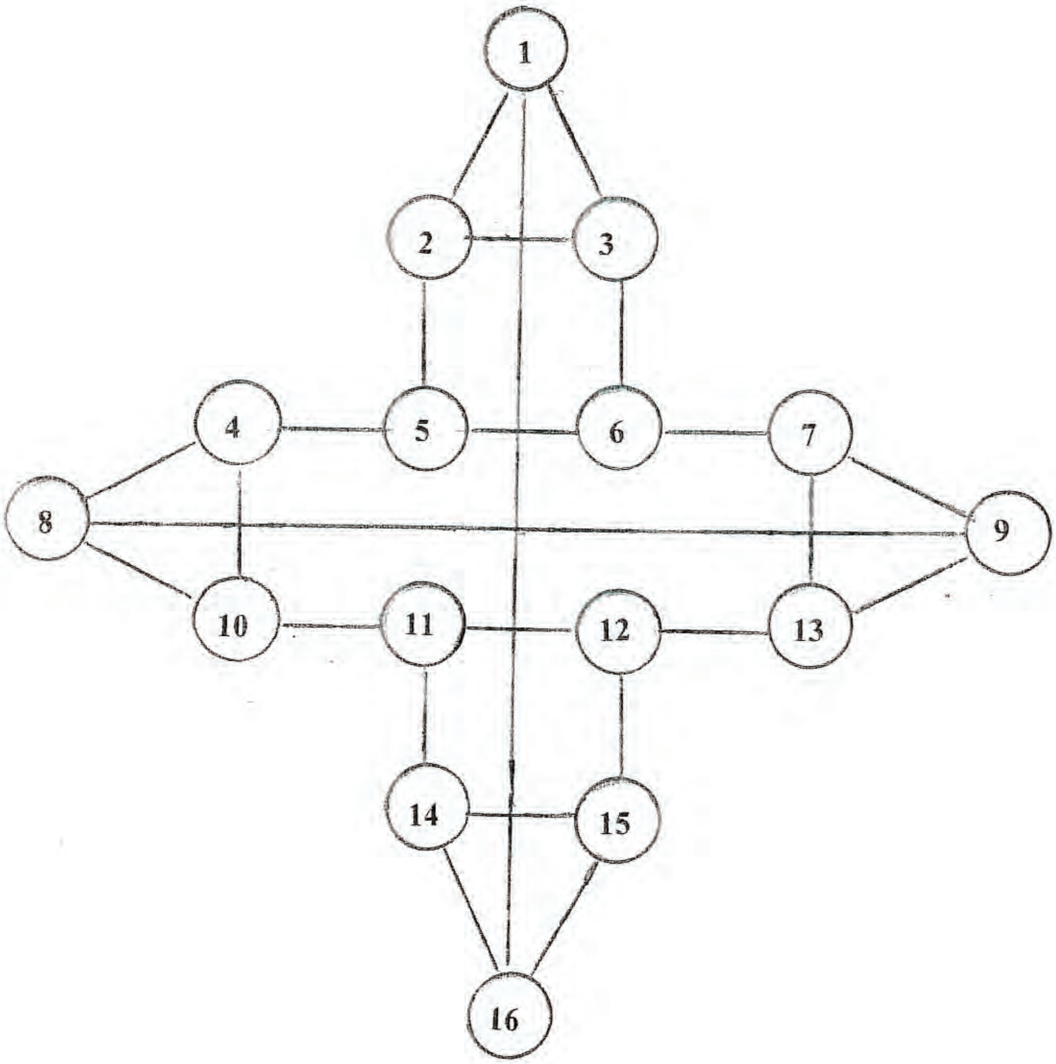
In either case, players can draw from the remaining pieces in the "kitty" if they cannot place one of their own. The onus is always on the second player to note misplays by the first player. The last player to be able to play wins.

There are 10 diagonals in any 5x5 and version (2) of the games will allow two more mutual orthogonals to reach the maximum four. Once again (2) will yield an ordinary magic square with constant 65 by labeling the shapes 0, 5, 10, 15, and 20 then adding the numbers to them in each entry.

Some solutions follow at the end of this article.



There are many more puzzles using the three order 4 tokens. For instance mathematicians call graphs in which each node has three edges on it Cubic Graphs. For 16 nodes 4207 non-isomorphic (essentially different) cubic graphs exist. Although the graphs are fundamentally different, the 16 tokens are rich enough to form somewhat challenging puzzles. In each of the following graphs place the tokens so that any connected tokens have a symbol in common. This is called a "hit" puzzle. It may also be possible in most cases to form a solution where connected tokens have no symbol in common, a "keepaway" puzzle.



THE CROSS

Cubic "Hit" Graph

Figure 1

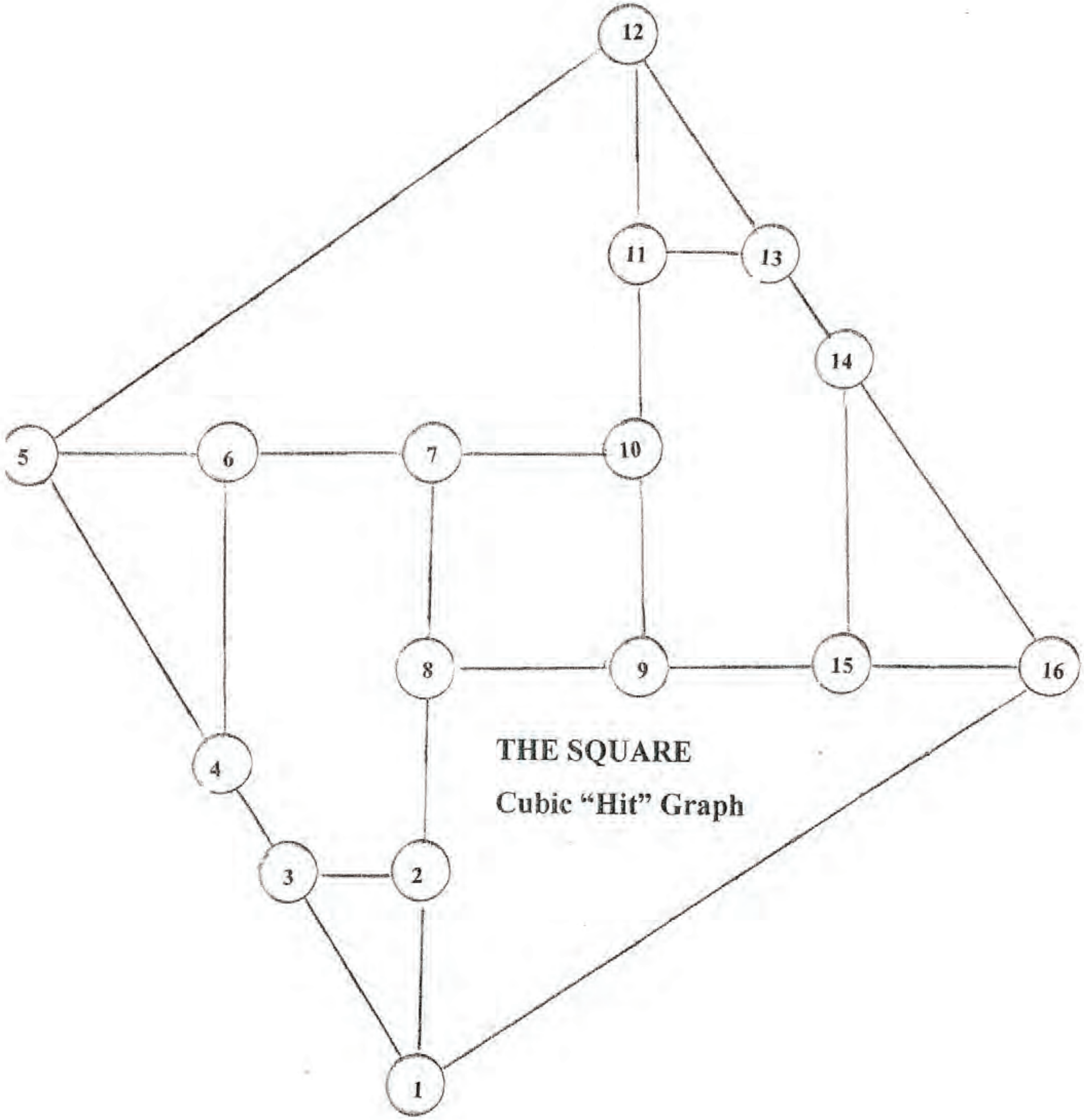


Figure 2

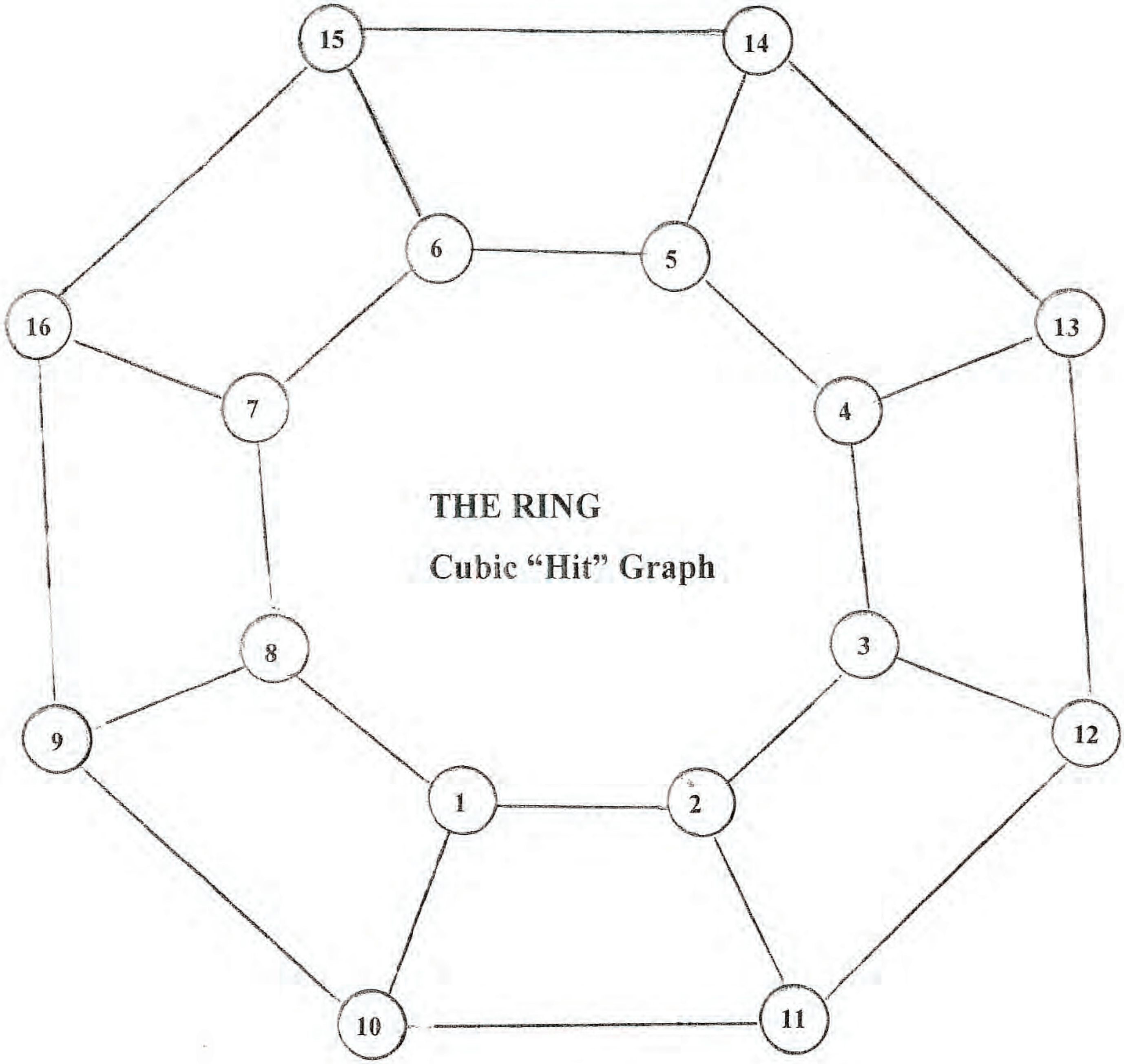
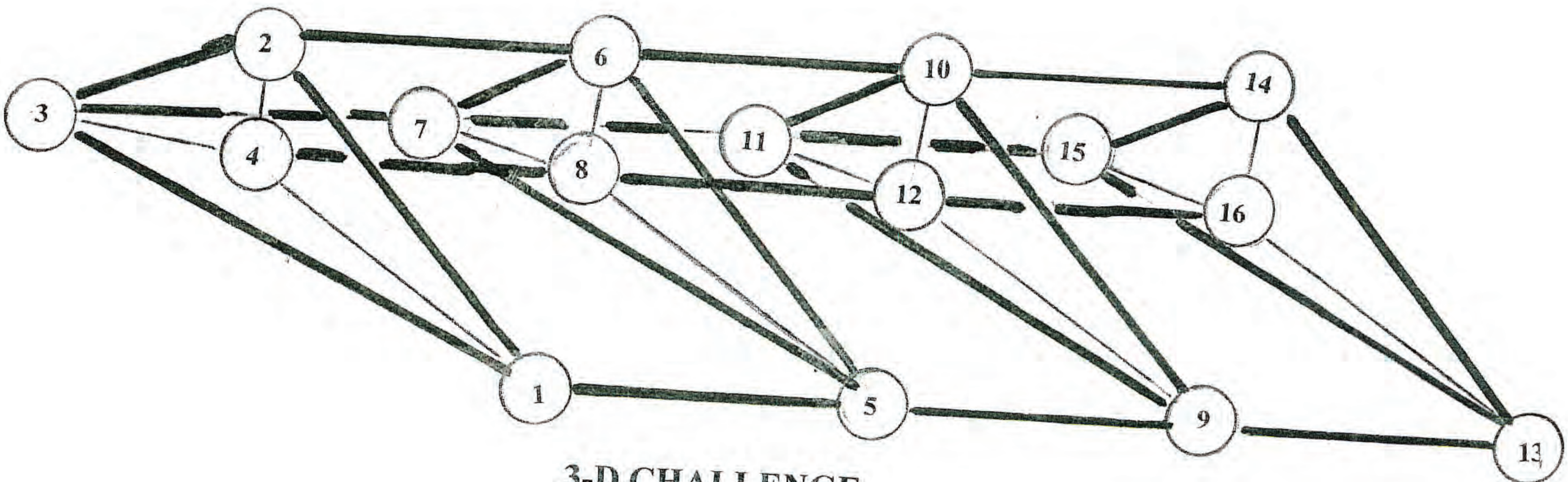


Figure 3



3-D CHALLENGE

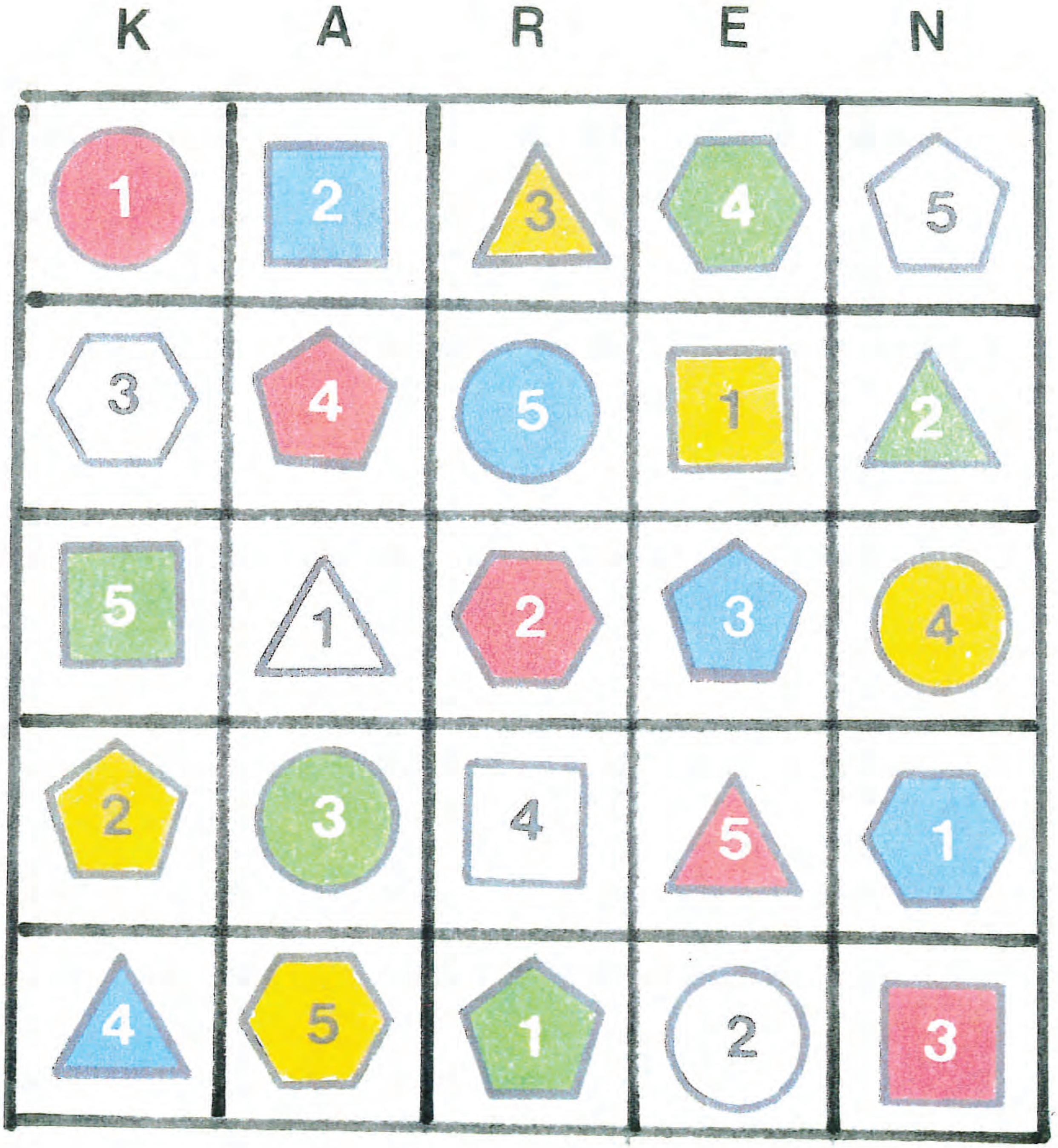
Play "Keep-Away" on

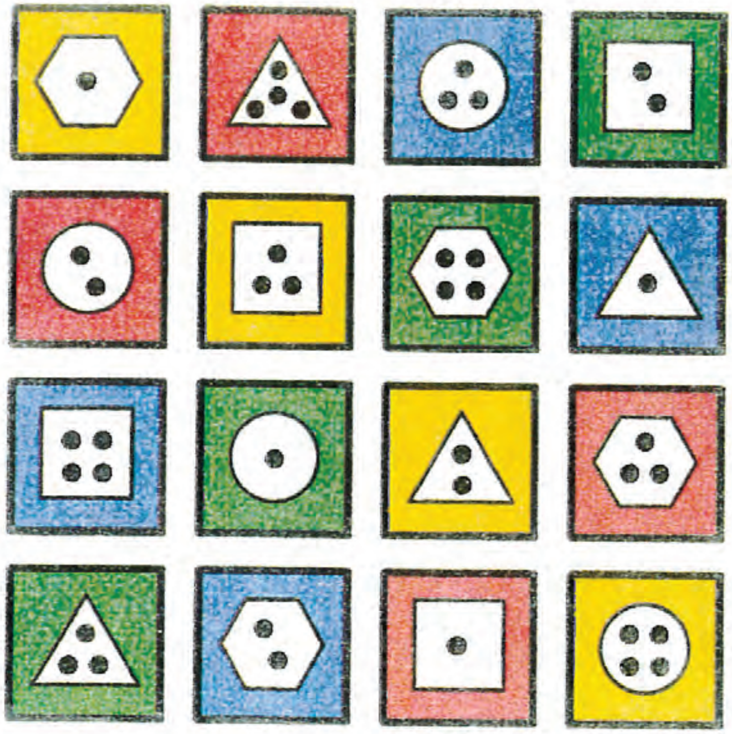
All Lines

Figure 4

ANSWERS

We define a knight sweep as chess knight moves from a corner of a square and the center of a 5x5. Trace for example in the playing card example how knight sweeps for the values and the suits evolve. In our other square examples likewise. For the 5x5 solution note the five colored diagonals. The five opposite diagonals can be labeled with the letters of KAREN.



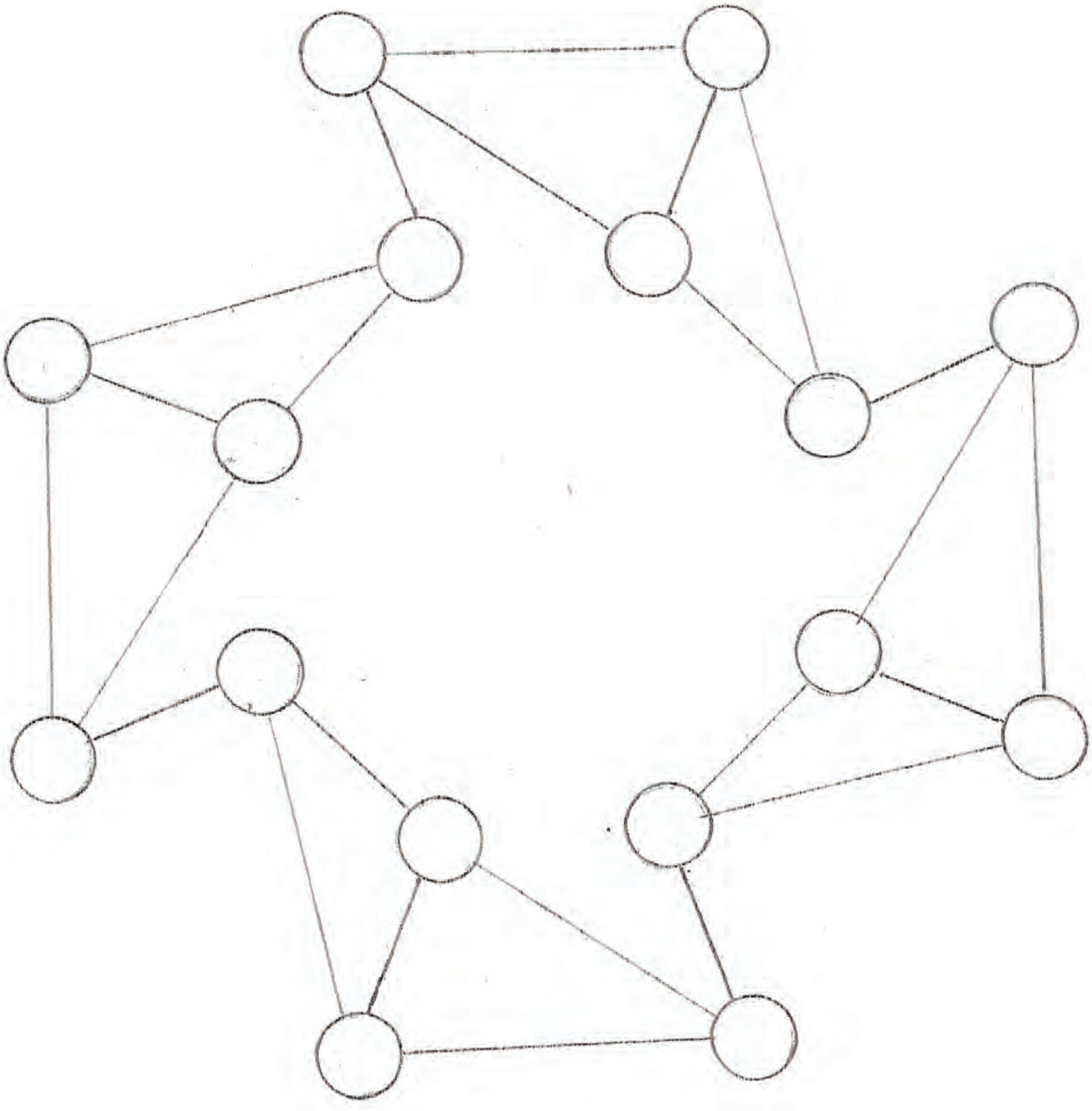


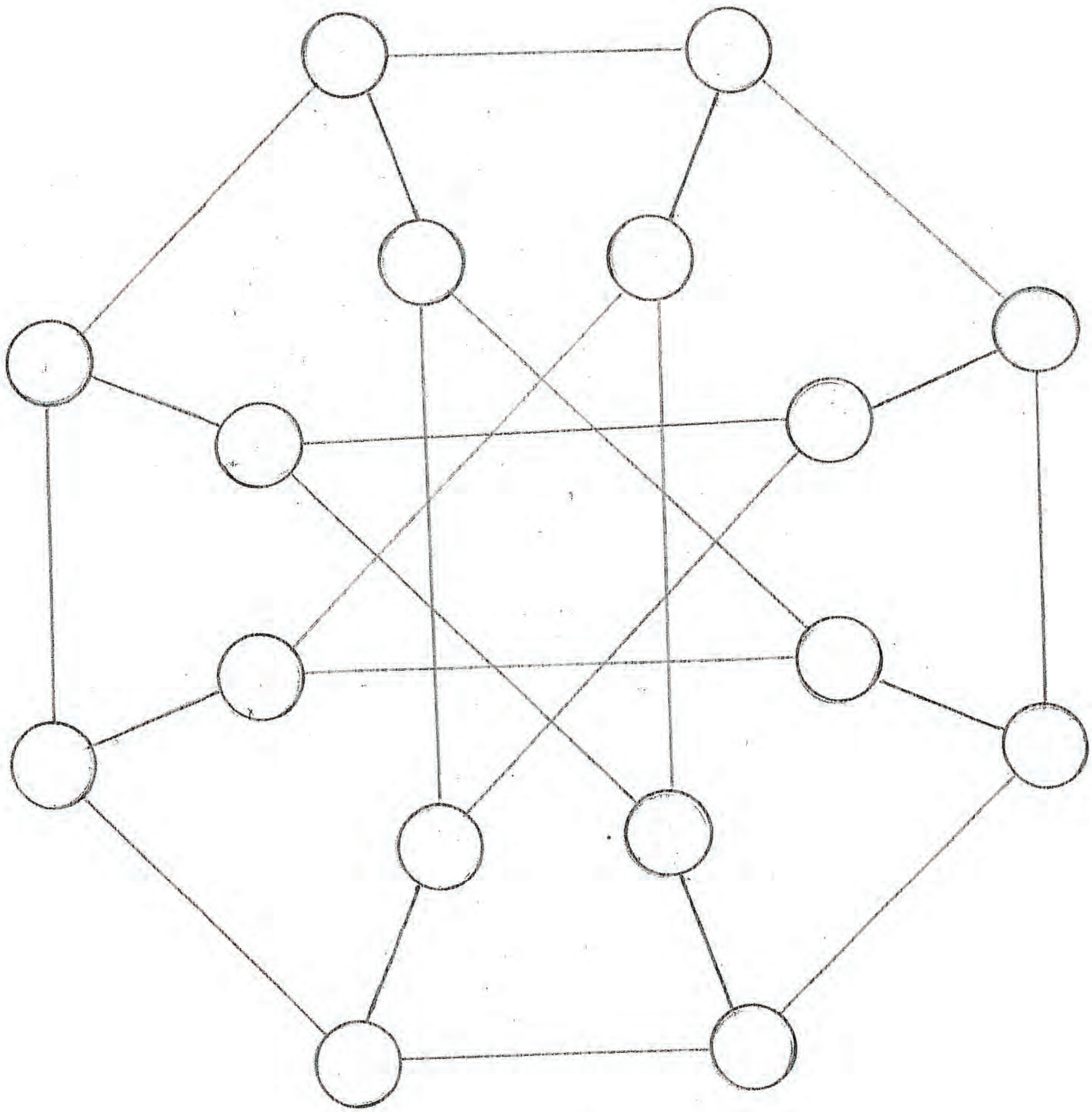
Figures

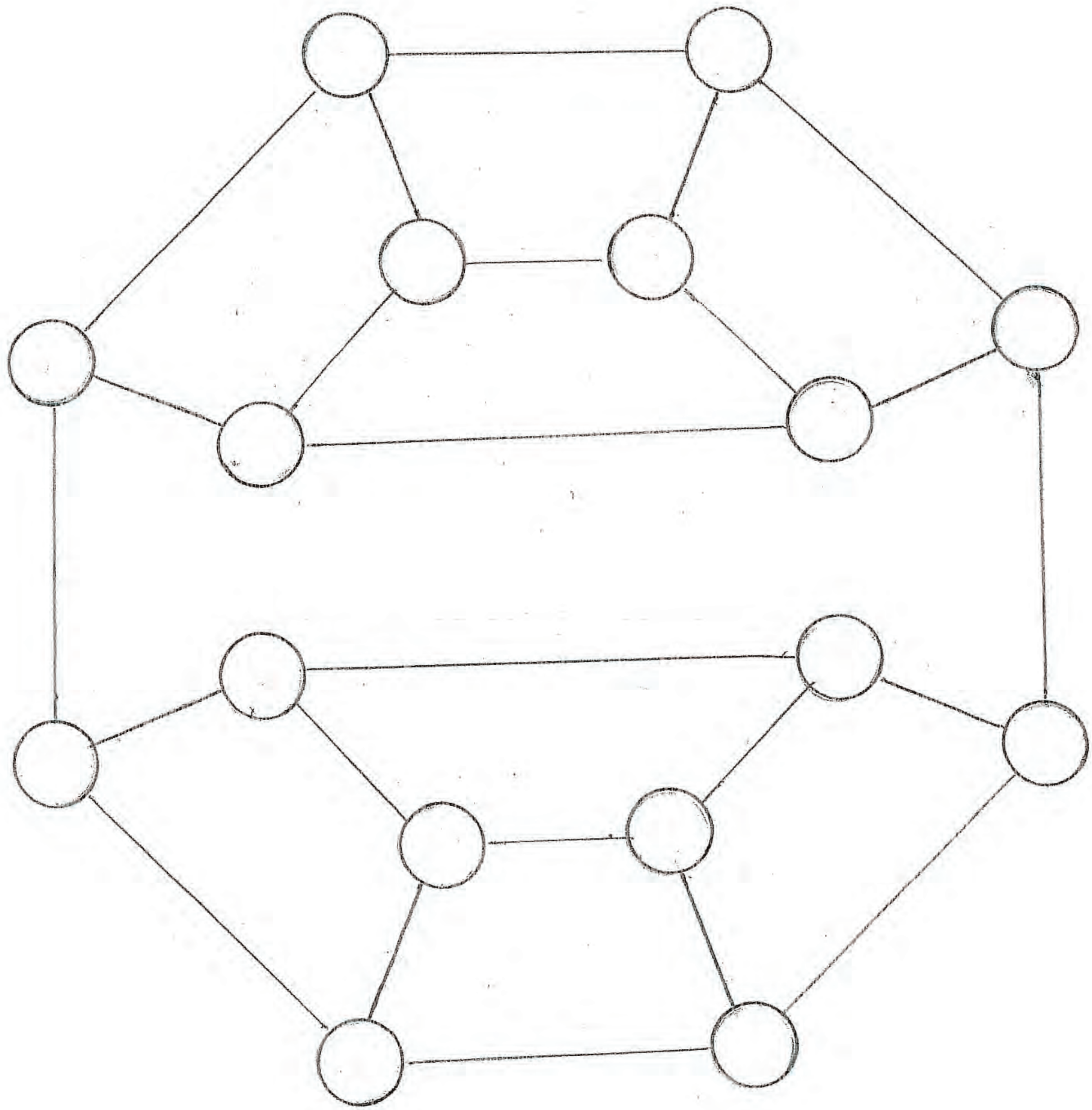
(1)	4	3	14	7	13	2	15	10
(2)	10	15	12	9	8	6	1	7
(3)	3	8	7	4	10	15	14	11
(4)	3	6	13	12	5	4	11	14

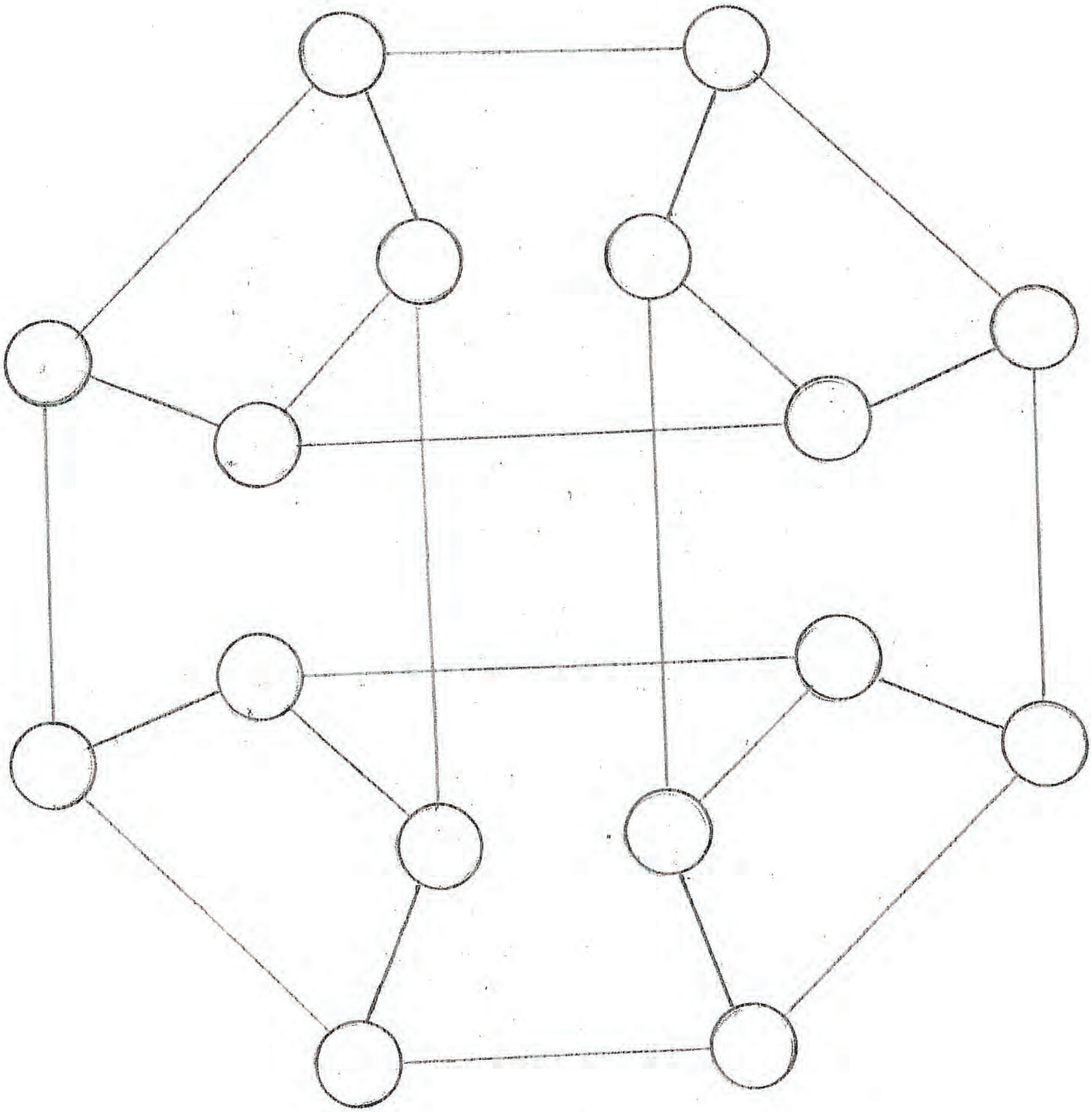
(1)	8	12	11	9	16	6	5	1
(2)	4	3	5	2	11	13	16	14
(3)	12	9	16	13	2	5	6	1
(4)	9	16	7	2	8	1	10	15

More cubic 16 puzzles or two person games. Either "HIT" or "KEEPAWAY"









A

GATHERING FOR GARDNER

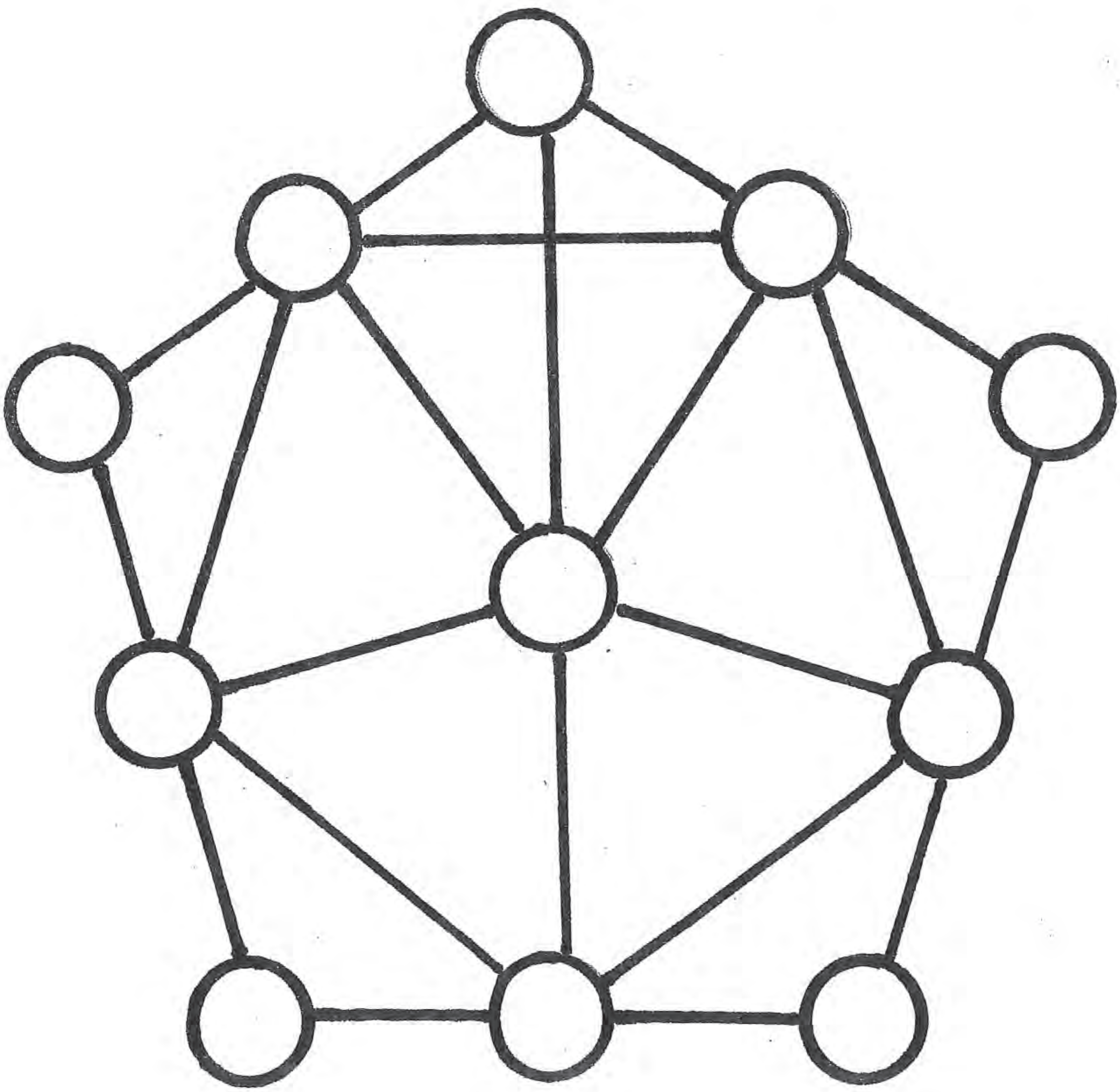
Puzzle-Game

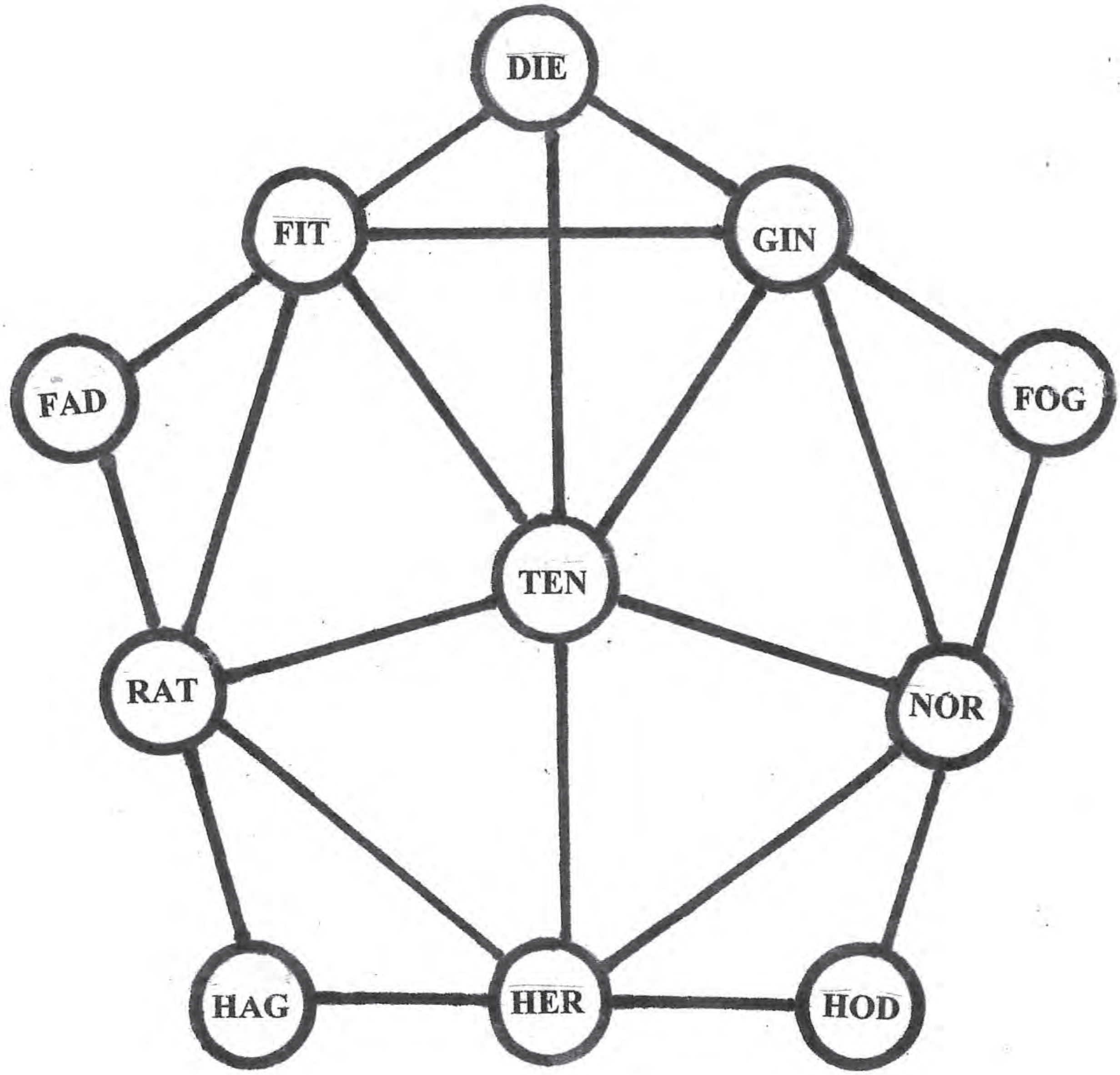
By Jeremiah Farrell and Chris Morgan

Each different letter of "GATHERING FOR GARDNER" is used exactly three times in the following words: DIE, FAD, FIT, FOG, GIN, HAG, HER, HOD, NOR, RAT, TEN.

The Puzzle. Place the words on the nodes so that every connected node has a letter in common.

The Game. The players alternately place their distinctive tokens on the nodes of a completed puzzle. The first player to cover three words with a common letter is declared the winner.




DIE
FIT
GIN
FAD
FOG
TEN
RAT
NOR
HAG
HER
HOD

Balance Puzzles

You either love them or curse them

Paper for the souvenir book
by Rik van Grol

For G4G13
April 11 - 15, 2018

Balance Puzzles

You either love them or curse them

by Rik van Grol, NL
Rvgrol@hotmail.com

A well-known balance puzzle is the Columbus Egg*. The object is to balance the egg on its tip. Impossible so it seems! By using other senses than just vision, such as hearing and feeling, and by logical thinking, some patience and above all perseverance the solution can be found. Careful manipulation of the egg centralizes the weight of the egg and then it can be balanced on its tip. Many people lack some of the above mentioned qualities and will never solve these puzzles; consequently they dislike them or even curse them. Some even say that they are not puzzles. If they would experience solving them they would realize that these are indeed puzzles and they would love them.*

Introduction

What do I mean with a balance puzzle? When you google for the term "balance puzzle" the search results do not lead you towards the puzzles I mean. The balance puzzle, or weighing puzzle, google provides is a logic puzzle about balancing items, often coins. Other balance puzzles will also appear. To find the puzzles I mean you should google for the term "*Columbus Egg puzzle*". Many of the balance puzzles this paper is about are egg-like objects, but not all of them.

So what is a balance puzzle? Best is to take the egg balance puzzle as an example. The object of the egg balance puzzle is to put the egg on its tip. With a regular egg this will result in the egg tipping over (not always – see the tale about Columbus), but with the egg balance puzzle there is a way to manipulate the egg in such a way that it will indeed stand on its tip.

A famous example is the *Columbus Egg* presented at the World Fair in Chicago in 1893. This metallic souvenir egg contains a ball that can be manoeuvred in such a way that it falls down a tube and ends up in the tip of the egg on which it can then be stood upright.

So, what are typical properties of a balance puzzle?

- They are single piece puzzles in that they are not meant to be taken apart.
- The puzzle needs to be manipulated in such a way that something internally is changed in order for the object to be balanced.
- Most balance puzzle do not have handles or levers.
- They can only be manipulated in a 3D-space: e.g. lifted, tilted, rotated, spinned.
- Most balance puzzles provide no clues as to whether or not you are heading in the right direction to solve it.
- With these puzzles you are, as it were, "left in the dark".

These properties, that most balance puzzles have, are exactly the properties that make you either love these puzzles or curse them. To solve a balance puzzle you need to use other senses than with most mechanical puzzles. Instead of depending on visual clues you now depend on sound, feeling and your creative ability to crawl into the mind of the designer. Many people, and also puzzlers, do not like to be left in the dark. Balance puzzles can be very frustrating, and unlike secret opening boxes (that share similar properties) they generally lack a satisfying AHA feeling. Personally I am in-between love and hate. I hate balance puzzles until I have solved them, then I love them...

In this paper I will start with some anecdotal history about egg balance puzzles. This will be followed by an overview of balance puzzles. Then I will talk about the different types of mechanisms used in balance puzzles, and how to solve them.

Afterwards you can decide for yourself whether to like or to curse them...

Anecdotal history

The oldest egg balance puzzle I know of is the *Columbus Egg*, see Figure 1. I am sure there must have been earlier puzzles, but not produced in the quantity as this one. I also do not have any record of other such puzzles from an earlier age. If a reader does, I would be very much interested.

The *Columbus Egg* was presented at the World Fair in Chicago in 1893 as a souvenir. Wikipedia says the following about this event [1]:

> *The World's Columbian Exposition (the official shortened name for the World's Fair: Columbian Exposition also known as the Chicago World's Fair and Chicago Columbian Exposition) was a world's fair held in Chicago in 1893 to celebrate the 400th anniversary of Christopher Columbus's arrival in the New World in 1492.*

Wikipedia also mentions an “Egg of Columbus” in relation to the World Fair, but this is not our puzzle egg. The mentioned egg is a metal egg that spun inside an electric field. Quite a novelty at that time… I have found no record of our puzzle egg being mentioned in relation to the World Fair, but it must be strongly related to the same celebration as the puzzle depicts Columbus and the period 1492-1892. In *The Book of Ingenious & Diabolical Puzzles* [2] Jerry Slocum mentions the Egg of Columbus as made for the Columbian Exposition. Professor Hoffmann in *Puzzles Old and New* [3] also mentions a *New Egg of Columbus*, but this is not our egg.

So, there are quite a few Columbus Eggs... What is it about Columbus and eggs anyway? This has to do with a tale, and the clue of the tale is related to solving these puzzles. It is a tale from (according to Wikipedia) the historian Girolamo Benzoni, who wrote [4]:

> *Columbus being at a party with many noble Spaniards, where, as was customary, the subject of conversation was the Indies: one of them undertook to say: "Mr. Christopher, even if you had not found the Indies, we should not have been devoid of a man who would have attempted the same that you did, here in our own country of Spain, as it is full of great men clever in cosmography and literature." Columbus said nothing in answer to these words, but having desired an egg to be brought to him, he placed it on the table saying: "Gentlemen, I will lay a wager with any of you, that you will not make this egg stand up as I will, naked and without anything at all." They all tried, and no one succeeded in making it stand up. When the egg came round to the hands of Columbus, by beating it down on the table he fixed it, having thus crushed a little of one end; wherefore all remained confused, understanding what he would have said: that after the deed is done, everybody knows how to do it; that they ought first to have sought for the Indies, and not laugh at him who had sought for it first, while they for some time had been laughing, and wondered at it as an impossibility.*

This out-of-the-box-thinking is exactly what is needed for solving many puzzles, especially these puzzle eggs. Quite often, maybe even always, egg balance puzzles are categorized under dexterity puzzles. And, yes, they certainly need some dexterity for solving, but by applying logic, deduction, creative thinking it can become much more a puzzle that can be solved at will.

Overview of balance puzzles

An small overview of some balance puzzles is available on the internet, on Rob’s Puzzle pages [5]. It starts off with the *1893 Columbus Egg puzzle* but also shows several of the others presented below. Partly due to a link on his pages I was led to a number of patents on egg balance puzzles. The U.S. Patent Office devotes an entire sub-class to "Balancing Ovoids" (ccl/273/154). Most of the patents are from the time around the *World's Columbian Exposition* in 1893. Rob’s pages also demonstrated that there are several variants of the *1893 Columbus Egg puzzle* – some with an inscription: “World's Fair Souvenir”. My copy of *Columbus Egg* does not show these words.

No.	Name	Date	Balance object	Solution type	Puzzle / trick	Description
1	*Columbus Egg Puzzle*	1893	ball	logic & path	puzzle	Metal egg shaped balance puzzle. Internally there is a ramp that will let the ball fall into a tube towards the tip. Made by P.M. Baumgardner & Co, USA. Puzzle in my possession.
2	*Fall Guy*	1951	sand	trick	trick	This is a small trick in the shape of a man. This is more a magic trick than a puzzle. After holding it upside-down for about 20 seconds you can turn it around and balance it on its feet. Then after about 15 seconds it will fall over. From *Four Guys Products Inc.* , New York , USA. Puzzle from the Lilly Library, Bloomington, IN, USA.
3	*Magic Egg Puzzle*	?	ball	logic & path	puzzle	Plastic egg-shaped balance puzzle. Internally there is a ramp that will lead the ball to a central resting position, which will allow you to balance the egg. Patent US 1,763,814. Puzzle from the Lilly Library, Bloomington, IN, USA.
4	*Magic Egg*	?	sand	trick	trick	A.k.a. L'Oeuf Enchanté, Trick-ei, "Ei des Columbus". This is an egg-shaped balance puzzle from Pussy, Germany. This is more a magic trick than a puzzle. After holding it tip-side-up for 20-25 seconds you can turn it around and balance it on its tip. Art-Nr. 80 2100. Then after about 10 seconds it will fall over. Puzzle in my possession.
5	*The Trick*	?	sand	trick	trick	This is a trick in the shape of a doll. This is more a magic trick than a puzzle. After holding it upside-down for a while you can turn it over and balance it, head in the air. Then after a while it will fall over. From TOBAR Norfolk UK. Puzzle from James Dalgety.
6	No name	?	ball	logic & path	puzzle	Yellow-red plastic egg-shaped balance puzzle. Internally there is a trench that will lead the ball to a central resting position which will allow you to balance the egg. Origin unknown. Puzzle in my possession.
7	*The X super puzzle*	1984	n.a.	twisty	puzzle	A.k.a. Columbus puzzle(?). This is an egg shaped balance puzzle, but unlike others it has two moving parts. The bottom half of the egg can rotate in relation to the top, and it has a sliding button with three positions. There is also a small window, behind which there are five disks visible that each can take ten positions. One of the ten positions shows red through the window. Sequences of rotations and different positions of the button are needed to get all five disks to show their red position. Then a weight is unlocked, which will allow you to balance the egg on its tip. Origin: Japan. Patented by Morichika Hatakeyama and Koichi Minami. Patent US 4,489,944. Puzzle from the Lilly Library, Bloomington, IN, USA.
8	*L'UOVO DI COLOMBO*	1990's	ball	trick	puzzle	Plastic egg-shaped balance puzzle. Internally there is a tube. No ramp or anything else to help. Object is to get the ball into the tube, which requires dexterity. Patented by Sileno Lavorini, Pat no. 0336/676303. Made in Italy. Puzzle in my possession.
9	*Tower of Pisa*	2000	sand	logic	puzzle	This wooden tower of Pisa is a slanted tube. Internally it has cavities with sand. By moving the sand around, balance can be reached. The puzzle also contains a ball as decoy. Exchange puzzle from Tatjana Matveeva (Russia) at IPP 20, in 2000, in LA, USA. Puzzle in my possession.
10	*Dice*	2002	ball	logic & path	puzzle	This is a die that is to be stood upright on a flattened corner. Internally there is a ball and a central stem on whch the ball can be positioned to balance the dice. Dexterity is expected, but with the right movement you always succeed. Exchange puzzle from Jacques Zeimet (France) at IPP 22, in 2002, in Antwerp, Belgium. Puzzle in my possession.
11	*Clock*	2008	ball	logic	puzzle	This is a disc that looks like a clock that is to be stood upright. Internally there is a ball, and several moving objects (the latter are fixed to their position). The ball must be moved between the objects to a location in order to balance the puzzle. Exchange puzzle from Jacques Zeimet (France) at IPP 28, in 2008, in Prague, Czech Republic. Puzzle in my possession.
12	*Rik's Egg Balance 2010*	2010	two balls	logic & path	puzzle	2D-egg-shaped balance puzzle. This is a flattened egg made of wood. Internally there is a ramp by which the two balls need to be transported to the tip, one at a time. Exchange puzzle from Rik van Grol (The Netherlands) at IPP 30, in 2010, in Osaka, Japan. Puzzle in my possession.
13	*Rik's Egg Balance 2011*	2011	ball	logic & path	puzzle	2D-egg-shaped balance puzzle. This is a flattened egg made of wood. Internally it is mostly empty. The provided stand needs to be used to release the ball such that the ball drops to the tip. Solution is symbolically depicted on the outside of the puzzle. Exchange puzzle from Rik van Grol (The Netherlands) at IPP 31, in 2011, in Berlin, Germany. Puzzle in my possession.
14	*Rik's Egg Balance 2013*	2013	sand	logic & path	puzzle	2D-egg shaped balance puzzle. This is a flattened egg made of wood. Internally there are several chambers, partially filled with sand. The sand must be manipulated such that it is transported to the tip. Solution is symbolically depicted on the outside of the puzzle. Exchange puzzle from Rik van Grol (The Netherlands) at IPP 33, in 2013, in Narita, Japan. Puzzle in my possession.
15	*Ze Balancing Egg*	2015	ball	logic & dexterity	puzzle	Wooden egg-shaped balance puzzle. Traditional balance puzzle, but with a very subtle ramp. Exchange puzzle from Stephen Chin (Australia) at IPP 35, in 2015, in Ottawa, Canada. Puzzle in my possession.

1. ***Columbus Egg Puzzle***

2. ***Fall Guy***

3. ***Magic Egg Puzzle***

4. ***Magic Egg***

5. ***The Trick***

6. **(no name available)**

7. ***The X super puzzle***

8. ***L'UOVO DI COLOMBO***

9. ***Tower of Pisa***

10. ***Dice***

11. ***Clock***

12. ***Rik's Egg Balance 2010***

13. ***Rik's Egg Balance 2010***

14. ***Rik's Egg Balance 2010***

15. ***Ze Balancing Egg***

Mechanisms of balance puzzles

The mechanisms of balance puzzles and their solution type are closely related. In the table above different solution types are listed. The solution types relate directly to the method used to transfer the moving weight to a position where balance can be achieved. This is generally done by centralizing the weight. The solution types listed are:

- **Path** — With path I mean that there is a distinct place or position to start (to position the weight) and then there is a specific path or movement to make, after which the weight is centralized and the egg can be balanced.
- **Trick** — The eggs classified as trick are not puzzles that you can solve. They are more attributes of a magician. A magician will make you believe the “puzzle” can be solved (the egg can be stood upright), but this is an illusion. The object seems to balance indefinitely, but in reality it does so for only a short period of say ten seconds or so. Before the balance is lost, the magician will pick up the egg and hand it to the audience. The audience will try and fail. The trick-eggs, in my view, do not actually belong in this list as they are not puzzles, but I keep them in to show the contrast with real balance puzzles.
- **Twisty** — this relates to the fact that the puzzle itself can be altered, by twisting or shifting. Most balance puzzles have moving parts, but only internally and they cannot be controlled directly. With twisty puzzles you do have direct control, turning, shifting or otherwise altering the puzzle.
- **Logic** — A balance puzzle generally leaves you in the dark as to what can or needs to be done, but observation (feeling and hearing) combined with creativity and logic may and/or will help you find a solution. Logic is generally combined with **path** or **dexterity**.
- **Dexterity** — with dexterity the manipulation of the object —the egg— is meant: tilting, shifting, flipping, etc. When a puzzle has a high level of dexterity, it may take a lot of practise.

Most balance puzzle are characterized by combinations of the above.

Relating dexterity a further deliberation is required. Almost every puzzle requires a level of dexterity. Personally I would classify a puzzle as a dexterity puzzle if you fail more often than you succeed and if you cannot use logic to turn the odds.

Solving balance puzzles

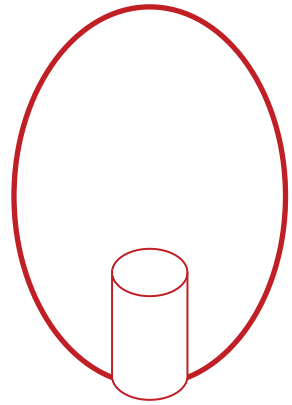
Solving a balance puzzle starts by investigating the puzzle. When you solve any puzzle you start by making some observations. In the case of balance puzzles your eyes are not given a lot of clues, so you need to rely on your other senses, mostly hearing and feeling, as smelling and tasting generally do not really help with puzzles...

At this stage the object is to determine the type of balance puzzle. Based on the suspected mechanism, or combinations of mechanisms, you start looking for further clues. If it is a well-known mechanism the task may be relatively straightforward, but if it is new then the problem is much more difficult. You need to imagine a new mechanism and “look” for clues. Looking in this context is, again, mostly feeling and hearing. This is the part that can be really satisfying or extremely frustrating. Satisfying if your suspicion was right and you find the path or logic and solve the puzzle. Frustrating if you cannot find the path, cannot imagine the new mechanism, cannot explain what is happening.

At this point two other qualities enter the equation: patience and perseverance. Admittedly, I do not always have enough of these qualities to solve a puzzle by myself. Let me give you two examples: one with some success and one with defeat.

L’UOVO DI COLOMBO – puzzle #8
When I purchased this puzzle and received it, I was very disappointed. I felt cheated. This was an impossible puzzle. Just a hollow egg with a tube in the tip and a ball. The only way to balance it would be to repeatedly flip the egg and to try and catch the egg with the tube – virtually impossible. I did manage to do it once or twice, but I could not deliberately repeat it. For years I cursed this “puzzle”, I didn’t consider it as a valid puzzle. Only recently while i was writing this article I got an

insight. I was trying to prove to myself that this indeed was not a valid puzzle, that it only is a game of chance; that no logic would help solving this puzzle. Then it hit me: this puzzle can be solved with logic. I should hold the puzzle upside down with the tube straight above the ball, than the only thing needed would be to move the tube down quicker than gravity, to catch the ball. After I got this idea I stood up tried it once and failed; tried it another time….success! Right now I cannot deliberately repeat it, but I demonstrated to myself that logic actually helped me to balance this egg. Still, the level of dexterity of this puzzle is very high, which still makes me qualify this as a bad puzzle. Or should I believe in a better solution and persevere?



Ze Balancing Egg – puzzle #15
This egg was a mystery to me. I initially thought this was a traditional balancing puzzle with some groove and a volcano to centralize the ball. You can feel the base of the volcano because the ball circles around it, but I could not feel any sign of a groove. I felt cheated, like with *L'UOVO DI COLOMBO*. I basically had given up, but in the back of my mind I thought this could not be true. It is an IPP puzzle, so there should be a solution. This puzzle was from IPP 35 in 2015, but the souvenir book of that IPP has not been distributed yet. So I contacted the organisers and asked for the solution. After I saw the solution it was still a challenge. My original suspicion was correct – it is a more or less traditional balancing egg. The groove is very hard to "feel". Thanks to a marking on the outside of the egg —very tiny and easily mistaken for random damage— I finally found the groove. But then unlike the traditional balance puzzle you are supposed to flip the ball into the volcano. Initially this disappointed me, but after a bit of thinking I found out that it should not be a "flip", but just a vertical toss while turning the egg upright. Almost always, but at least one out of two tries I succeed in solving the puzzle. Love it!

Balance puzzles — you either love them or hate them

I have read comments on the Internet from people talking about balance puzzles. They argue that balance puzzles are not really puzzles, but dexterity games. This suggests that solving balance puzzles requires mainly dexterity and no logical thinking. I hope to have shown that most balance puzzles do require logic and creative thinking (out-of-the-box thinking). The main difference is that you need to rely on sound and feeling instead of sight. So, balance puzzles can be a lot of fun and very satisfying once you have cracked the solution. Otherwise you will probably curse them and try to avoid them.

Acknowledgements

Thanks go to the Lilly Library in Bloomington, Indiana, USA where I have studied several of the balance puzzles. Thanks also go to James Dalgety. I found and studied several balance puzzles at the puzzle museum at his house. Thanks also go to Joop van der Vaart who has helped eliminating errors in this manuscript.

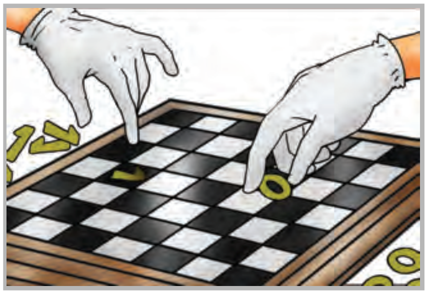
References

[1] https://en.wikipedia.org/wiki/World%27s_Columbian_Exposition.
[2] Jerry Slocum, *The Book of Ingenious & Diabolical Puzzles*, 1994, ISBN 0-8129-2153-4, p. 130.
[3] Professor Hoffmann, *Puzzles Old and New*, 1893, Chapter II, LII.
[4] Girolamo Benzoni, *History of the New World*, Venice, 1565, p. 12.
[5] http://robspuzzlepage.com/dexterity.htm.

FIVE PROBLEMS

emrehan@halici.com.tr

These problems are chosen from puzzleup.com 2017
(weekly puzzle competition prepared by Emrehan Halici)



01 - ODD-EVEN

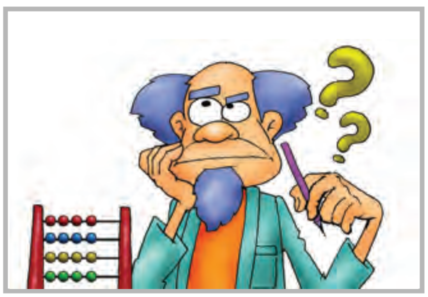
You will play a game, where you place "0" and your friend places "1" on a 7x7 chess board. You will start the game and both of you will place numbers on the empty squares alternatively. When the entire board is full, sums of the numbers on each row and column are noted. Among the 14 sums, you will get a point for each odd sum and your friend will get a point for each even sum.

If both you and your friend play perfectly, what is the maximum points you can get?

02 - LOTTERY

In a lottery, every week 5 different numbers are randomly drawn from numbers between 1 and 30 (including 1 and 30).

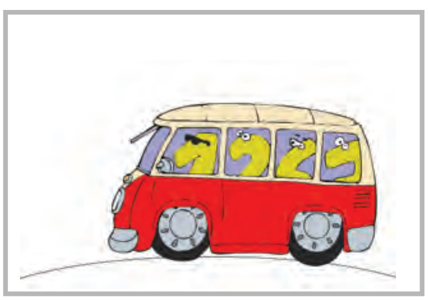
What is the probability of the 3 smallest numbers drawn this week being the same with the 3 smallest numbers drawn the previous week?



03 - TEN DIGIT NUMBER

A 10-digit number has distinct digits. Using all of its digits, two new numbers are created. The sum of these two numbers is 99999 and their multiplication is the same 10-digit number.

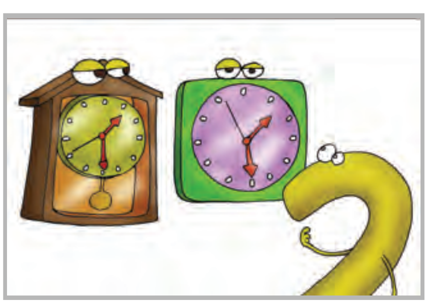
Find this 10-digit number.



04 - FOUR DIGITS

A number has distinct digits and for its any four consecutive digits, the multiplication of the two digits in the middle is greater than the sum of the four digits.

What is the greatest such number?



05 - CLOCKS

There are two analog clocks with hour, minute and second hands. One of them works correctly and the other one is broken, moving 20% faster. Both of them are set to 12:00 and observed until their second hands are at the same angle and the minute hand of the correct clock is at the same angle with the hour hand of the broken clock.

What is the time when this first happens after they are set?

Answers:
1)8, 2) 391/498771, 3)2495671308 (47931, 52068), 4)1956483720, 5)03:20

From Untouchable 11 to Hazmat Cargo

This article is a reprint of the article with the same title published in Game & Puzzle Design, vol. 3, no. 2, 2017, pp. 27-34, edited by Russ Williams and prepared for publication by Cameron Browne, and reprinted with permission.

Carl Hoff, Applied Materials

Untouchable 11 is a packing puzzle designed by Peter Grabarchuk. This paper describes Untouchable 11 and its 'untouchable' concept, and explores applying this concept to other hexomino packing puzzles. Every untouchable packing puzzle can be mapped to an equivalent conventional packing puzzle (in which pieces can touch), enabling the use of existing software tools for analysis. Exploring this puzzle space led to the creation of a new puzzle, Hazmat Cargo.

1 Introduction

UNTOUCHABLE 11 is a packing puzzle consisting of eleven pieces based on the eleven possible unfoldings of a cube, which themselves are a subset of the 35 *hexominoes*.[1] The goal is to place all eleven pieces onto a board such that no pieces touch, even diagonally at corners. The pieces can be rotated and flipped, but must be placed orthogonally onto the grid of the board. The puzzle offers three challenges:

1. *Easy* (9×17 board).
2. *Medium* (10×15 board, Figure 1).
3. *Hard* (12×12 board).

This paper describes how this idea of 'untouchable' packings has spread to other puzzles, and ultimately led to a new design of mine, described in a later section.

1.1 History

Untouchable 11, designed by Peter Grabarchuk,[2] first appeared on the gaming website SmartKit.com,[3] which sponsored the development of the associated app. In October 2008, it was launched with a contest[4] which gave a Smartkit t-shirt and the book *Puzzles' Express 3* [1] to the first person to solve all three challenges.

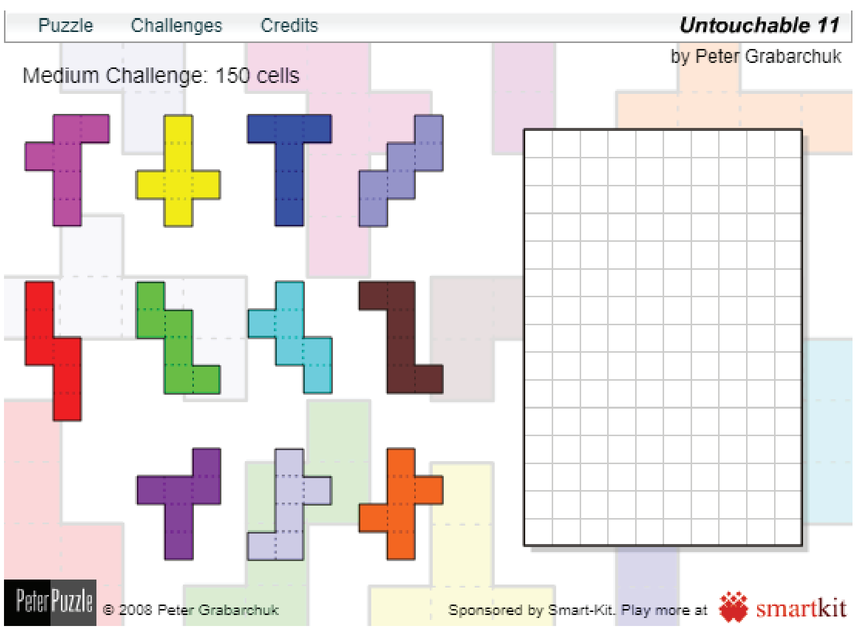


Figure 1. Screenshot of the medium (10×15) Untouchable 11 challenge.

[1] http://mathworld.wolfram.com/Polyomino.html
[2] http://www.grabarchukpuzzles.com
[3] http://smart-kit.com
[4] http://smart-kit.com/s1512

Hoff, C., 'From Untouchable 11 to Hazmat Cargo', *Game & Puzzle Design*, vol. 3, no. 2, 2017, pp. 27–34. © 2017

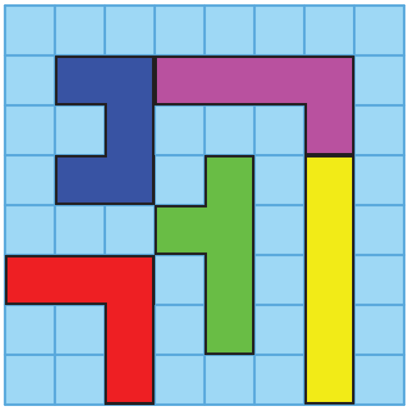


Figure 2. Solution to Golomb's problem.

The concept of a polyomino packing puzzle, in which no two pieces can touch even at a corner, appears to be original to the Grabarchuk family. In his book *Polyominoes* [2], Solomon Golomb asks what is the minimum number of pentominoes that can be placed on an 8×8 checkerboard such that none of the remaining ones can be added. The answer is five, and Figure 2 shows one such configuration. This sparse covering of the board seems to be a precursor to Grabarchuk's untouchable concept.

Kadon Enterprises, Inc.[5] also has a few games using similar concepts. Squint, a logic game played on a 9×12 grid, using their Quintillions set (1980). The goal is to make the last move by leaving no space on the grid for the opponent to place another quint (their brand name for pentomino).

Players in turn select a quint from the common pool and place it on the grid. The first quint must cover one of the board's corner squares. Later quints must be placed so that at least one of their corner points touches a corner point of any of the quints already on the board, and no sides may touch. Figure 3 shows such an arrangement.

This rule that corners must touch and sides may not touch results in a similarly sparse covering of the board. It also appears in the well-known game Blokus (2000) as a restriction on each player's own pieces.

Cornered is a similar logic game played using the Sextillions set. In that game, the pieces (the 35 hexominoes plus one duplicate) are divided between two players. In turn, players select one of their own pieces and place it on a 15×15 grid. A player's own pieces may not touch each other, not even diagonally at corners. A piece may touch opponent's pieces only at corners (no sides), but are not required to touch. The last player to put a piece on the board wins.

[5] http://www.gamepuzzles.com
[6] http://www.gamepuzzles.com/g4g11cubes.pdf

Figure 3. Squint example.

The only other puzzle I am aware of which uses the eleven unfoldings of a cube is a puzzle Kate Jones presented as her exchange gift at the 11th Gathering for Gardner. She named this puzzle 11 Magic Cubes.[6] Other than using the same pieces, it bears little resemblance to Untouchable 11.

2 Solving

In 2008, I solved the easy and medium challenges by hand. After days of struggling with the hard challenge, the closest I came to solving it is shown in Figure 4.

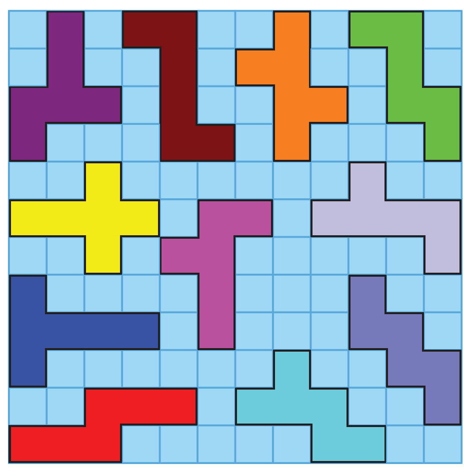


Figure 4. Near-solution to the hard challenge.

At this point, Peter was contacted and asked if the solution was unique. It turned out that the initial challenges were solved by Grabarchuk family members without the aid of computer algorithms. Peter knew of only two solutions to the hard challenge, and the total number of solutions was an unknown at that time. So now there were two puzzles to solve: I still needed to solve the hard challenge, and — more interestingly — to count the total number of solutions!

Unable to find a solver capable of solving these untouchable packing problems, I created my own, shown in Figure 5. Algorithms for solving packing puzzles typically use a recursive backtracking search [3]. Knuth describes how to efficiently implement this type of search in his paper 'Dancing Links' [4]. Matt Busche also has an article[7] suggesting how to combine a number of relevant strategies and ideas, including those developed by de Bruijn [5] and Fletcher [6].

Figure 5. The author's Untouchable 11 solver.

My Untouchable 11 solver uses several of these strategies. The source code is in Quick Basic 4.5 and is available.[8] The code works and found all seven solutions to the hard challenge of Untouchable 11, but it took over 24 days to complete its search. The output of that initial search is available,[9] but be warned that it contains solutions.

However, before the 24-day search was completed, it became apparent that the puzzle could be mapped to a conventional (touching) packing puzzle. This would allow the use of many other existing solvers which are much more efficient.

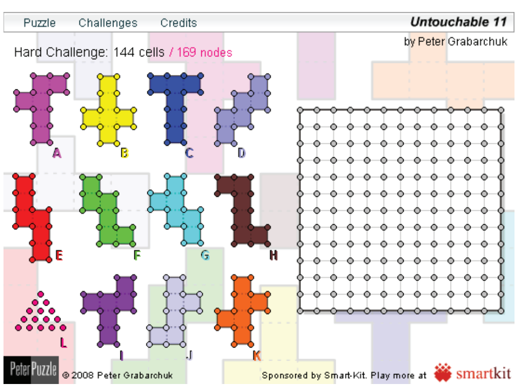


Figure 6. Mapping to a touching packing puzzle.

The idea is to map each original piece to a new piece defined by squares centred at vertices of the original piece, and increasing the width and height of the playing grid by one square. Figure 6 shows the original 12×12 challenge viewed this way: an equivalent task is to place the vertices onto the 13×13 grid of vertices. This results in exactly fifteen empty vertices.

In effect, this thickens each piece by wrapping it in an additional half-square wide layer. This additional part of each piece neatly fits into the required gaps between pieces in the original version of the puzzle. Each resulting piece is one square higher and one square wider. Figure 7 shows how two original pieces become two touching thicker pieces under this mapping.

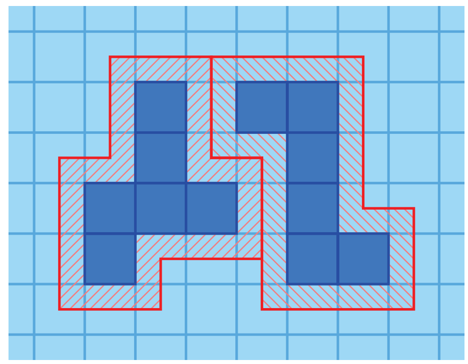


Figure 7. Half-unit thickening of pieces.

The fastest of the polyomino solvers that were readily available in 2008 was Gerard Putter's Polyomino Solver.[10] Once the hard challenge was mapped to its conventional touching equivalent and fed into this solver, the seven solutions were all found in under an hour. This work was completed before my 24-day search finished running.

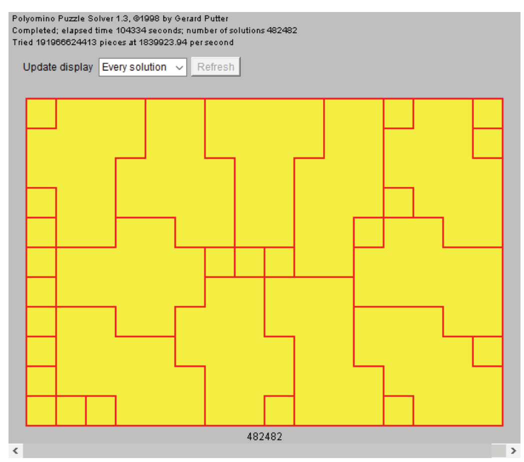


Figure 8. Result from Gerard's Polyomino Solver.

[7] http://www.mattbusche.org/blog/article/polycube
[8] http://wwwmwww.com/gapd/Untouch.TXT
[9] http://wwwmwww.com/gapd/SOLUTION_Finished.TXT
[10] https://gp.home.xs4all.nl/PolyominoSolver/downloadsolver.htm

The latter results confirmed the count and solutions found with Gerard's solver. Figure 8 shows output from Gerard's solver for the medium challenge. (We will not spoil the solution to the hard challenge here!) It found 482,482 solutions in 104,334 seconds (roughly 29 hours).

3 New Challenges

With a general solver, the first space to explore was additional rectangular boards as new challenges for these eleven original pieces. Table 1 shows these results. The 'Empty' column gives the number of empty cells in the mapped version, i.e. number of untouched vertices in the original version.

Board	Solutions	Name	Empty
12×12	7	Hard	15
11×13	33		14
10×15	482,482	Medium	22
9×16	174		16
9×17	65,516,235	Easy	26
8×18	15		17
7×21	60,327		22
6×24	8		21

Table 1. Solution counts for Untouchable 11 challenges.

Five new challenges were found that all fall between the medium and hard challenges in terms of difficulty. It was also proven that one entire row of the easy challenge, the 9×17 board, could be left empty, because the 9×16 board is solvable. Untouchable 11 now consisted of eight total challenges and received the Gamepuzzles Annual Polyomino Excellence Award for 2015.[11] Figure 9 shows the trophy.

A physical version of Untouchable 11 was created as the author's exchange puzzle for the 2017 *International Puzzle Party* (IPP37) in Paris, France. This puzzle included all eight challenges. The pieces were made of laser-cut acrylic by Sculpteo.[12] The board was 3D printed in Polyamide using selective laser sintering, SLS, by i.Materialise.[13] Figure 10 shows the final product.

Figure 10 does not show a solution, as two pieces touch at corners. A state with a single corner touch is known as a *near-solution*. These were counted for the original Untouchable 11 hard challenge in November, 2016, and 3,092 near solutions were found. This count was later verified by Landon Kryger in December 2016.

Figure 9. Gamepuzzles Annual Polyomino Excellence Award for 2015.

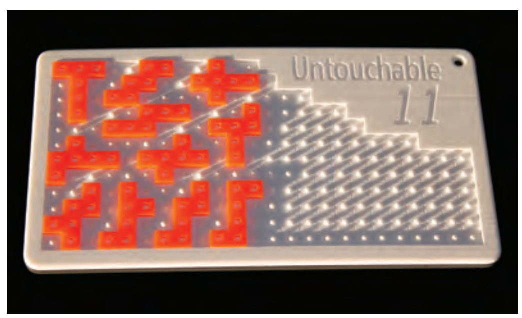


Figure 10. Carl Hoff's IPP37 exchange puzzle.

4 Widening the Search

The search for a set of eleven hexominoes which can be placed on a 12×12 board with a single unique solution was started in 2012. That work was done by creating modified code for each subset and running it through Gerard Putter's Polyomino Solver.

[11]http://www.gamepuzzles.com/gape15.htm
[12]https://www.sculpteo.com
[13]https://i.materialise.com

As each subset had to be coded by hand, this was slow tedious work, and the work was put on hold when a set with just two solutions was found. That set uses one hexomino which is not an unfolding of the cube. It was shared with Peter Grabarchuk and resulted in the release of Untouchable 11: Master Challenge[14] in March 2012, shown in Figure 11. This work was initially prompted by the need for an exchange gift[15] for the 10^{th} Gathering for Gardner, G4G10.

The search resumed late in 2016 with the assistance of programmers Brandon Enright and Landon Kryger. Landon had created a new, efficient solver which could test all possible subsets of a given size from a master set on a given board, to find puzzles with unique solutions.

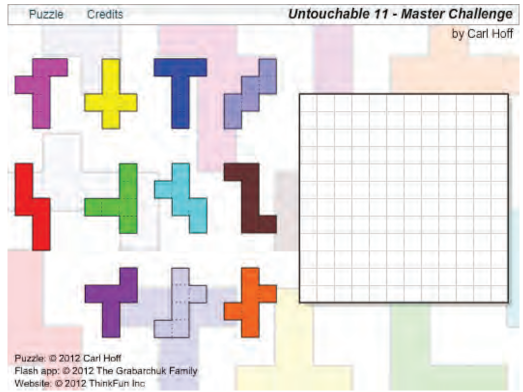


Figure 11. Untouchable 11: Master Challenge.

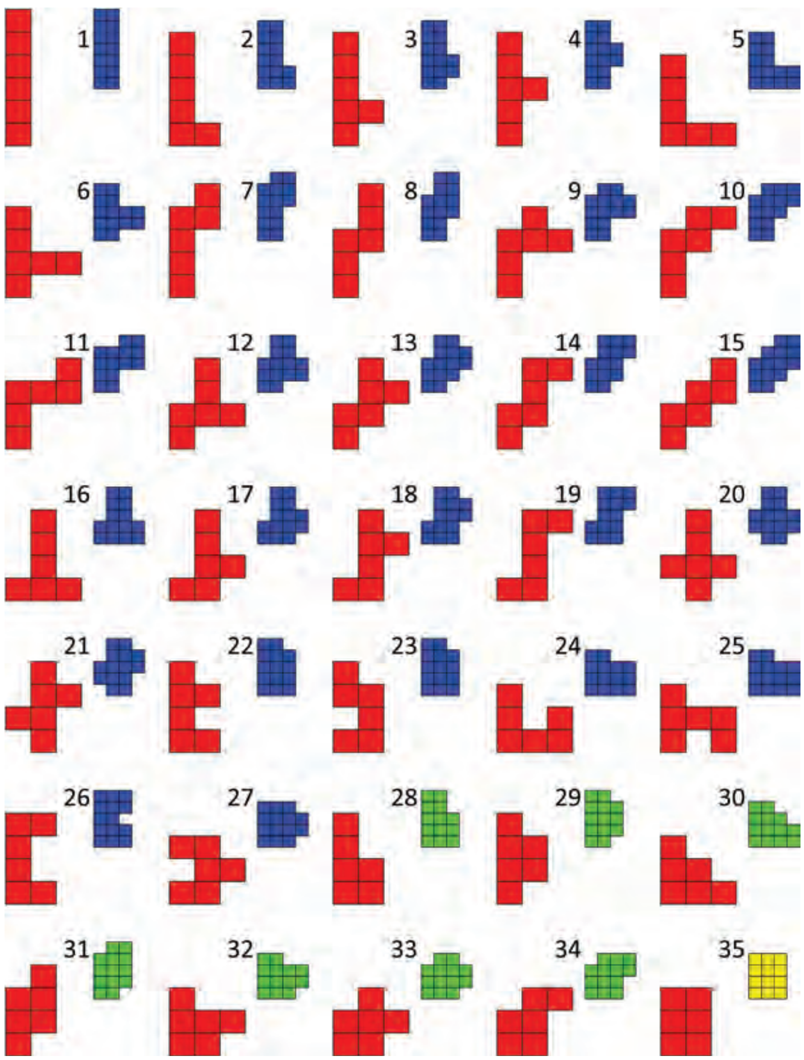


Figure 12. The 35 hexominoes and their vertex duals.

[14]http://www.puzzles.com/PuzzleClub/Untouchable11MasterChallenge
[15]http://wwwmwww.com/gapd/U11MasterChallenge.pdf

Board	N	Empty	Subsets	Tested	Search%	Single%	0 Solns	1 Soln	>1 Soln
5×5	2	8	210	210	100.0%	18.5714%	124	39	47
6×6	2	21	210	210	100.0%	0.0000%	0	0	210
6×6	3	7	1,330	1,330	100.0%	6.0902%	1,074	81	175
7×7	3	22	1,330	1,330	100.0%	0.0000%	0	0	1,330
7×7	4	8	5,985	5,985	100.0%	8.6717%	4,365	519	1,101
8×8	4	25	5,985	5,985	100.0%	0.0000%	0	0	5,985
8×8	5	11	20,349	20,349	100.0%	4.2017%	4,750	855	14,744
9×9	6	16	54,264	54,264	100.0%	0.0792%	199	43	54,022
9×9	7	2	116,280	116,280	100.0%	0.0439%	116,213	51	16
10×10	7	23	116,280	116,280	100.0%	0.0000%	0	0	116,280
10×10	8	9	203,490	203,490	100.0%	5.0980%	79,601	10,374	113,515
11×11	9	18	293,930	293,930	100.0%	0.0003%	0	1	293,929
11×11	10	4	352,716	107,010	30.3%	1.9325%	100,236	2,068	4,706
12×12	11	15	352,716	352,716	100.0%	0.0020%	49	7	352,660
12×12	12	1	293,930	293,930	100.0%	0.0024%	293,920	7	3
13×13	14	14	116,280	116,280	100.0%	0.0000%	116,280	0	0
14×14	16	0	20,349	20,349	100.0%	0.0000%	20,348	0	1

Table 2. Summary of search results. **N** indicates number of pieces.

The first thing to decide on was the master set that would be used: as shall be shown, there is no reason to include all 35 hexominoes, and a smaller set of candidates would mean a shorter search time. Figure 12 shows the complete set of 35 hexominoes and their vertex duals, created by mapping each vertex to a square, i.e. the thicker versions of each piece. 27 vertex duals have fourteen squares (shown in blue), but seven have thirteen squares (shown in green), and one has only twelve (yellow). We decided to use only the first 21 hexominoes as the master set. The hexominoes 22 through 35 were removed from consideration for the following reasons:

Hexominoes 28-35 have fewer than fourteen squares in their dual versions, so they seem easier to place. Hexominoes 22-35 can all be contained in a 3×3 or a 4×2 box. These are all more compact than the original eleven unfoldings of a cube, so they seem easier to place.

Hexominoes 22 and 23 both map to the same vertex dual polyomino. Any set containing both could never have a single solution, since those two pieces could always swap positions, so at least one must be excluded. Hexominoes 24 and 25 also both map to the same vertex dual polyomino. Hexomino 26 is unsuitable, as no vertex dual has a protruding square which could fit in the small gap on its right side. Therefore, any solution containing this piece produces a second solution with this piece rotated 180°.

After we selected the master set, the results shown in Table 2 were generated after many months of CPU time. We found seven sets of eleven hexominoes with unique solutions on the 12×12 board. Also note that there are seven sets of twelve hexominoes which also have unique solutions on the 12×12 board.

Table 3 shows all 11-piece and 12-piece sets with unique solutions. These are excellent puzzles left for the reader to solve. It may seem counter-intuitive, but the 12-piece sets are much easier to solve than the 11-piece sets. This is due to the availability of only a single empty node, which allows one to backtrack much sooner, thus simplifying the search.

Set	Hexominoes											
A	9	10	12	13	14	15	16	17	18	20	21	
B	8	9	10	11	13	15	17	18	19	20	21	
C	8	9	10	11	12	13	15	17	18	20	21	
D	8	9	10	11	13	15	16	17	18	20	21	
E	1	8	9	11	12	13	15	17	18	20	21	
F	1	11	12	13	14	15	16	17	19	20	21	
G	4	8	9	12	13	15	16	17	18	20	21	
a	1	2	3	4	5	6	7	8	10	11	13	14
b	1	2	3	4	5	6	7	8	10	11	13	18
c	1	2	3	4	5	6	9	11	12	13	18	21
d	2	3	4	5	6	7	9	10	11	13	16	18
e	1	2	4	5	6	8	9	10	11	12	14	18
f	1	2	3	4	5	7	8	10	11	12	14	16
g	1	2	3	5	6	7	8	9	11	14	16	19

Table 3. 11 (A–G) and 12 (a–g) piece hexomino sets.

During this search, the need arose to design a puzzle for the IPP37 design competition. The initial candidates were the 11-piece sets with unique solutions seen above. But it was thought that these may be too difficult to be fully appreciated by most puzzlers, so smaller boards and fewer pieces were also considered. The search was focussed on square boards, because a compact board (with a lower perimeter-to-area ratio) will typically maximise difficulty. A good example of this is the Eternity Puzzle,[16] whose nearly circular board is shown in Figure 13.[17]

[16]http://www.mathpuzzle.com/eternity.html

[17]Figure derived from: http://www.archduke.org/eternity/solution/index.html

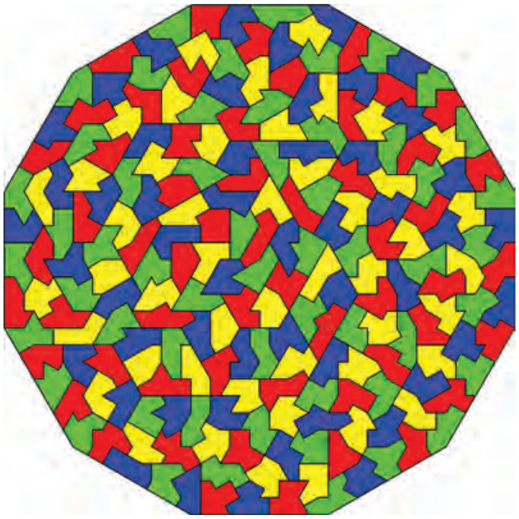


Figure 13. Eternity Puzzle solution by Alex Selby. Eternity pieces by Christopher Monckton © 1999.

Out of the 293,930 sets tested for the 9-piece puzzle on an 11×11 board, only a single set was found to have a unique solution. This set was explored and found to be very interesting for the following reasons:

1. It has only a single solution.
2. It uses a square board.
3. It has 3,761 near-solutions.

This single set has more near-solutions than the 3,092 near-solutions of the Untouchable 11 hard challenge, which has seven actual solutions. This puzzle also contained eighteen empty cells in its mapped version, three more than the original Untouchable 11 hard challenge. This is the largest number of empty cells found in any puzzle of this type with a unique solution to date.

The nine hexominoes found in this set are 8, 9, 12, 13, 15, 17, 18, 20, and 21. This puzzle was the one chosen for the design competition. In keeping with the untouchable theme, the pieces are physically designed to resemble groups of six industrial drums containing hazardous materials.

The board was designed to suggest a barge. The goal is to pack the nine groups of six hazmat drums onto the barge, an 11×11 array, such that no two pieces touch, not even at corners. (Any contact could lead to a catastrophic chemical reaction!) Figure 14 shows the puzzle submitted to the competition. All components were designed in SolidWorks,[18] and 3D printed in steel or polyamide by i.Materialise or Shapeways.[19]

[18]http://www.solidworks.com/
[19]https://www.shapeways.com/

5 Open Questions

Here are two open hypotheses, neither of which have been proven:

1. The 9-piece set used in Hazmat Cargo is the only 9-piece subset of the hexominoes to have a single solution on the 11×11 board.
2. All other 9-piece subsets have multiple solutions on the 11×11 board; there are none with no solutions.

There are $\binom{35}{9} = 70,607,460$ possible 9-piece subsets of the 35 hexominoes. Of these, only 293,930 have been searched, i.e. only about 0.42%. The sets that have been searched contain the hardest-to-place pieces.

Since they all have solutions, it is believed that adding easier-to-place pieces to the mix will not result in sets without solutions, or other sets with just a single solution. Still, neither hypothesis can be asserted with certainty. Please contact the author if you are able to prove either hypothesis.

There is also the question of what fun and interesting puzzles may exist in the space of untouchable hexomino packing puzzles with rectangular boards. That is the next task slated for Kryger's solver. If the piece sets are expanded to include other polyominoes and the board shapes are not restricted to just squares or rectangles, then there are even more possibilities.

6 Conclusion

While Hazmat Cargo did not win any awards at the design competition, it did receive numerous compliments, including the thematic barge and hazmat drums. Several commented that the physical design fit the untouchable concept perfectly. It was fun to design and took on a significantly different aesthetic than my previous designs.

Aside from the simple pleasure of designing a new puzzle, the lesson here is to take a new look at the puzzles you have enjoyed. In this case it was Peter Grabarchuk's Untouchable 11, which introduced a new concept to polyomino packing puzzles. This concept proved to open a very vast and interesting area which proved worthy of exploration. Five new challenges were added to the original Untouchable 11 puzzle. The Untouchable 11: Master Challenge was created and resulted in a new app being released and enjoyed. And the exploration resulted in a very difficult 9-piece puzzle named Hazmat Cargo.

Figure 14. The Hazmat Cargo puzzle.

Acknowledgements

Thanks to Peter Grabarchuk for his efforts in creating Untouchable 11, his permission to use that puzzle as my exchange puzzle at IPP37, and his blessings on writing this article. Thanks to Brandon Enright and Landon Kryger for all the assistance they have provided. Using Gerard Putter's Polyomino Solver alone, it would have taken me twenty years to search the 352,716 subsets needed for just the 11-piece 12×12 puzzle, while Kryger's solver reduced that time to a couple of months. Thanks to Gerard Putter for sharing his polyomino solver and Jaap Scherphuis for sharing his polyform puzzle solver[20], both of which were used in this study, with Jaap adding new functionality to his solver at my request. Thanks to Kate Jones for providing the information on Squint and Cornered. Please contact me for solutions to challenges presented in this paper.

References

[1] Homa, H., Grabarchuk, P. and Grabarchuk, S., *Puzzles' Express 3*, New York, Peter Puzzle, 2008.

[2] Golomb, S., *Polyominoes*, 2nd edition, Princeton, Princeton University Press, 1994.

[3] Norvig, P. and Russell, S. J., *Artificial Intelligence: A Modern Approach*, 3rd edition, Upper Saddle River, Pearson, 2009.

[4] Knuth, D., 'Dancing Links', *Millennial Perspectives in Computer Science*, November, 2000, pp. 187–214.

[5] de Bruijn, N. G., 'Programmeren van een Pentomino Puzzle', *Euclides*, vol. 47, 1972, pp. 90–104.

[6] Fletcher, J., 'A Program to Solve the Pentomino Problem by the Recursive Use of Macros', *Communications of the ACM*, vol. 8, no. 10, 1965, pp. 621–623.

Carl Hoff is an epitaxy process service engineer for Applied Materials currently supporting Global Foundries. Research interests include twisty puzzle design and theory, packing puzzles, and puzzle rings.
Address: 108 North Broadway, Malta, NY, 12020-5200, USA.
Email: carl.n.hoff@gmail.com

[20] https://www.jaapsch.net/puzzles/polysolver.htm

The Dezign-8 Puzzle[1]

Lyman Hurd[2]

Abstract This paper describes some of the intriguing properties of the Dezign-8 puzzle published by Kadon Enterprises. Sixty-four tiles are arranged in an 8x8 grid matching edges. All patterns formed this way have the property that the number of simple closed loops always equals the number of connected components. An upper bound on the number of components is derived and the various degrees of symmetry possible. are described

Keywords: puzzle, tiling.

Introduction

Created by Bill Biggs in 1959, Dezign-8[1] pictured in Figure 1, has 64 tiles representing the various ways a path can emerge from one, two, three or four sides of a square.

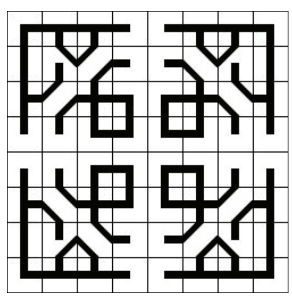


Figure 1: The Dezign-8 Puzzle

The solution in Figure 1 has eight connected components and eight "loops" (by which we mean connected components in the complement not including the outside.) The fact that these two counts are the same is not a coincidence as further shown below.

For the purposes of this paper, the types of tile are assigned names. All of the tiles are mirror symmetric except for the LEFT and RIGHT tiles which are each other's mirror images.

---

[1] Dezign-8 is a trademark of Kadon Enterprises, Inc. ©2000.

[2] Address correspondence to: Lyman Hurd lyman.hurd@gmail.com.

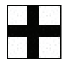
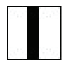
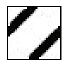
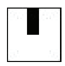
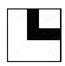
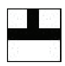
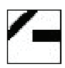
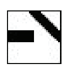
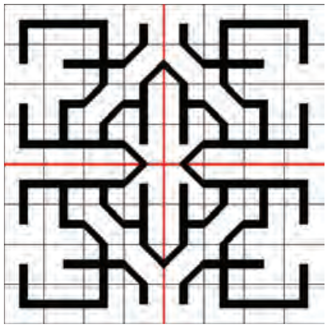
Tile	*Name*	*Count*	*Rotations*
	PLUS	4	1
	STRAIGHT	4	2
	DIAGONAL	8	2
	END	16	4
	CORNER	16	4
	TEE	12	4
	RIGHT	2	4
	LEFT	2	4
		64	

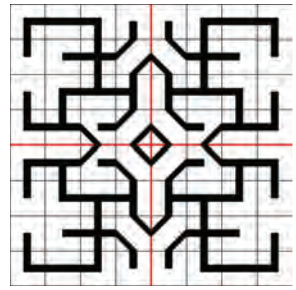
Table 1: Distribution of tiles.

Loops = Components

Figure 2 shows solutions with one component and one loop and two components and two loops. Other figures show solutions with varying numbers of loops and components but in every case the two are equal (e.g., Figure 1, 8 components, 8 loops, Figure 4, 11 components, 11 loops).



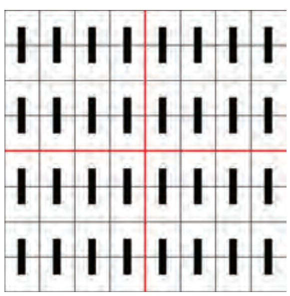
One loop, one component.



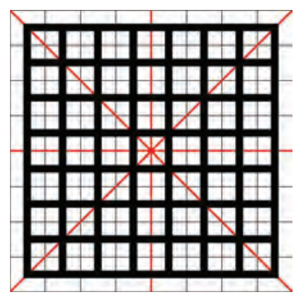
Two loops, two components.

Figure 3: More sample solutions.

The property that the number of loops equals the number of components has to be a property not only of these kinds of tiles, but of their frequency. For example, Figure 2 illustrates two different solutions with other sets of tiles violating this equality.



32 Components 0 Loops



1 Component 49 Loops

Figure 3: Counterexamples with different tile sets.

By Euler's Polyhedral Formula[3] all polyhedra (equivalently connected graphs drawn on a sphere) satisfies the following relationship among vertices. edges and faces:

$$V - E + F = 2$$

The plane can be considered a sphere with one point punctured, or equivalently we can consider the entire region outside the graph as comprising one face which leads to the equation for the plane:

$$V - E + F = 1$$

And finally, we note that for a connected graph component (C=1) and for a graph with more than one component one can add an edge and subtract a component without affecting V or F hence:

$$V - E + F - C = 0$$

This means that to show that loops = components , $F = C$, it suffices to show that $V = E$.

To show the relationship, one associates each solution with a graph by adding vertices to some of the tiles as illustrated in Table 2. On those tiles with vertices, each line from the vertex to the edge of the table represents half a graph edge since two such segments are required to join one vertex to another.

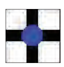
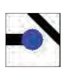
Tile	*Name*	*Count*	*Vertices*	*Edges*
	PLUS	4	1	2
	END	16	1	1/2
	TEE	12	1	3/2
	RIGHT	2	1	1/2
	LEFT	2	1	1/2
		Weighted Sum	36	36

Table 2: Tiles with vertices added.

Figure 3 shows a solution marked with its associated graph.

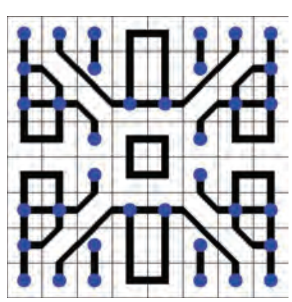


Figure 4: A solution with its associated graph.

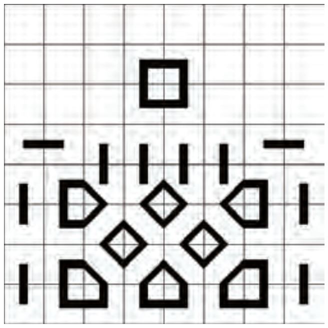
Note that the central square in Figure 4 has no vertices whatsoever. The CORNER and DIAGONAL tiles as well as the diagonal portion of the LEFT and RIGHT tiles can form closed loops, but in each case it is a simple loop contributing one loop and one component simultaneously and therefore having no effect on the difference: $Loops - Components$.

Note that with the current set of tiles, adding up the total number of edges is, as required equal to the total number of vertices. Using the labeling in Table 1 and collecting terms one reaches Equation 1 which gives necessary and sufficient conditions for a combination of these tiles to satisfy the Euler property:

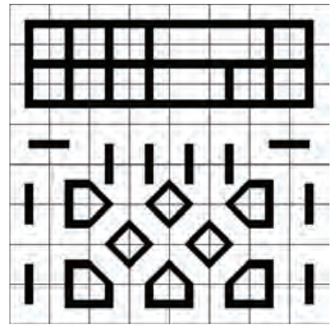
Equation 1: $N_{TEE} + 2\,N_{PLUS} = N_{END} + N_{LEFT} + N_{RIGHT}$

Maximal Solutions

A solution will be called "maximal" if it exhibits as many components (or loops) as possible for a given set of tiles. The first question that arises is what this maximal number is. The most efficient way to form a connected component is to connect two END tiles together. Alternatively one can join four corner tiles. For the purposes of this enumeration the corners provided by CORNER tiles are topologically equivalent to the diagonal lines of the DIAGONAL, RIGHT and LEFT tiles. Setting aside for the moment the PLUS, TEE and STRAIGHT tiles and trying to form as many components as possible from the remaining tiles, one can form 19 connected components as illustrated on the left of Figure 3. While this is an upper bound, it only is achievable if the tiles we did not use can be incorporated into a full solution. It is apparent that on their own there is no way to form the remaining PLUS. TEE, STRAIGHT tiles into an additional component. Such as extension is illustrated on the right of Figure 3.



(a) 19 Components with CORNER, DIAGONAL, END tiles.



(b) Solution extended using all tiles.

Figure 5: Maximal number of components.

Summarizing, an upper bound on the number of components achievable with a tile set is given by:

$$(N_{CORNER} + 2N_{DIAGONAL} + N_{LEFT} + N_{RIGHT})/4 + (N_{END} + N_{LEFT} + N_{RIGHT})/2$$

simplified to the following formula for an upper bound, U for the number of components:

Equation 2: $\frac{1}{4}(N_{CORNER} + 2N_{DIAGONAL} + 3N_{LEFT} + 3N_{RIGHT} + 2N_{END}) = U$

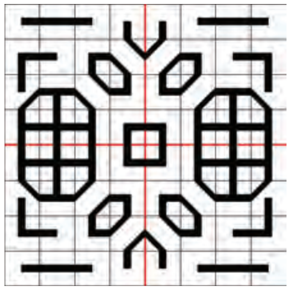
Combining Equation 1 and Equation 2 one can derive a formula that only depends on pieces with "corners" (whether straight as in CORNER pieces or slanted as in DIAGONAL).

Equation 3: $\frac{1}{4}(N_{CORNER} + 2N_{DIAGONAL} + N_{LEFT} + N_{RIGHT} + 2N_{TEE} + 4N_{PLUS}) = U$

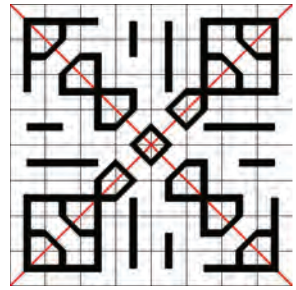
What this bound implies is that in a solution with the maximum number of components and loops, every corner has to form the corner of its own loop. This constraint restricts the form of such a solution and should make searching for such solutions much faster.

Symmetries

As has been noted above, solutions can be left-right and top-bottom symmetrical or can be symmetrical in both diagonals. Figure 6 shows examples of each type.



Orthogonal Symmetry



Diagonal Symmetry

Figure 6: Different types of symmetry.

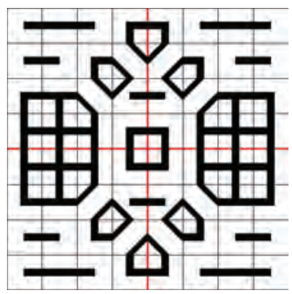


Figure 7: Daniel Austin's orthogonally symmetric maximal solution.

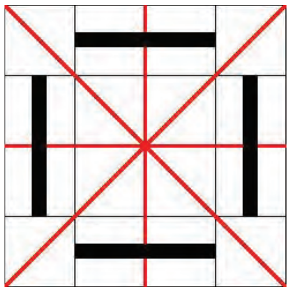
Question 1: Figure 7 shows an orthogonally symmetric maximal solution discovered in 2015 by Daniel Austin. What is the largest number of components that can be achieved for a diagonally symmetric solution?

Question 2: Being maximally symmetric introduces a number of constraints on a solution. In searching for an orthogonally symmetric solution, the author worked independently only to discover that he had rediscovered this solution. Is this maximal orthogonally symmetric solution unique (apart from a trivial 90 degree rotation)?

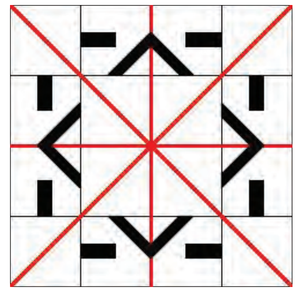
Dihedral Symmetries

Left-right and diagonal symmetries cannot be achieved simultaneously because having a diagonal and an orthogonal axis of symmetry implies that the solution is rotationally symmetric and this is not possible with the default set of tiles.

As illustrated in Figure 8, a solution with all eight dihedral symmetries would require the presence of a multiple of eight STRAIGHT tiles and a multiple of four RIGHT and LEFT tiles, whereas the original set has four of the former and two each of the latter.



Each STRAIGHT piece forces seven more.



Each LEFT or RIGHT piece forces three more.

Figure 8: Tiles forced by dihedral symmetry.

However, by altering the default set of tiles by making the adjustments shown in Table 2, a set of tiles can be arranged that satisfies Equation 1, and therefore maintains the property that *Loops* = *Components* while allowing a fully symmetrical solution seen in Figure 9.

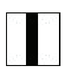
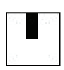
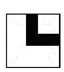
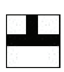
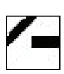
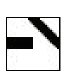
Tile	*Name*	*Count*	*Delta*
	STRAIGHT	8	+4
	END	8	-8
	CORNER	20	+4
	TEE	8	-4
	RIGHT	4	+2
	LEFT	4	+2

Table 2: Distribution of new tile set.

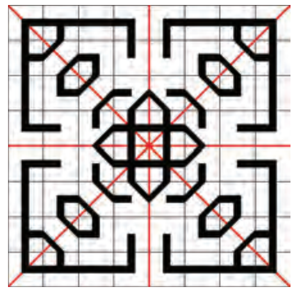


Figure 9: An alternate tile set admitting all eight dihedral symmetries.

Question 3: How many components/loops can be formed with this new set of tiles? Note that the upper bound equation yields 19 in this case as well, but is this achievable?

Enumerating Solutions

When searching for positions, it is also noted that the DIAGONAL and PLUS are interchangeable in any pattern and in any pattern the DIAGONAL pieces can be rotated 90 degrees without affecting any other pieces.

In the orthogonally symmetric case, the pattern is determined by one quadrant which contains on PLUS and two DIAGONALS. The DIAGONALS can be oriented in any of four ways and the PLUS can take any of the three positions giving twelve distinct positions by permuting these pieces.
The same argument applies to the TEE and LEFT/RIGHT pieces. For the orthogonally case the tiles used comprise one LEFT (or RIGHT and three TEE tiles yielding three possibilities as the asymmetric piece can take any of the four positions (orientation is fixed).Combined with the observations above, when searching for solutions, by arranging the DIAGONAL pieces are PLUS pieces and the LEFT/RIGHT and TEE pieces, every solution of this simplified puzzle can be rearranged to form a total of 48 different solutions.

In the general case where we do not enforce symmetry constraints and instead allow the LEFT, RIGHT and TEE pieces to be permuted, the PLUS and DIAGONAL pieces to be permuted and the DIAGONAL pieces to be oriented symmetry, each solution can be rearranged in:

$$\binom{16}{4}\binom{12}{4}2^8 = 1,820 \times 495 \times 256 = 230,630,400$$

different ways.

Acknowledgements

Thanks to Kate Jones and Kadon Enterprises for fruitful discussions as well as supporting the use of their puzzle as the topic of this paper. Their website is a treasure trove of mathematically oriented puzzles.

References

[1] Kadon Enterprises, Inc., Tilings and Designs, Page 2 of 4, www.gamepuzzles.com/tiling2.htm#D8.

[2] Malkevitch, Joseph. "American Mathematical Society." Euler's Polyhedral Formula, American Mathematical Society, Dec. 2004, www.ams.org/publicoutreach/feature-column/fcarc-eulers-formula.

Four Logical Deduction Problems from Famous Motion Pictures

Justin Kalef

Introduction

Stumped while trying to come up with a suitable gift for the Gathering, I procrastinated by switching back and forth among some movies on television. Imagine my delight when it dawned on me that, seen in the right light, some famous scenes from these films contain quite remarkable logic puzzles. I reproduce those scenes here for your solving pleasure. True, the scenes may be slightly different from how some cinema lovers will remember them: I blame the lapses on my own faulty memory.

If you're stumped while trying to figure any of these out and would like a hint, or if you would like to check your answer, please feel free to contact me at jkalef@philosophy.rutgers.edu.

Puzzle 1

Sam Spade, private investigator and master of logical deduction, has his hands full dealing with a gang of four criminals (Kasper Gutman, Brigid O'Shaughnessy, Joel Cairo, and Wilmer Cook). He knows that one of them always tells the truth, one of them always lies, and the other two alternate between true and false statements (that is, if they make a true statement, the next statement they make is false, and vice versa). One of them has the gun that killed Spade's partner, Miles Archer. Spade needs the gun to give to Police Detective Tom Polhaus, who will be arriving soon. When he asks these four characters about the gun, the following conversation ensues.

Spade: All right, Cairo, cough up the gun if you've got it.

Cairo: Excuse me, sir, but Miss O'Shaughnessy has it.

O'Shaughnessy: No, Sam, it's Cairo who has it.

Cairo (trying to contain his anger): Mr. Spade, Miss O'Shaughnessy tells nothing but lies.

O'Shaughnessy (getting angry in turn): Why, I've never told a lie in all my life!

Spade (turning to Gutman): I take it you can help resolve this lovers' quarrel? If I'm pretending to trust you now, that is.

Gutman (laughing): By gad, sir, I'm a man who always speaks truthfully.

Spade (to Cook): And how about you? You've been awfully quiet. Can you vouch for Gutman's truthfulness?

Cook: He always tells the truth. Ain't you been listening?

Spade: Oh, yeah, he assures me he's honest. But if I'm not sure about someone's honesty, I shouldn't take his word for it, and I shouldn't trust his gunsel on the subject, either, should I?

Cook: [speaks two words, the first a short guttural verb, the second "you."]

Spade: People lose teeth talking like that.

Gutman (smiling at the situation): You'll find that Wilmer here has the gun, sir.

Cook: [stands up, staring wildly and open-mouthed at Gutman, not saying a word].

Spade: All right, fellows. No need to start breaking up the furniture over this. It's pretty clear now who has the gun and who's been lying about it.

Who has the gun that shot Miles Archer?

Puzzle 2

Imagine visiting the Rocky Mountains, at a grand, luxurious hotel whose employees are all, to say the least, unusual. Half the employees are sane, and have been all their lives: these employees believe everything that is true and disbelieve everything that is false. The other half are insane, and have been insane all their lives: they believe everything that is false and disbelieve everything that is true. Moreover, half the employees are chronic liars: every statement they make is false, or so they believe. The other half are absolutely honest: every statement they make is true, or so they believe. It's not possible to tell, from looking at a hotel employee, whether he or she is sane and honest (and hence always truthful), sane and dishonest (and hence always untruthful), insane and honest (and hence always untruthful), or insane and dishonest (and hence, inadvertently, always truthful). You also happen to know that exactly one employee is the caretaker.

You enter the grand ballroom and see people dressed up for a roaring twenties costume evening. As you watch, you notice a very proper-looking English employee inadvertently spilling a tray of drinks onto a quite scruffy-looking American employee. As the employee who spilled the drinks apologizes and tries to lift the stain out of the scruffy employee's clothing, the following conversation ensues between them:

Torrance: Look, Mr. Grady: you believe that I believe that you believe that I'm the caretaker.

Grady: Sir?

Torrance: (smirking) Mr. Grady, you're the caretaker of this hotel.

Grady: I'm sorry to differ with you, sir; but you're the caretaker here. You've always been the caretaker.

Torrance: (smiling after a confused pause) Mr. Grady, I'm not insane.

Grady: I hope you don't mind my saying so, sir, but I am fully sane. I should know, Mr. Torrance. I've always been sane.

What can be deduced about these two employees? And which of them, if either, is in fact the caretaker?

Puzzle 3

In the late 1980s, Detective Kimball, a private investigator, was hired to look into the disappearance of Paul Allen, a vice president of the Wall Street firm Pierce and Pierce. Before meeting with the other vice presidents, Kimball learned that they had all earned their MBAs at either Yale (in which case they belong to the elite Walrus club) or Harvard (in which case they belong to the secretive Boden club). He also learned that members of the Walrus club take a lifelong oath to always make true statements if their business cards have a lettering type that contains an R in its name, and to always make false statements otherwise. Each Boden club member, by contrast, swears to only make true statements if his business card's lettering type doesn't contain an R in its name, and to always make a false statement otherwise. The only confounding factor is that a few members of either club earn VIP status, in which case they have to do the opposite of what they promised in their oaths. All such VIP members are able to make Friday night reservations at Dorsia, a fashionable Manhattan restaurant. It is impossible for anyone who is not a VIP member to make such a reservation.

Detective Kimball's conversation with the other vice presidents goes as follows:

Detective Kimball: Thank you all for taking the time to meet with me. Let's start with you, Mr. Bateman. Where were you on the evening of Friday, October 16th, the night Paul Allen disappeared?

Patrick Bateman: Let's see... I was returning some videotapes that night.

Timothy Bryce (smirking): What are you going to tell us next, Bateman? That Phil Collins' ëSussudio' is a new peak of professionalism?

Patrick Bateman: Bryce, it is a new peak of professionalism. It's a great, great song, and a personal favorite.

David van Patten: Paul Allen made a reservation at Dorsia that night. He was the only one of us who could get one.

Detective Kimball (turning to Carruthers): I forget now, Mr. Carruthers. Did you tell me in our pre-interview that Mr. van Patten was also able to get such a reservation?

Luis Carruthers: I'm not the sort of person who could have said that.

Patrick Bateman: Here's Paul Allen's business card. Note its tasteful thickness. And that lettering...

Detective Kimball (turning to van Patten): Like the lettering on your card, Mr. van Patten?

Marcus Halberstram: No, van Patten's card has Romalian type, or something else with an R in it.

Detective Kimball: Do you and Mr. van Patten know each other well, Mr. Halberstram?

David van Patten: Not really. I went to Harvard. Halberstram is part of that Yale thing.

Marcus Halberstram: Actually, van Patten did his MBA at Yale.

Craig McDermott: No he didn't, Marcus, you nitwit. He went to Harvard.

Detective Kimball: And how do you know that, Mr. McDermott?

Craig McDermott: Because I was there with him. We were in the same year.

Detective Kimball: Okay, let's cut to the chase. What's going on with Paul Allen? Where is he?

Patrick Bateman: I had to kill him last week because of his business card. It even had a watermark.

Luis Carruthers: Patrick, don't even joke about such a thing.

Craig McDermott: Actually, he's in London. A friend of mine just had lunch with him there yesterday.

At this point, Detective Kimball logically deduced what had happened to Paul Allen and was able to close his case. What did he deduce?

Puzzle 4

(Don) Vito Corleone, the head of an underworld family, has to keep his wits about him. There has been an attempt on his life by Philip Tattaglia, the head of a rival family, presumably over a dispute about whether Corleone should use his political influence to help support drug dealing. He has just recovered and learned that his eldest son, Santino, has been killed in an ambush. To end the escalating violence, he calls a truce and meets with the heads of all five families (the other three families are Stracci, Cuneo, and Barzini).

Corleone no longer knows which family heads are involved in which criminal enterprises. But he does know that the heads of families that are involved in drugs but not gambling only make false statements, as do the heads of families that are involved in gambling but not drugs. Heads of families that are not involved in drugs or gambling only make true statements, as do heads of families that are involved in both drugs and gambling.

Corleone welcomes everyone to the meeting. Then, he listens as the heads of the families speak as follows:

Victor Stracci: The Tattaglia family, the Barzini family, and the Cuneo family are all in the drug business.
They need your support, Don Corleone.

Philip Tattaglia: None of us are involved in gambling, though. Of the five families, only yours is involved in that, Don Corleone.

Emilio Barzini: Don Corleone, what can I say? I was never behind the attack against you.

Carmine Cuneo: I had nothing to do with the attack against you, Don Corleone.
And don't worry about Stracci. Stracci had nothing to do with that attack.

Victor Stracci: That's right, Don Corleone. We Straccis earn money from drugs. We earn money from gambling. But I never acted against you, directly or indirectly.

Philip Tattaglia: I alone acted against you, Don Corleone. Nobody here directed me to do it.

After some further discussion, Don Corleone makes his peace with Tattaglia and embraces him, ending the meeting. But unlike the heads of the other families, Don Corleone is a great master of deductive reasoning. On the way home from the meeting, in a private conversation with his adopted son, he speaks disparagingly of Tattaglia, saying "He never could've outfoxed Santino. But I didn't know until this day that it was ___________ all along who directed the attack against me."

Fill in the blank:

Who was the mastermind behind the attack on Don Corleone, and how did Don Corleone know?

Crypto Word Search

Tanya Khovanova

G4G 13

A	B	C	D	E	F	G
H	C	I	F	B	B	C
D	I	J	K	L	A	J
C	I	F	M	A	C	K
N	O	O	N	F	B	I
F	J	O	P	P	Q	G
H	F	A	R	K	J	B

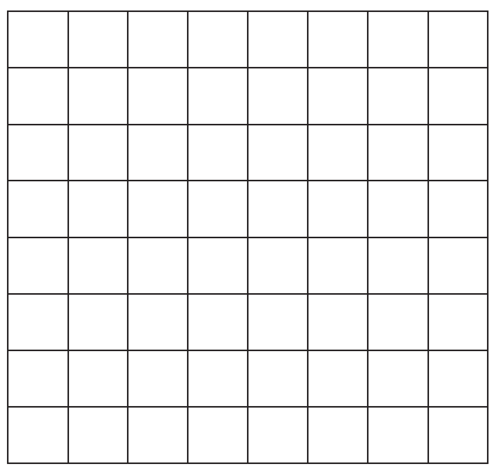
ART	IDEA	MAGIC	MATH	NOTE
PI	PROBLEM	PUZZLE	RIDDLE	TRICK

Two Tiling Problems

Anany Levitin

A Questionable Tiling

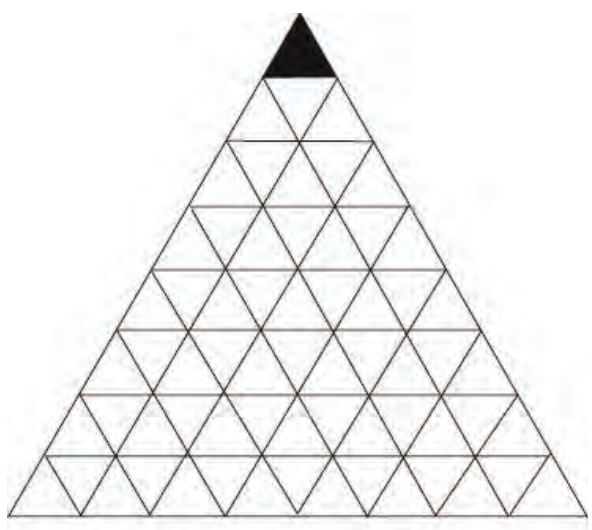
Is it possible to tile an 8×8 board with dominoes (2×1 tiles, which can be placed either horizontally or vertically) so that no two dominoes form a 2×2 square?



A solution can be found in *Algorithmic Puzzles* by Anany Levitin and Maria Levitin, Oxford University Press, 2011, p. 90.

Trapezoid Tiling

An equilateral triangle is partitioned into smaller equilateral triangles by parallel lines dividing each of its sides into 2^n equal segments where n is a positive integer. The topmost equilateral triangle is chopped off, yielding a region like the one shown below for $n = 3$. This region needs to be tiled with trapezoid tiles made of three equilateral triangles of the same size as the triangles composing the region. (Tiles need not be oriented the same way, but they need to cover the region exactly with no overlaps.) Design a divide-and-conquer algorithm for this problem.



A solution can be found in *Algorithmic Puzzles* by Anany Levitin and Maria Levitin, Oxford University Press, 2011, pp. 163−164.

Symmetrix Puzzles

Andy Liu

Symmetrix puzzles are a new craze where pieces are put together to form symmetric figures. They may be rotated or reflected, but may not overlap. In this article, we analyse a three-piece puzzle designed by Vladimir Krasnoukhov of Russia. It consists of a very large $30° - 60° - 90°$ triangles, a similar triangle which is much smaller, and a trapezoid with two right angles and two angles of measures 60° and 120° respectively. These are shown in Figure 1.

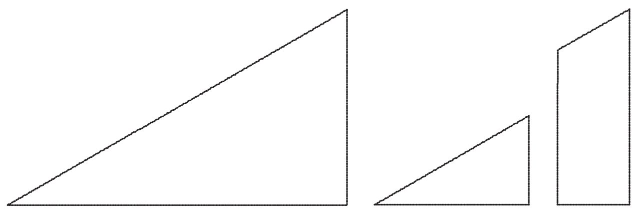


Figure 1

The most probable motivation for this puzzle is the equilateral triangle partitioned into six congruent triangles, as shown in Figure 2 on the left. Two of these triangles are discarded, while three of the remaining ones are combined into a large triangle, as shown in Figure 2 on the right.

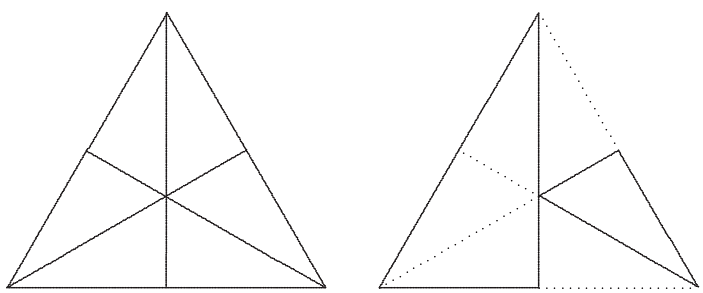


Figure 2

Despite their difference in size, these two pieces can be put together to form a symmetric figure, in two different ways, as shown in Figure 3.

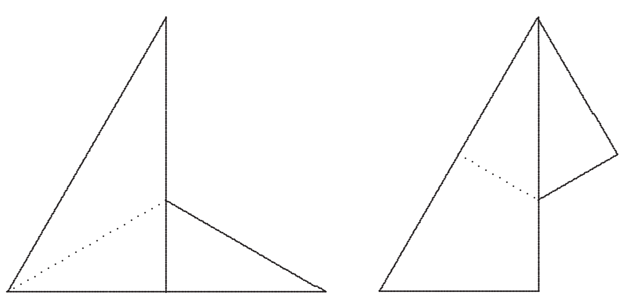


Figure 3

As such, this would not have been much of a puzzle. In a crafty move, the large triangle is further enlarged as shown in Figure 4, and a new trapezoidal piece congruent to the the enlargement is introduced.

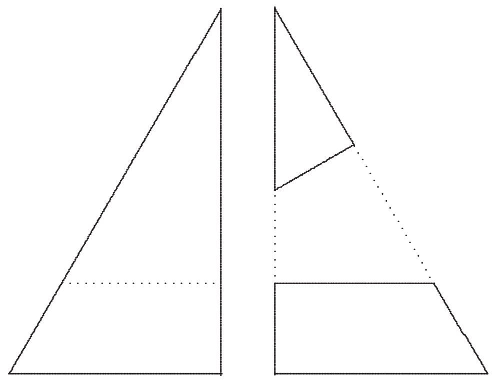


Figure 4

For the puzzle to work, the height of the trapezoid can be chosen arbitrarily. However, if it is too large, then we have two large pieces of more or less the same size, and the psychological impact of one very large piece versus two relatively small ones is lost. This height is chosen to be one-third of that of the original equilateral triangle.

In Figure 4 on the right, we subtract the two small pieces from the large piece, leaving behind a symmetric shape. This leads to the first of two symmetric figures that can be constructed with these three pieces, as shown in Figure 5.

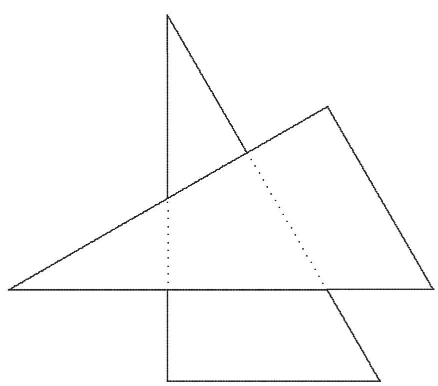


Figure 5

The two small pieces may be subtracted from the large piece in another way, as shown in Figure 6 on the left. This leads to the second solution of the puzzle, as shown in Figure 6 on the right. These two solutions are based on the same idea as those in Figure 3.

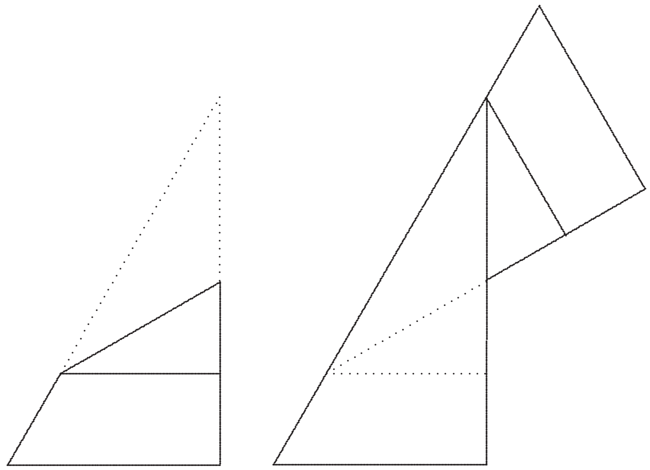


Figure 6

In another crafty move, Alan Tsay of Canada replaces the smaller triangle by an even smaller similar triangle, whose shortest edge is equal in length to the height of the trapezoid. This time, there is only one way in which the two smaller pieces may be subtracted from the large piece in order to leave behind a symmetric shape. This is shown in Figure 7 on the left.

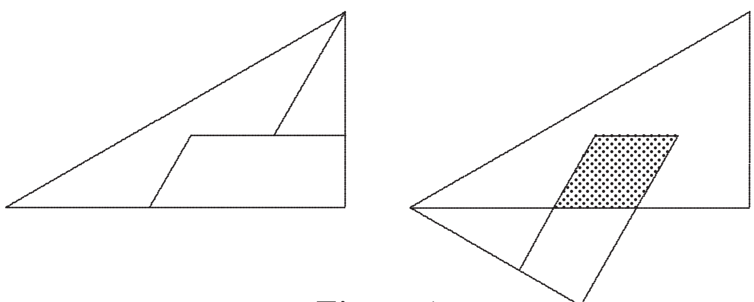


Figure 7

However, when we move the two smaller pieces to the other side, we discover that the trapezoid overlaps the large triangle in a rhombus. This is shaded in Figure 7 on the right.

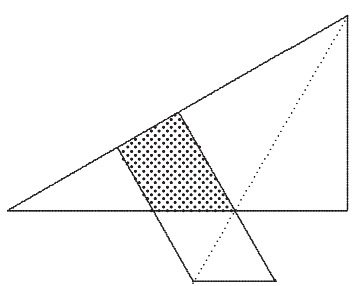


Figure 8

In Figure 8, we subtract the smaller triangle from the large one. Then we take the *symmetric difference* between the trapezoid and what is left of the large triangle, by removing their *intersection*, which is shaded. The symmetric difference consists of a kite left over from the large triangle, and a rhombus from the trapezoid. They have a common axis of symmetry.

This time, we obtain the desired solution shown in Figure 9. Note that this could have emerged had we reflected the two smaller pieces in Figure 7 on the left across the longer leg of the large triangle.

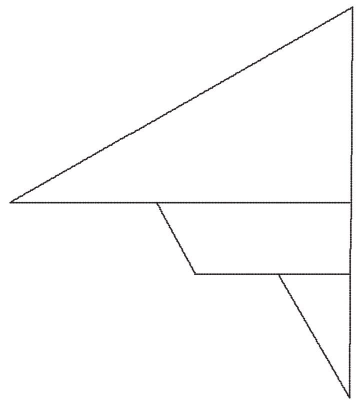


Figure 9

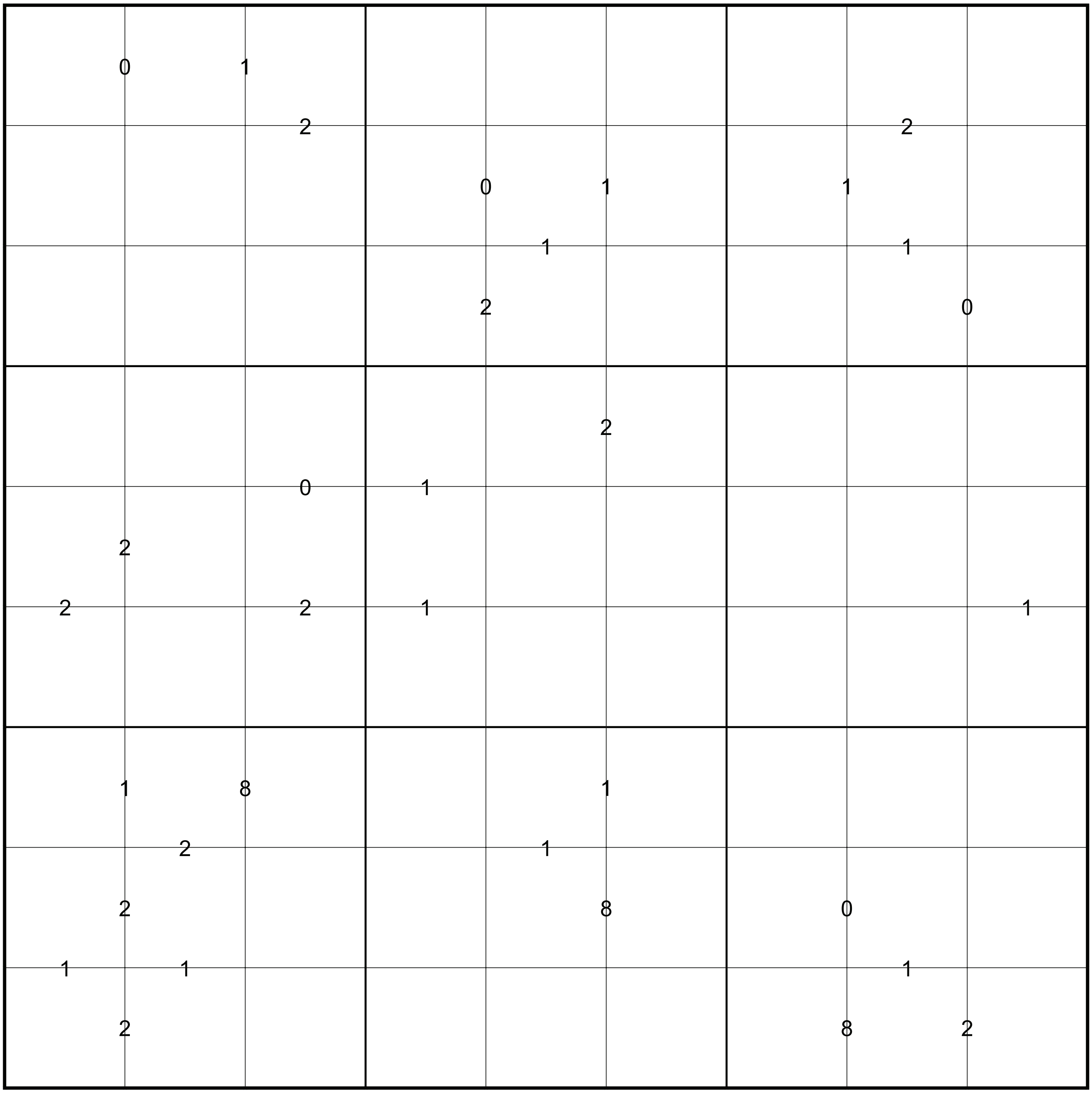
The similarity between the components of these two puzzles may be exploited in many ways. In a small group presentation, one small triangle may be substituted surreptitiously for the other. In a large group presentation, the two puzzles may be handed out to participants seated in alternating columns.

It should be pointed out that subtraction is also a form of taking the symmetric difference. In this case, the intersection happens to be identical to the smaller piece. It may be argued that finding the symmetric differences is not any easier than finding the symmetric figures themselves. Nevertheless, it does give us some additional things to look for, and broadens the avenue of approach to the problem. A good starting point is forming the union of two pieces with some aspect of symmetry.

G4G132018 Sudoku #1

by David Nacin

Fill in the cells so that each row, column, and three by three square cage contain each of the symbols 0,1,2,3,4,8,G the same number of times that they appear in the title.
When adjacent cells in a cage both contain numbers, the difference in some order is given as a clue between them if and only if that difference is 2,0,1 or 8.

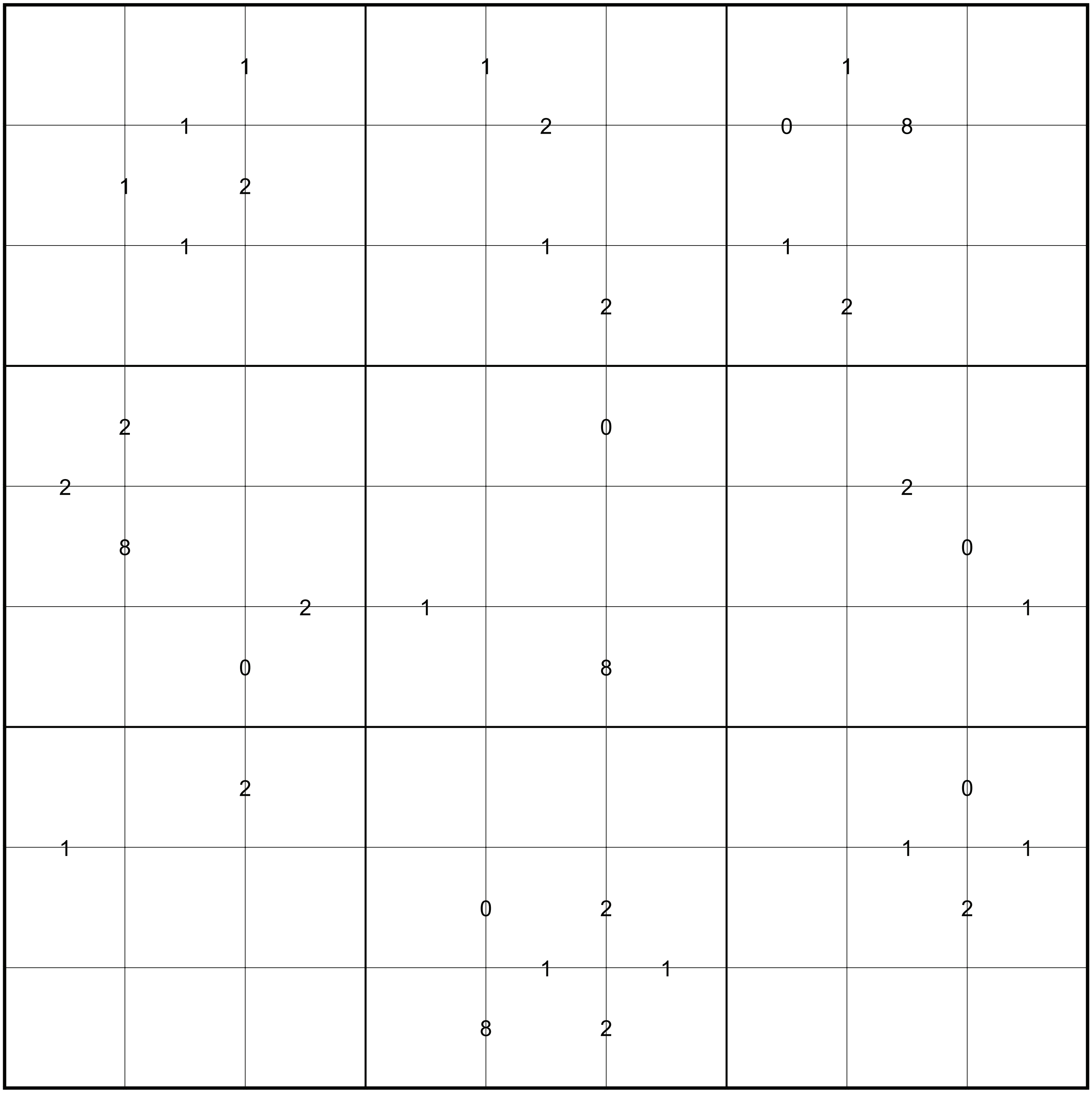


For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

G4G132018 Sudoku #2

Fill in the cells so that each row, column, and three by three square cage contain each of the symbols 0,1,2,3,4,8,G the same number of times that they appear in the title. When adjacent cells in a cage both contain numbers, the difference in some order is given as a clue between them if and only if that difference is 2,0,1 or 8.

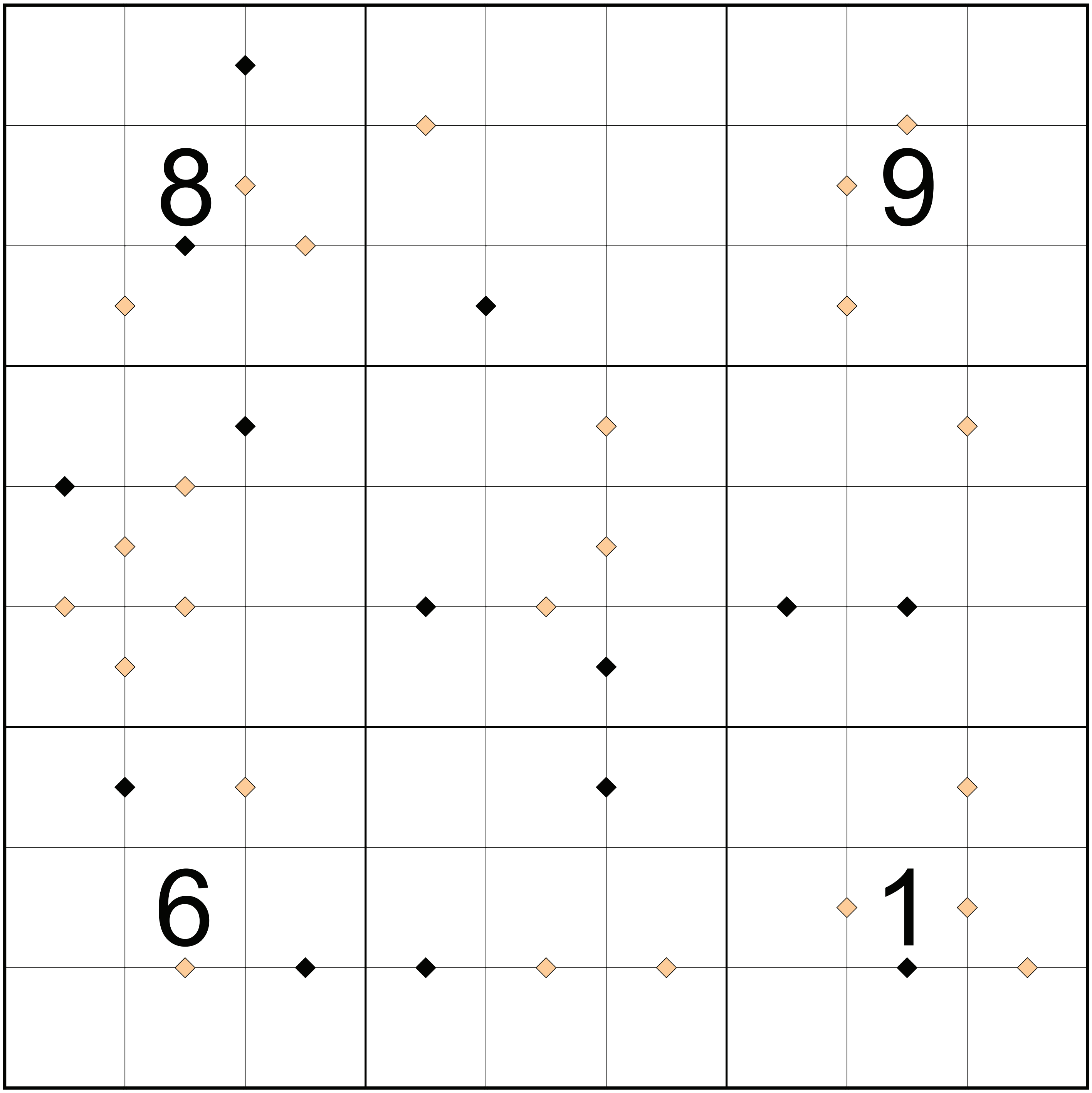


For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

G4G Four Puzzle #1

Fill in the cells so that each row, column, and three by three square cage contain each of the numbers 1 though 9 exactly once. A diamond appears between adjacent cells in a cage if, and only if, the distance between the numbers is four or more. The diamond is black if and only if it is exactly four.

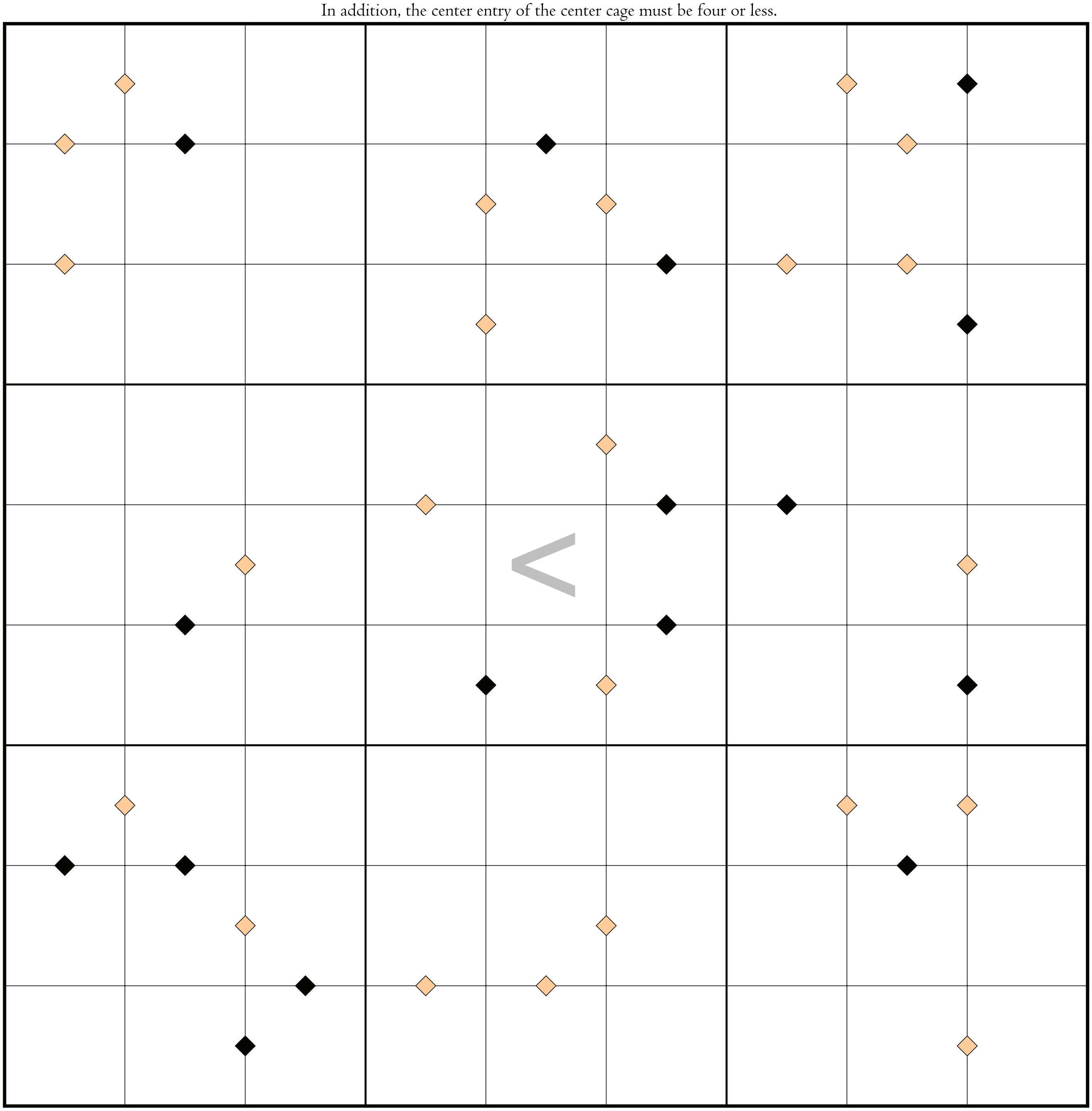


For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

G4G Four Puzzle #2

Fill in the cells so that each row, column, and three by three square cage contain each of the numbers 1 though 9 exactly once. A diamond appears between adjacent cells in a cage if, and only if, the distance between the numbers is four or more. The diamond is black if and only if it is exactly four. In addition, the center entry of the center cage must be four or less.

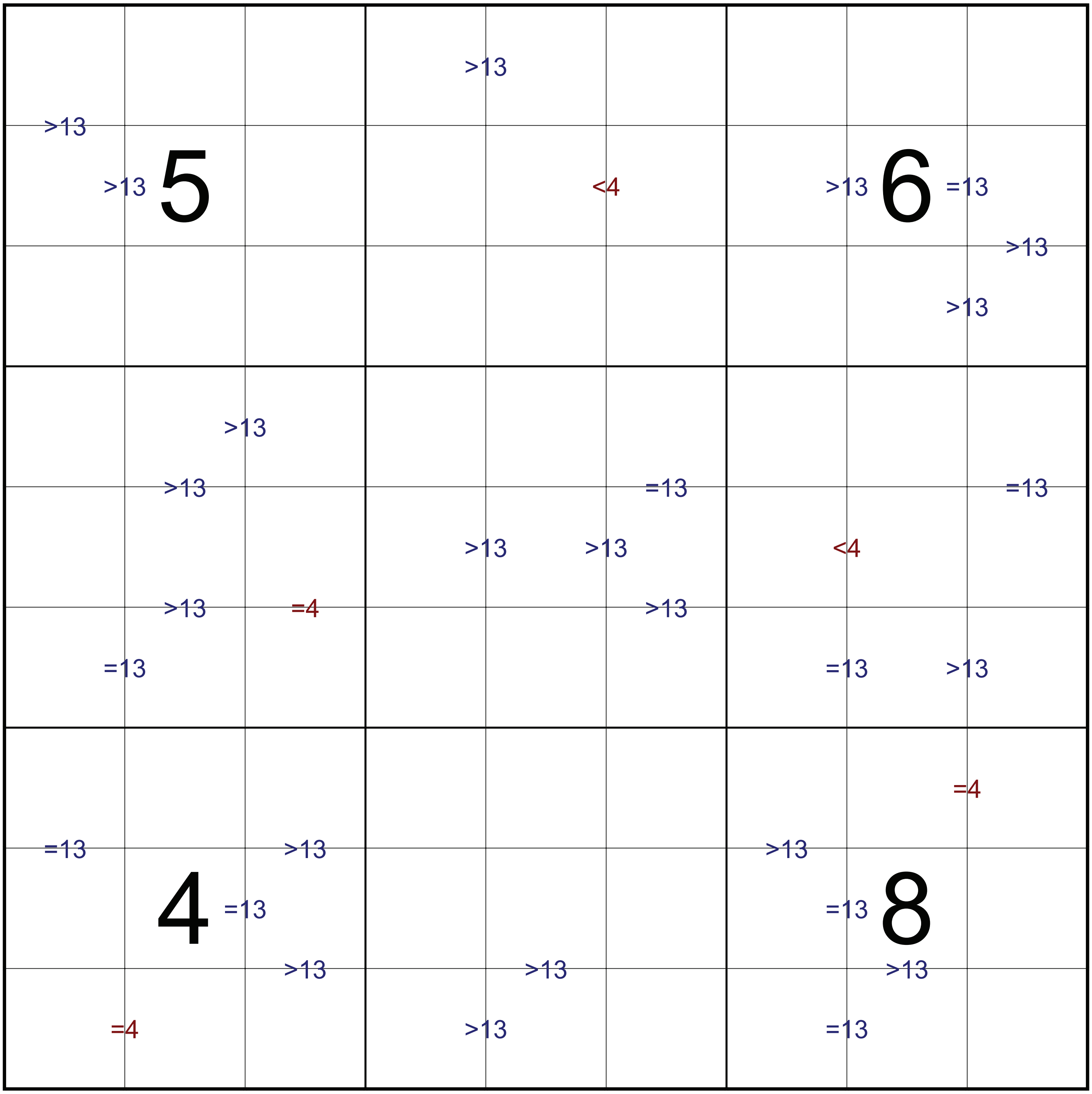


For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

G4G13 High-Low Sudoku #1

Fill in the cells so that each row, column, and three by three square cage contain each of the numbers 1 through 9 exactly once. A clue between adjacent cells in the same cage is given if and only if the sum of those entries is either thirteen or more or four or less.

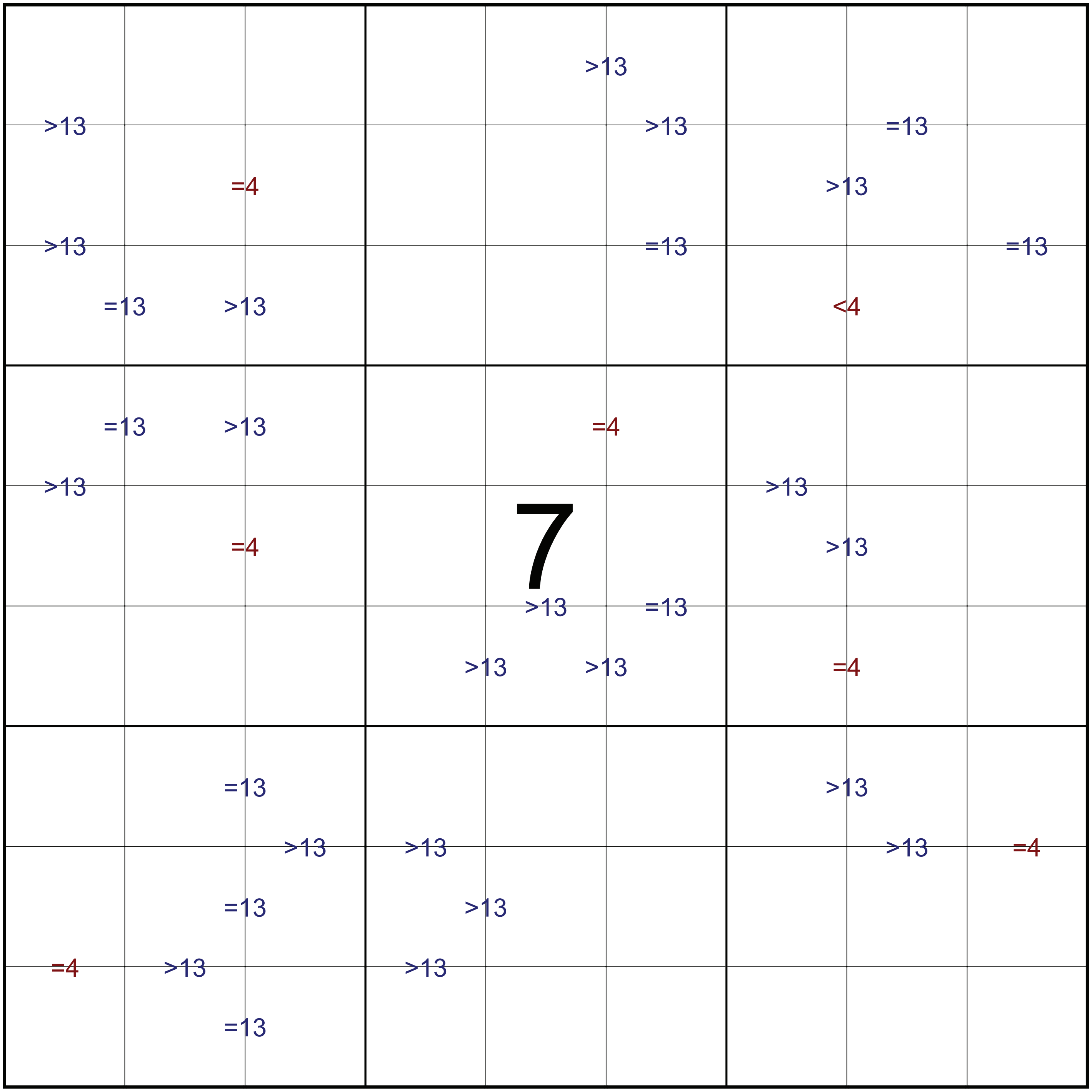


For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

G4G13 High-Low Sudoku #2

Fill in the cells so that each row, column, and three by three square cage contain each of the numbers 1 through 9 exactly once. A clue between adjacent cells in the same cage is given if and only if the sum of those entries is either thirteen or more or four or less.

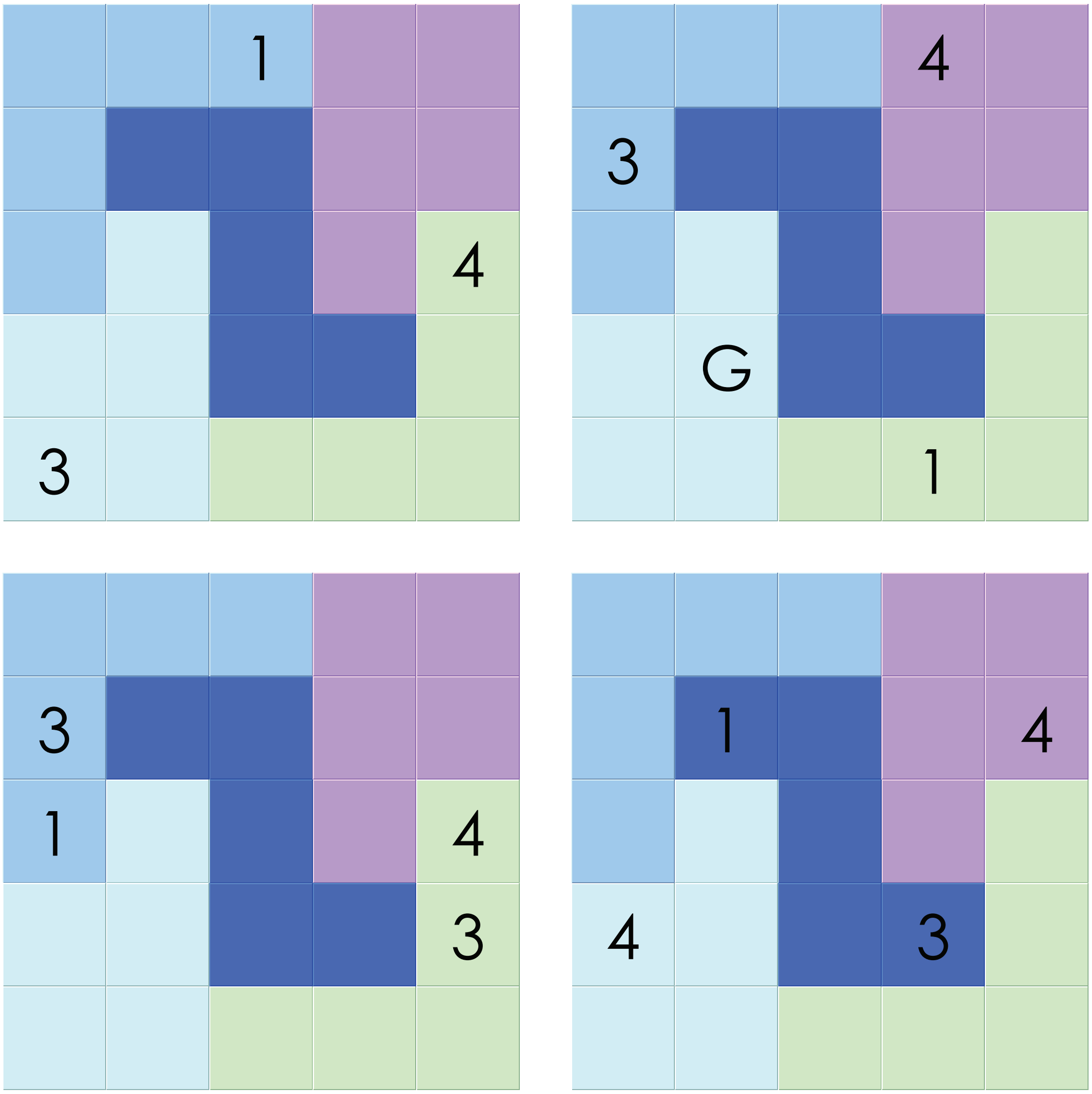


For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

G4G13 Latin Squares

Fill in the cells with the letters G,4,G,1 and 3 so that each number appears exactly once and the letter G appears exactly twice in each row, column and colored region.



For more puzzles visit Quadratablog.blogspot.com

Puzzle by David Nacin

Knight Mazes

Mike Naylor
Matematikkbølgen / Amborneset Center for Mathematics Creativity
7125 Vanvikan Norway
Email: abacaba@gmail.com

Abstract

Knight mazes are a set of squares on a square lattice upon which a chess knight may move. We examine elements of mazes which can be both attractive and puzzling, and discuss two methods of creating mazes.

Knight Mazes I - Elements

A chess knights sits alone on a small island in a peaceful pond. Across the pond, a trophy awaits on another island (Figure 1). A moment's reflection may reveal that the scene is a puzzle – the knight is free to hop from island to island, moving as a chess knight does, with the goal of reaching the trophy. The route to the trophy is riddled with topological surprises, and you are invited to try out the puzzle before reading further.

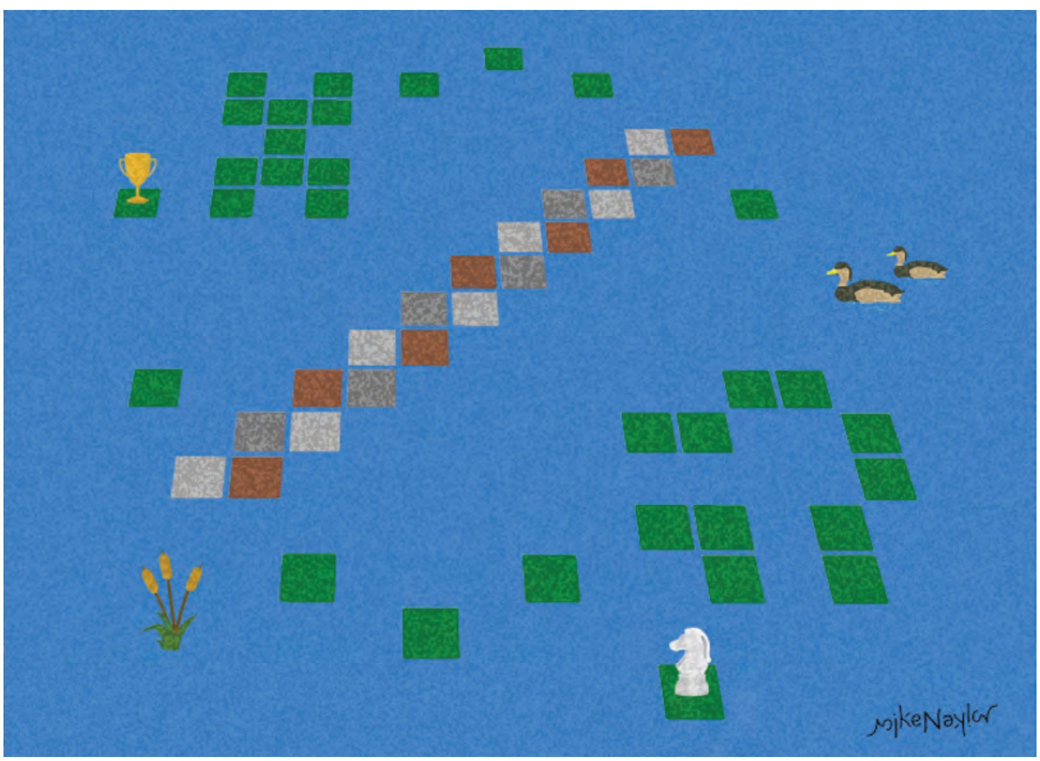


Figure 1: *Knight Maze I - Relax*

The image is an example of a knight maze and is composed of 3 main elements. The first shape to the lower right is a *double loop-the-loop* (see also Figure 2). The knight must hop in a counter-clockwise loop and travel two times around the loop before leaving the area. The central diagonal area is a *triple braided ladder* (see also Figure 3) with the squares are colored in three different colors to help the puzzle-solver distinguish between the three routes. The knight must travel up, down, and up the ladder again before moving on. Squares to side on the top and bottom facillitate switching between the three interwoven pathways. The final element is a *double-Y* (see also Figure 4) which contains a mix of possibilities for jumping, making for a enjoyable and puzzling finish to the maze.

Figure 2: *double loop*

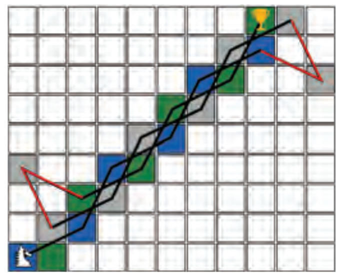


Figure 3: *triple braided ladder*

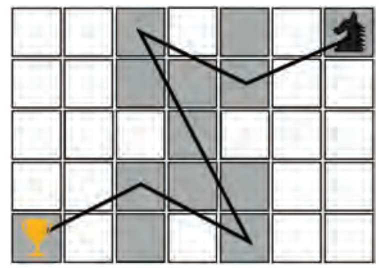


Figure 4: *double-Y*

Knight maze elements are fun to design, and it can be challenging to create shapes that are both attractive and interesting to solve. Figure 5 shows the design of an element based on a square. A knight can travel in a loop of 8 positions that form the outside of a square. By removing one of these positions from the route we break the loop, creating starting and ending points on the square that can be connected to outside positions.

Figure 5: The square

Figure 6: *The square as a maze element*

The movement of the knight allows independent paths to cross and weave around each other. Figure 6 show two octagonal paths beside each other with no connection between them. If the paths are dupicated and shifted down one square, we now have 4 octagons. The octagons are connected pairwise – it is possible to jump between a pair of octagons but not possible to jump to the other pair of octagons. Figure 7 shows the two independent sets of squares colored accordingly. This shape can be used as the basis for a puzzling maze such as the one shown in Figure 8. The knight starts on one color and the goal is on the other color. An extra square is added in a subtle position allowing for transition from one set of colors to the other. Can you discover the square that is the key to solving this maze? (Hint: look for a break in symmetry.)

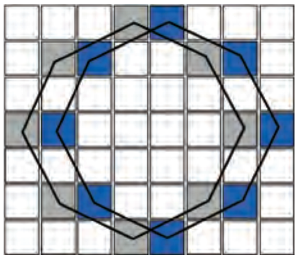


Figure 6: *two octagons*

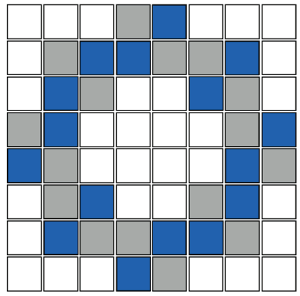


Figure 7: *independent paths*

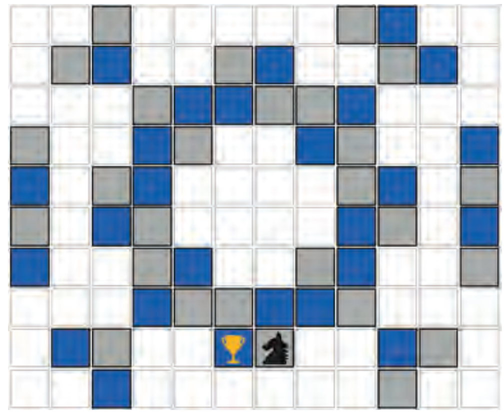


Figure 8: *Maze built on four octagons*

Knight Mazes II - Destructive construction

Figure 9 shows *Knight Maze II - Danger!*, an artwork maze with a rather difficult solution. The reader is encouraged to attempt a solution before reading the details of its construction and thus the key to solving this puzzle.

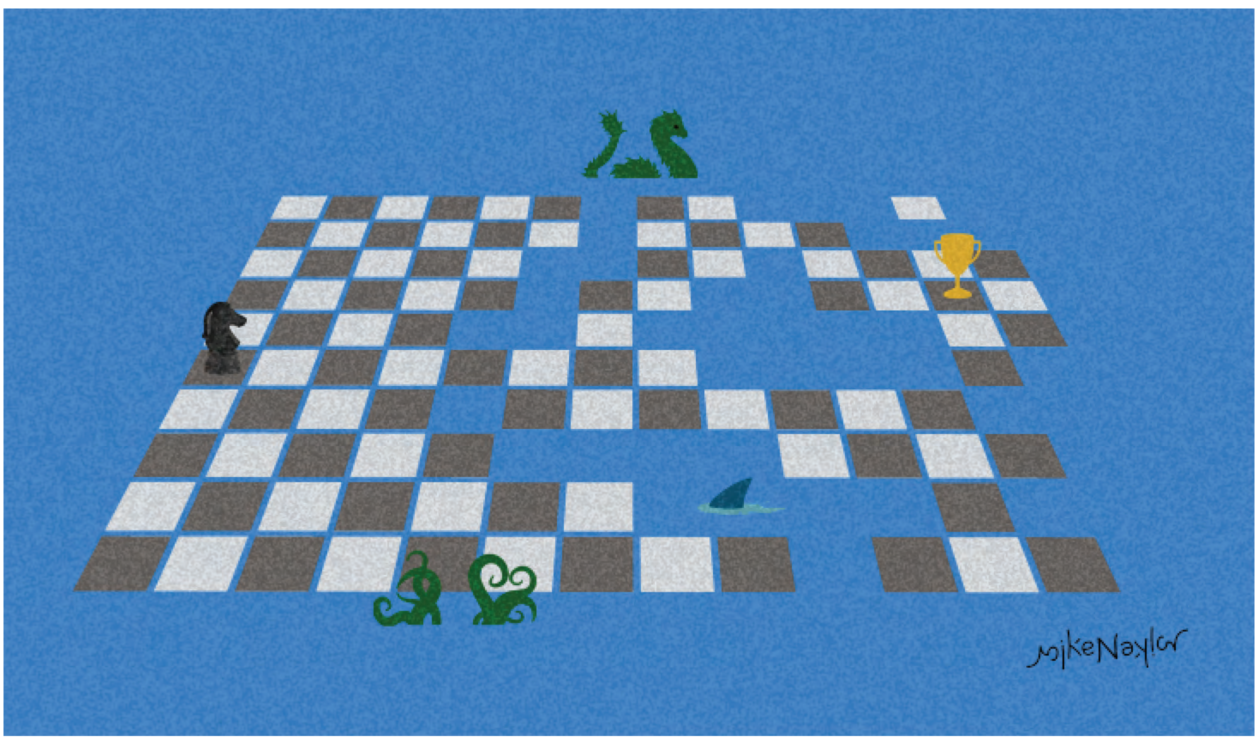


Figure 9: *Knight Maze II – Danger!*

While the earlier knight's mazes are built constructively, adding squares to create interesting routes, we can make a difficult puzzle by building *destructively*, removing squares in order to limit the knight's movements.

In the middle of a blank board, a knight has 8 positions it can move to (Figure 10). If we block all of these positions, a knight is trapped in the center. We then remove one of these blocks and then block all of the positions the knight can move to from this new position as shown in Figure 11. A knight on one of the two squares in the center is free to move back and forth between the two, but cannot go further anywhere else on the board.

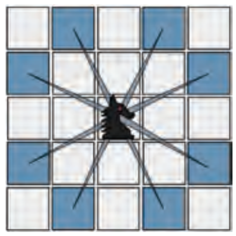


Figure 10: blocked

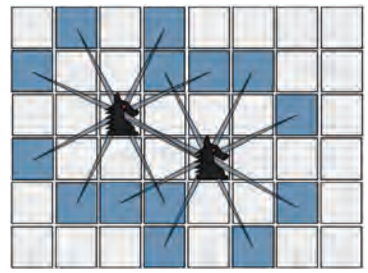


Figure 11: the start of a forced path

By stringing together a chain of such elements, we can create a long pathway with no exits or entrances. Figure 12 shows a chain of seven linked but isolated positions (marked with black dots). The colored squares are both sufficient and necessary to isolate the path.

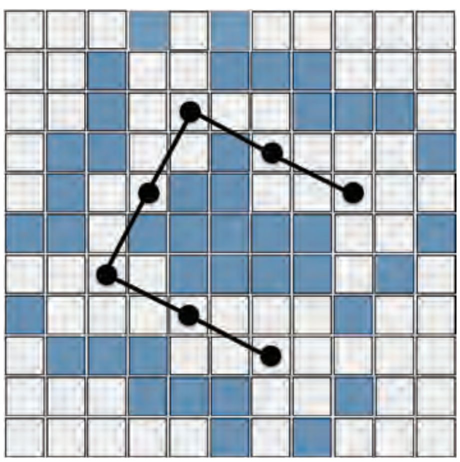


Figure 12: Seven linked but isolated positions

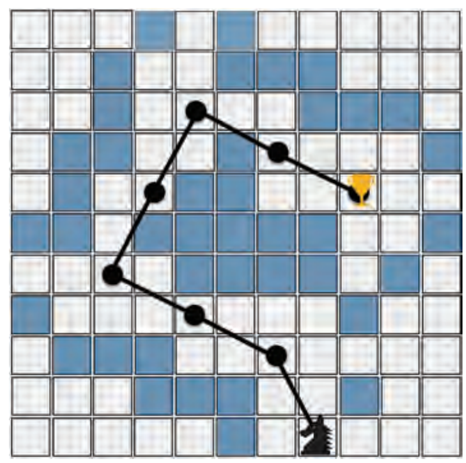


Figure 13: Implementing the chain in the maze

To complete the maze, we add the goal at one end of the chain and remove one of the squares blocking access to the other end of the chain (Figure 13). The chain is now open to the rest of the board. This is the pattern used to create *Knight Maze II – Danger!* The board is extended to the left with plenty of open squares to give a feeling of freedom, but the goal can be reached only by first coming to the key position at the start of the chain.

Endgame

Knight mazes can be fun and surprising. They are also ideal for garden mazes or in public spaces, where participants can hop from tile to tile. A human-size knight maze is currently being built at Amborneset Math Creativity Center in Norway.

Sudoku Ripeto and Custom Sudoku Sampler

by Miguel Palomo

How to play

Complete the board so that each row, each column and each 3 x 3 subsquare contains the symbols:

· M A R T I N · ·

Custom Sudoku — Very easy

·		A		·	R		M	T
R	I		·	**M**	N			
	·		·	**A**	T			
		N	·	**R**	A			M
·				**T**		A		R
A	·		M	**I**	·	T		N
·	R	·	A	**N**			T	
			R	·	M	·	·	A
	A		T	·	I	R	N	

Solution

·	·	A	I	·	R	N	M	T
R	I	T	·	**M**	N	·	A	·
N	·	M	·	**A**	T	I	R	·
I	T	N	·	**R**	A	·	·	M
·	M	·	N	**T**	·	A	I	R
A	·	R	M	**I**	·	T	·	N
·	R	·	A	**N**	·	M	T	I
T	N	I	R	·	M	·	·	A
M	A	·	T	·	I	R	N	·

· G A R D N E R ·

Custom Sudoku — Easy

R	D						R	E
	·	N	E	R	R	·		
				N	D	·		R
	R				·	G	N	
·	**G**	**A**	**R**	**D**	**N**	**E**	**R**	**·**
·	N		G	R	A			
N			R	G	·	A	E	
G			N	A		R		·
	A	·	D		R	R		N

Solution

R	D	·	A	·	G	N	R	E
A	·	N	E	R	R	·	D	G
R	E	G	·	N	D	·	A	R
D	R	R	·	E	·	G	N	A
·	**G**	**A**	**R**	**D**	**N**	**E**	**R**	**·**
·	N	E	G	R	A	D	·	R
N	·	R	R	G	·	A	E	D
G	R	D	N	A	E	R	·	·
E	A	·	D	·	R	R	G	N

G · 4 · G · 1 · 3

Custom Sudoku

Medium

		1	G		G		3	·
		G				G	4	
	G					G	1	·
·			G	·				G
G	**·**	**4**	**·**	**G**	**·**	**1**	**·**	**3**
			1	3	·	·		4
	G		·	·				1
		·				3		·
G	3	·	·	4		·	·	

Solution

4	·	1	G	·	G	·	3	·
·	·	G	3	1	·	G	4	·
3	G	·	4	·	·	G	1	·
·	1	3	G	·	4	·	·	G
G	**·**	**4**	**·**	**G**	**·**	**1**	**·**	**3**
·	·	G	1	3	·	·	G	4
·	G	·	·	·	3	4	G	1
1	4	·	·	G	G	3	·	·
G	3	·	·	4	1	·	·	G

· A T L A N T A ·

Custom Sudoku

Difficult

	·	N					·	A
A		·			·		N	
T	T				L	·		A
L					·	N		
·	**A**	**T**	**L**	**A**	**N**	**T**	**A**	**·**
	T	·	·		A			
T	·			·	A			N
·		L		N		A		T
A	N	A	T		T	·	A	

Solution

A	·	N	A	T	T	L	·	A
A	L	·	A	A	·	T	N	T
T	T	A	N	·	L	·	A	A
L	A	A	T	T	·	N	·	A
·	**A**	**T**	**L**	**A**	**N**	**T**	**A**	**·**
N	T	·	·	A	A	A	T	L
T	·	T	A	·	A	A	L	N
·	A	L	·	N	A	A	T	T
A	N	A	T	L	T	·	A	·

M C · E S C H E R

Custom Sudoku

Very difficult

	H					R	·	
C		C	·					E
		R	E					C
H		C			M			
M	**C**	**·**	**E**	**S**	**C**	**H**	**E**	**R**
	E	S			·			C
						S		
	C		M		S			
R		M		·	H			

Solution

E	H	E	S	C	C	R	·	M
C	M	C	·	H	R	E	S	E
S	·	R	E	M	E	C	H	C
H	R	C	C	E	M	·	E	S
M	**C**	**·**	**E**	**S**	**C**	**H**	**E**	**R**
E	E	S	H	R	·	M	C	C
C	E	H	R	C	E	S	M	·
·	C	E	M	E	S	C	R	H
R	S	M	C	·	H	E	C	E

2 3 4 5 6 5 4 3 2

Sudoku Ripeto

Very easy

3	5		2	3		6	2	4
6		3	2	4	3	5	4	5
	4		5			2	3	
2			4	2	3	5		
2	**3**	**4**	**5**	**6**	**5**	**4**	**3**	**2**
	4	5	3		4	3	2	
4	3	2			2	3		5
3	2	4		5	5	2	6	3
	5	6	3	3		4	4	2

Solution

3	5	5	2	3	4	6	2	4
6	2	3	2	4	3	5	4	5
4	4	2	5	5	6	2	3	3
2	6	3	4	2	3	5	5	4
2	**3**	**4**	**5**	**6**	**5**	**4**	**3**	**2**
5	4	5	3	2	4	3	2	6
4	3	2	6	4	2	3	5	5
3	2	4	4	5	5	2	6	3
5	5	6	3	3	2	4	4	2

F I B O N A C C I

Custom Sudoku

Easy

B		C	A	**F**			I	
F	N		C	**I**			B	
O		A	C	**B**	I	C		F
I		F		**O**	C	I		N
C		O		**N**		A		B
C		N		**A**	I	F	C	
	C		O	**C**			F	
	O	I	N	**C**	F			C
				I	B	N	O	C

Solution

B	C	C	A	**F**	N	O	I	I
F	N	I	C	**I**	O	C	B	A
O	I	A	C	**B**	I	C	N	F
I	A	F	B	**O**	C	I	C	N
C	I	O	F	**N**	C	A	I	B
C	B	N	I	**A**	I	F	C	O
N	C	B	O	**C**	A	I	F	I
I	O	I	N	**C**	F	B	A	C
A	F	C	I	**I**	B	N	O	C

7 5 6 8 4 8 6 5 7

Sudoku Ripeto

Medium

5	6		8		8	4		
4			5	5			7	
6			6	7	4		8	
	7	5	7		5		4	6
7	**5**	**6**	**8**	**4**	**8**	**6**	**5**	**7**
8		4			5		5	
		6			7	5	8	
	4		6	5		8	7	7
	5	8	4		7		6	6

Solution

5	6	7	8	7	8	4	6	5
4	8	7	5	5	6	6	7	8
6	8	5	6	7	4	7	8	5
8	7	5	7	6	5	8	4	6
7	**5**	**6**	**8**	**4**	**8**	**6**	**5**	**7**
8	6	4	7	6	5	7	5	8
6	7	6	5	8	7	5	8	4
5	4	8	6	5	6	8	7	7
7	5	8	4	8	7	5	6	6

P R I M A L I T Y

Custom Sudoku

Difficult

			T	**P**			M	A
	I	M	A	**R**		I		
A			Y	**I**				
L		I	P	**M**		R		
	R	Y		**A**				I
P	I			**L**				
I	T			**I**				
		L	M	**T**				P
R			L	**Y**	P		I	

Solution

Y	L	R	T	**P**	I	I	M	A
T	I	M	A	**R**	L	I	P	Y
A	P	I	Y	**I**	M	L	T	R
L	A	I	P	**M**	Y	R	I	T
M	R	Y	I	**A**	T	P	L	I
P	I	T	I	**L**	R	Y	A	M
I	T	P	R	**I**	A	M	Y	L
I	Y	L	M	**T**	I	A	R	P
R	M	A	L	**Y**	P	T	I	I

1 2 2 3 3 3 2 2 1

Sudoku Ripeto

Very difficult

		2			2	3		2
	3				3		2	
3		3	2			1		
2	3		2			3	1	3
1	**2**	**2**	**3**	**3**	**3**	**2**	**2**	**1**
3			2	1		2	2	
3	1				1			
2	2				1	2		1
	2	1				2		

Solution

1	2	2	1	3	2	3	3	2
2	3	2	1	3	3	1	2	2
3	1	3	2	2	2	1	3	2
2	3	2	2	1	2	3	1	3
1	**2**	**2**	**3**	**3**	**3**	**2**	**2**	**1**
3	3	1	2	1	2	2	2	3
3	1	3	2	2	1	3	2	2
2	2	3	3	2	1	2	3	1
2	2	1	3	2	3	2	1	3

· S C I E N C E ·

Custom Sudoku

Very easy

N	·	E		S	I	·	C	
S	E						·	E
			·	·				S
I			C	C	E	E		·
·	**S**	**C**	**I**	**E**	**N**	**C**	**E**	**·**
C	E	E	·	·	S	C		N
E	C	·	S	N	·	E	C	I
E		S	C	E			N	C
·	N	C	E	I	C	S		E

Solution

N	·	E	E	S	I	·	C	C
S	E	·	N	C	C	I	·	E
C	C	I	·	·	E	N	E	S
I	·	N	C	C	E	E	S	·
·	**S**	**C**	**I**	**E**	**N**	**C**	**E**	**·**
C	E	E	·	·	S	C	I	N
E	C	·	S	N	·	E	C	I
E	I	S	C	E	·	·	N	C
·	N	C	E	I	C	S	·	E

3 4 5 6 5 6 5 4 3

Sudoku Ripeto

Easy

4	3	5	6		5	3	5	4
	5	4		5	3			5
	3		5		3	6	5	4
	5	5		4	4			3
3	**4**	**5**	**6**	**5**	**6**	**5**	**4**	**3**
	4		5	3	5	4		6
	6	3	5	5		4		
		4		3		6		6
	6		4	6	4	5		5

Solution

4	3	5	6	6	5	3	5	4
6	5	4	4	5	3	3	6	5
5	3	6	5	4	3	6	5	4
6	5	5	3	4	4	5	6	3
3	**4**	**5**	**6**	**5**	**6**	**5**	**4**	**3**
3	4	6	5	3	5	4	5	6
4	6	3	5	5	6	4	3	5
5	5	4	3	3	5	6	4	6
5	6	3	4	6	4	5	3	5

M · A · G · I · C

Custom Sudoku

Medium

			C	**M**	G	·	·	
	·			**·**	·			M
				A			·	
·			I	**·**		G	M	C
·	I			**G**	·	A		·
			A	**·**			·	
		·	·	**I**	M			A
I		G		**·**		M		
	·	M		**C**	A			

Solution

A	·	·	C	**M**	G	·	·	I
G	·	I	·	**·**	·	C	A	M
C	M	·	·	**A**	I	·	·	G
·	·	A	I	**·**	·	G	M	C
·	I	C	M	**G**	·	A	·	·
M	G	·	A	**·**	C	I	·	·
·	C	·	·	**I**	M	·	G	A
I	A	G	·	**·**	·	M	C	·
·	·	M	G	**C**	A	·	I	·

9 9 7 5 5 5 7 9 9

Sudoku Ripeto

Difficult

	9	9	7		5	9		9
5	5	9			9	7		
	5		9	9	9	5	7	
5	7		9	7		9	5	
9	**9**	**7**	**5**	**5**	**5**	**7**	**9**	**9**
					9			
5	9					5		5
9	7		5			9	9	7
		7	5	9		9	9	

Solution

7	9	9	7	5	5	9	5	9
5	5	9	7	5	9	7	9	9
7	5	9	9	9	9	5	7	5
5	7	5	9	7	9	9	5	9
9	**9**	**7**	**5**	**5**	**5**	**7**	**9**	**9**
9	9	5	9	7	9	5	5	7
5	9	9	9	9	7	5	7	5
9	7	5	5	9	5	9	9	7
9	5	7	5	9	7	9	9	5

Custom Sudoku

Very difficult

		L	·			·	Z	E
				Z	S		·	Z
L		S					U	
·	**P**	**U**	**Z**	**Z**	**L**	**E**	**S**	**·**
	E	Z		·		P		
S		E		L	·			
U	L	·	E	S				Z
P			U					S

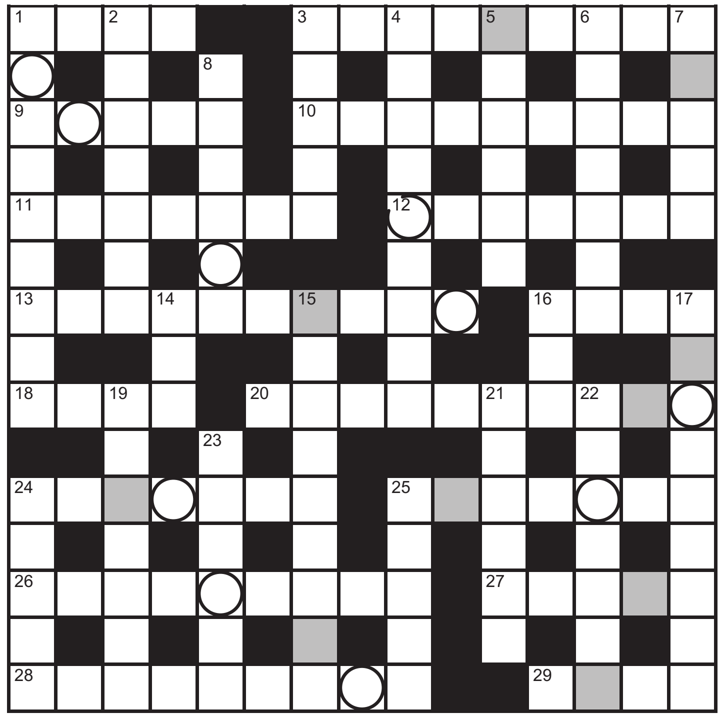
Solution

Z	S	L	·	U	P	·	Z	E
E	·	P	L	Z	S	U	·	Z
Z	U	·	Z	E	·	S	L	P
L	Z	S	·	P	E	Z	U	·
·	**P**	**U**	**Z**	**Z**	**L**	**E**	**S**	**·**
·	E	Z	S	·	U	P	Z	L
S	Z	E	P	L	·	Z	·	U
U	L	·	E	S	Z	·	P	Z
P	·	Z	U	·	Z	L	E	S

The Ambidextrous Puzzle

a G4G cryptic crossword by Henri Picciotto

Italicized clues consist of definitions for two ambidextlousry related words. Which of the two you enter into the diagram will become clear if you make sure that when the diagram is filled, you can shade in three additional squares, and circle two additional letters, so that all 13 shaded squares, and all 12 circled letters respectively spell relevant two-word phrases if read from left to right in each row, top row to bottom row.



Across

1 *Accommodated a flock (4)*
3 Café is toxic: head of management requests protective gear (4,5)
9 Affected one thousand new wave rockers' comeback (5)
10 Commercial time interrupted by guys and dudes, for a change (9)
11 *Officer and medical investigator (7)*
12 *Lit relatives (7)*
13 To dine poorly, ingest LSD, facing backwards to show real commitment (10)
16 Early man's a mother (4)
18 *Peasant personality (4)*
20 Imploring search engine to include rotten cheese (10)
24 Most significant (optimal) outside irregular gig (7)
25 *Nonstandard and brave (7)*
26 Confines Henry in (oops!) non-union establishments (4,5)
27 Digression from unorthodox ideas (5)
28 Fraudulent course by Rolling Stone (9)
29 *Fresh skin (4)*

Down

1 Murders Vietnamese leader with audio equipment on fateful date (9)
2 *Admired and knocked down (7)*
3 *Weak talent (5)*
4 Heck! Bee on broken part of skull (9)
5 Arab city switching final pair from the middle of the road (6)
6 *Driven and strengthened (7)*
7 Full of chopped dates (5)
8 Unspoiled in Sweden: iceberg (6)
14 Is equivalent to stiffer core (3)
15 Blackjack: Cheating Tony's left inside at end of game (6-3)
16 Tree has deteriorated (3)
17 Most powerful, outwardly goth connections in New England university (9)
19 *Move backwards, missing limbs (7)*
21 *Gather together and apprehend (6)*
22 Visualize one crazy enigma (7)
23 Reaches exit, having failed test (4,2)
24 Derek, seen over reference work, expressed disapproval (5)
25 Call 5 was tagged, apparently (5)

How to solve cryptic crosswords: go to www.picciotto.org/hot and scroll down to **Cryptics, How To**

ALICE IN WONDERLAND

For G4G13

Presented by Emmanuelle Malte Salvatore, Todd Wilk Estroff

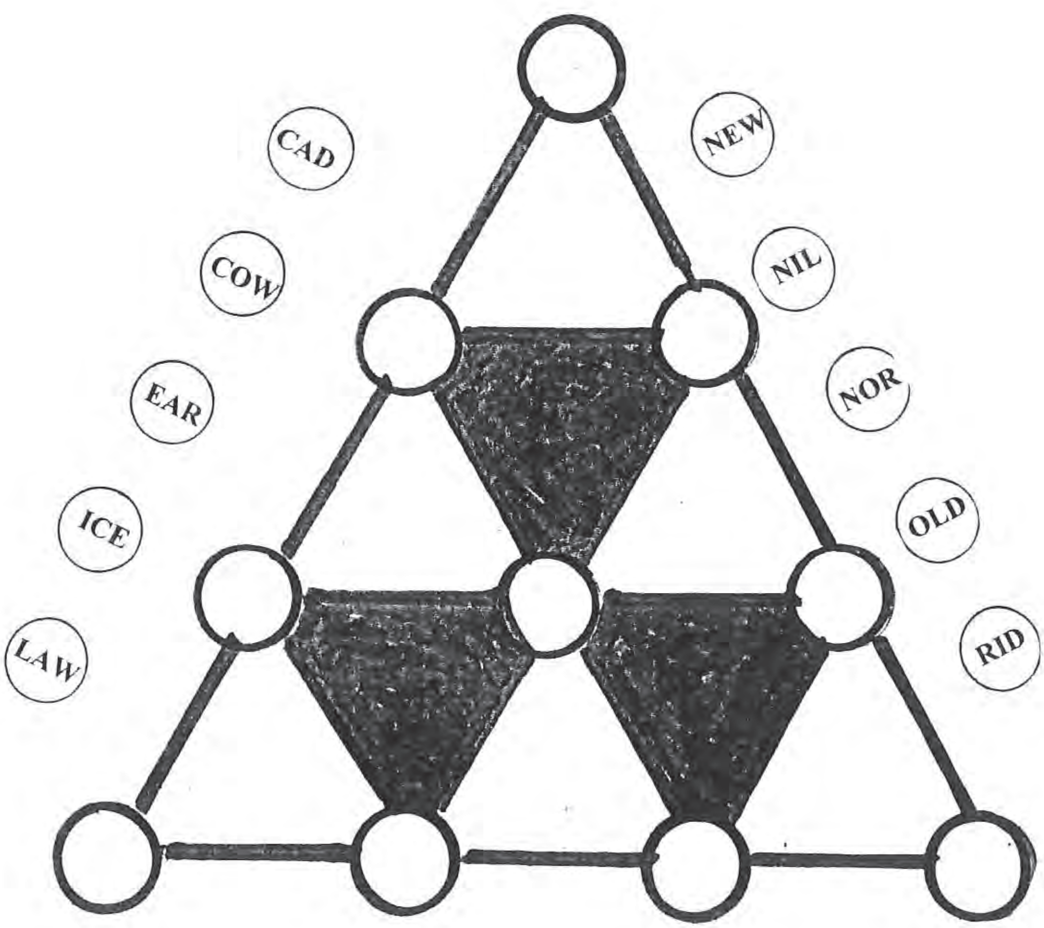
and Jeremiah Farrell

Each of the ten different letters in the title is used exactly three times to form the words in the circles. Martin Gardner's famous work *The Annotated Alice* was first published in 1960 and we honor him in this essay.

Puzzle 1. Place the ten words on the diagram so that each corner of ten equilateral triangles contain words with a common letter.

Puzzle 2. Place the letters in the nodes so that 10 triangles spell the given words.

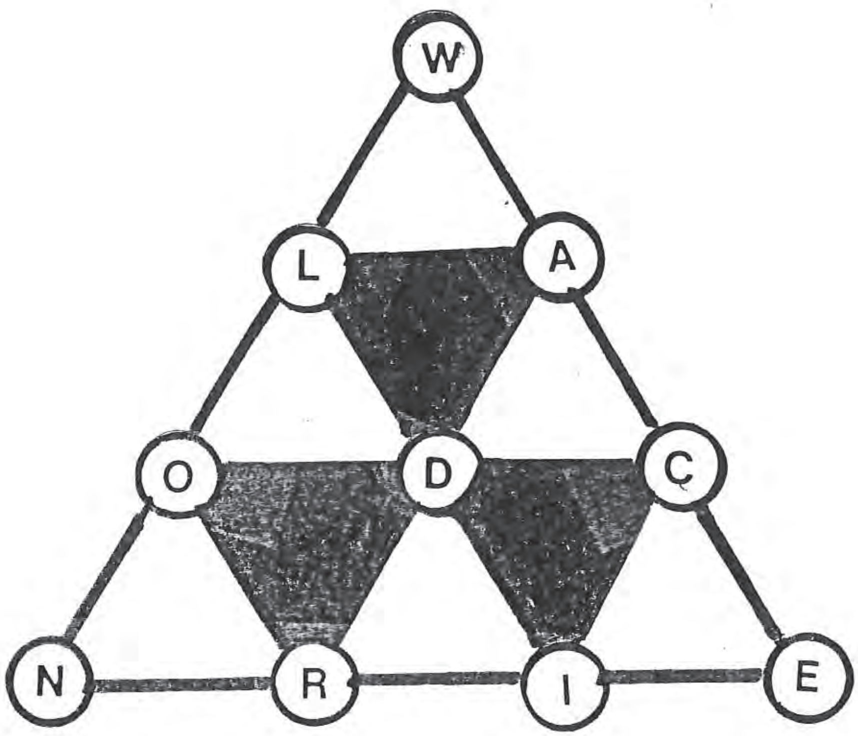
A Game. Two players alternately select words and the first to select three words with a common letter wins.

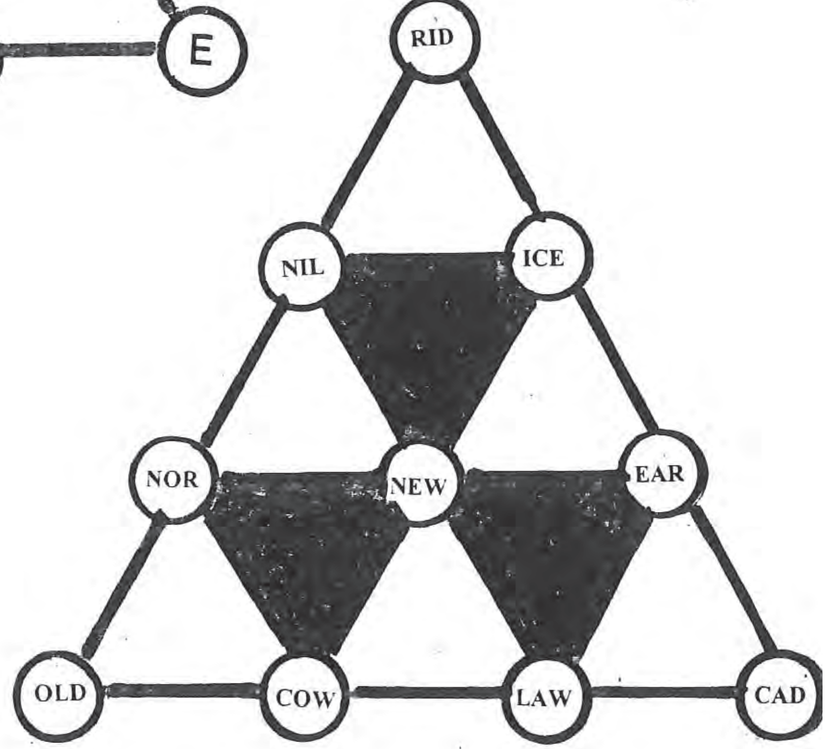


Hint: Each word has exactly three other words with which it shares no letter. Four of the words on their three contain a common letter. Where on the diagram must these four words go?

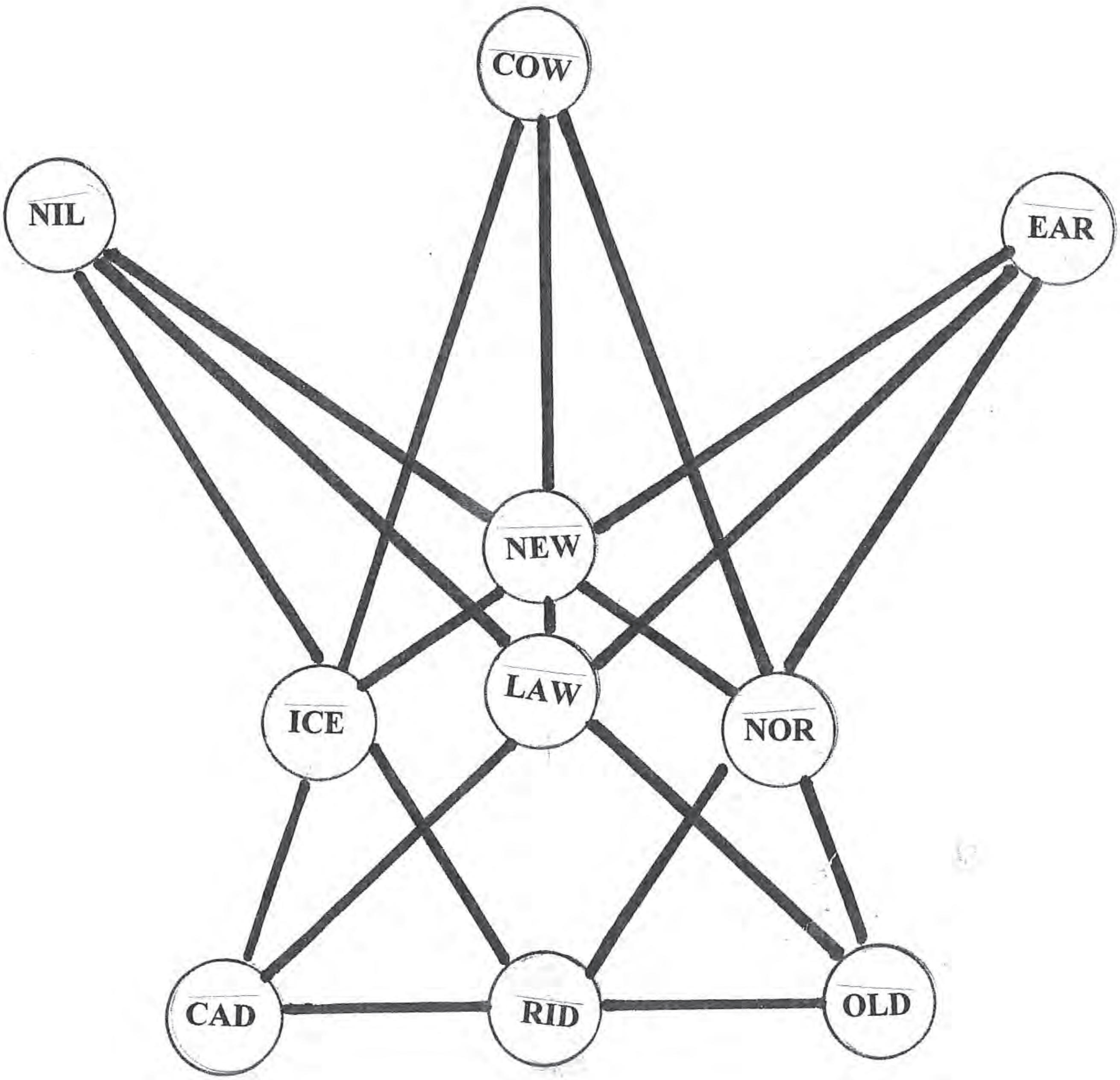
SOLUTIONS: As usual with symmetric configurations, the nodes and lines may often be interchanged.

For the Game, first can win by forcing second to waste a turn by choosing a node that doesn't help him.





In addition to the triangle puzzle, the words can be placed on the nodes of the following diagram so that the ten lines collect three words with a common letter. This diagram is called "Fool's Cap" in the article "Configuration Games" by Jeremiah Farrell, Martin Gardner and Thomas Rodgers. It appears in *Tribute to a Mathemagician*, AK Peters, 2005. Edited by B. Cipra, E.D. Demaine, M. L. Demaine and T. Rodgers.



Edgy Puzzles

Karl Schaffer
karl_schaffer@yahoo.com

Countless puzzles involve decomposing areas or volumes of two or three-dimensional figures into smaller figures. “Polyform” puzzles include such well-known examples as pentominoes, tangrams, and soma cubes. This paper will examine puzzles in which the edge sets, or "skeletons," of various symmetric figures like polyhedra are decomposed into multiple copies of smaller graphs, and note their relationship to representations by props or body parts in dance performance.

The edges of the tetrahedron in Figure 1 are composed of a folded 9-gon, while the cube and octahedron are each composed of six folded paths of length 2. These constructions have been used in dances created by the author and his collaborators. The photo from the author’s 1997 dance “Pipe Dreams” shows an octahedron in which each dancer wields three lengths of PVC pipe held together by cord at the two internal vertices labeled *a* in the diagram on the right. The shapes created by the dancers, which might include whimsical designs reminiscent of animals or other objects as well as mathematical forms, seem to appear and dissolve in fluid patterns, usually in time to a musical score.

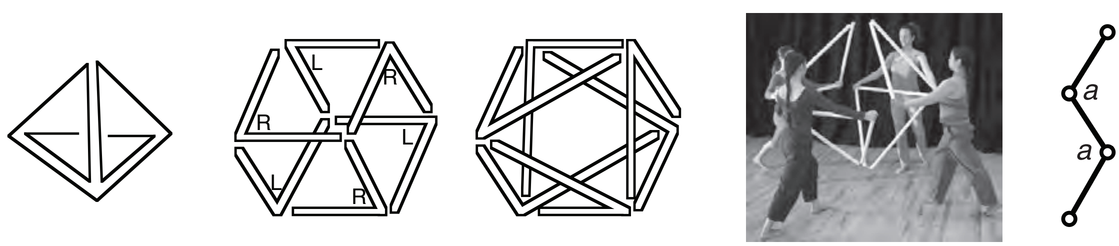
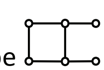
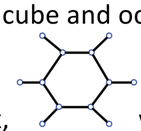
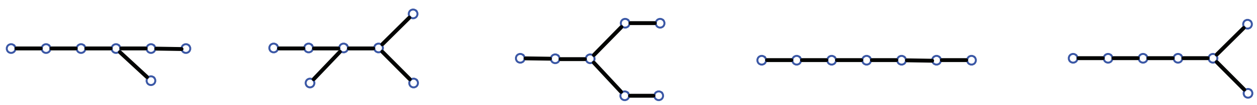


Figure 1. PVC pipe polyhedral skeletons used in dances

The desire in the dance company co-directed by the author to incorporate polyhedra into dance works led to these constructions, and to similar designs with loops of rope, fingers and hands, and the bodies of dancers. Just as mathematical concepts often suggest artistic explorations for those involved in the interplay between these fields, performance problems may suggest mathematical questions, in this case involving finding efficient and symmetrical ways to construct the skeletons, or edge sets, of the Platonic solids.

In one performance, we present an audience member with the puzzle of folding this shape  , constructed from PVC pipe sections which fold at the vertices, into a tetrahedron. In the 2009 music and dance concert *Harmonious Equations* [2] we gave ourselves the puzzle of folding one shape wielded by three dancers into a cube and octahedron, and came up with a PVC pipe hexagon with pendant edges at each hexagon vertex,  which also folds to form a doubled-edge tetrahedron. In [4] the authors showed classroom activities involving making polyhedra with PVC pipe, fingers, and loops of string; George Csicsery documented the latter two of these in a series of short films [1]. In various papers the author investigated modular constructions of the Platonic solids in a manner reminiscent of modular origami: in [5] the author showed how to construct the five Platonic solids with six loops of three colors, in [3] with length six PVC pipe modules, and with the bodies of six dancers, and in [6] constructions of the Archimedean solids and various plane tessellations with one six-edge tree.

In this paper, we will explore a variety of puzzles derived from constructions like those described above, in this case using multiple copies of small trees made from paper straws. Similar puzzles can also be created with simple paper diagrams. Here's a simple example of five graphs called trees, several of which fold at the vertices to give the skeleton or edge set of a regular tetrahedron (which ones?). Here $T_n(a,b,c)$, for example, indicates a tree with n edges and pendant edges of lengths a, b, and c. (Note: this notation may not uniquely specify a graph for larger examples than we are considering here.)



$T_6(1,2,3)$ $T_6(1,1,1,2)$ $T_6(2,2,2)$ $T_6(6)$ $T_6(1,1,4)$

Figure 2. Trees which might fold to a tetrahedron (which ones?).

Over the last twenty years, since we began incorporating such polyhedra into our dance works, the author has created a variety of such puzzles, and I imagined it might be a good idea to find a way to market physical examples of the puzzles. Recently, however, I had an epiphany and decided to try to answer the question, "What would Mary Laycock do?" Mary Laycock was a pioneer In the use of manipulatives and physical activities in math classes. She wrote a book, *Straw Polyhedra*, which is still in publication, in which she showed how to use straws and bobby pins to construct polyhedra very simply and inexpensively. So, I decided to find a way to construct physical edgy puzzles for very little money, as a kind of homage to Mary Laycock.

By the way, Mary Laycock was a follower of Zoltan Dienes, a math educator who created numerous whole body and dance class activities for elementary and middle school students. Dienes was the son of Valeria Dienes, a prominent Hungarian dancer and choreographer who invented a somewhat mathematical dance notation. She also brought her family to live in the commune established in Greece by Isadora Duncan's brother, at which dance was an integral part of the schooling of young Zoltan. So, there's a nice dance history connection here as well!

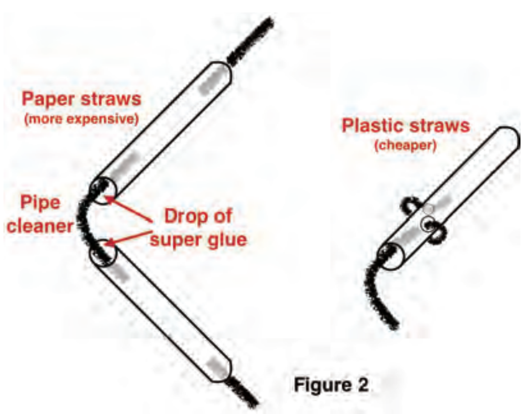


Figure 2

The simple construction method I've found most useful is to use paper straws for the edges, pipe cleaners to join them together at the vertices, and a drop of super glue to hold everything together (Figure 2). The pipe cleaners are flexible, yet hold their shapes, and the short pipe cleaner "tabs" at the ends allow the easy construction of three dimensional models.

I've found that paper straws are more expensive than plastic, but the glue does not hold to the plastic very well, and students playing with puzzles built with glue and plastic straws tend to pull them apart too easily! I have had some success punching holes in plastic straws and threading pipe cleaners through them, as shown on the right in Figure 2, but this is much more labor intensive than the method using paper straws. Glen Whitney (founder of MoMath) tells me that restaurants are beginning to replace plastic with paper straws, so we expect – or hope - that the price of paper straws will soon drop.

Puzzles

Below are a collection of puzzles that can be solved using the straw and pipe cleaner manipulatives or else using paper and pencil methods. Figure 3 shows how to use the paper and pencil puzzles to record a decomposition of the edges of the tetrahedron in Figure 3(b) using the tree in Figure 3(a). We will call that tree $T_6(1,2,3)$ since it has six edges and three sets of pendant edges of lengths 1, 2, and 3. Figure 3(b) is a puzzle diagram for the tetrahedron, and Figure 3(c) shows how we can draw over the diagram to solve the puzzle. Notice that we allow vertices to "overlap" or be identified, for example vertices A and B in the figure, but the edges must remain distinct.

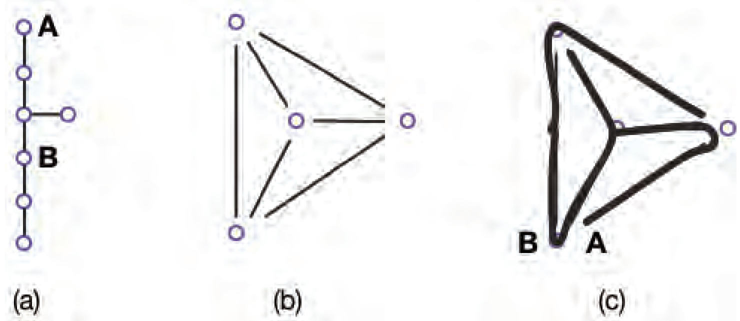


Figure 3

Alternatively, we might fold a regular tetrahedron out of a $T_6(1,2,3)$ tree made up of straws and pipe cleaners. A regular tetrahedron has edges that are all the same length, so the straw $T_6(1,2,3)$ will also have edges of equal length.

It turns out that $T_6(1,2,3)$ is a very versatile tree, as multiple copies of $T_6(1,2,3)$ will decompose the edges of each Platonic solid, each Archimedean solid, as well as each regular and semi-regular planar tessellation; see [6] for these decompositions. $T_6(1,2,3)$ will also decompose the edges of each of the Catalan solids, which are the duals of the Archimedean solids as well as the edges of many grid, cylinder, and toroidal graphs, some Johnson solids, and most duals of the semi-regular tessellations (contact the author for these solutions).

The grid graph P_m X P_n is the Cartesian product of the paths P_m and P_n with *m* and *n* vertices, respectively. The formal definition of the Cartesian product G X H of graphs G and H is the graph with vertices (u,v), where u and v are vertices in the graphs G and H, respectively; and with edges (u,v)(u'v'), where either u = u' and vv' is an edge in H, or v=v' and uu' is an edge in G.

Less formally, for the grid graph P_m X P_n we take a grid of *m* rows and *n* columns of vertices, with the vertices connected by edges in rectangular fashion (the graph in the upper left of Figure 4, for example, is P_3 X P_3). The graph C_n is the cycle with *n* vertices, and P_m X C_n is a "cylinder graph" or the skeleton of the *n*-prism. The graph C_m X C_n is known as a toroidal graph, since it is embeddable on the torus without edges crossing. In Figure 4 are a variety of somewhat easy decomposition puzzles; below each graph is the tree multiple copies of which will edge-decompose the graph. The bottom row shows the graphs P_2 X C_6 and P_2 X C_4. The edges of these graphs which extend out to the left we imagine connect to the rightmost pair of vertices. P_2 X C_4 is actually the cube.

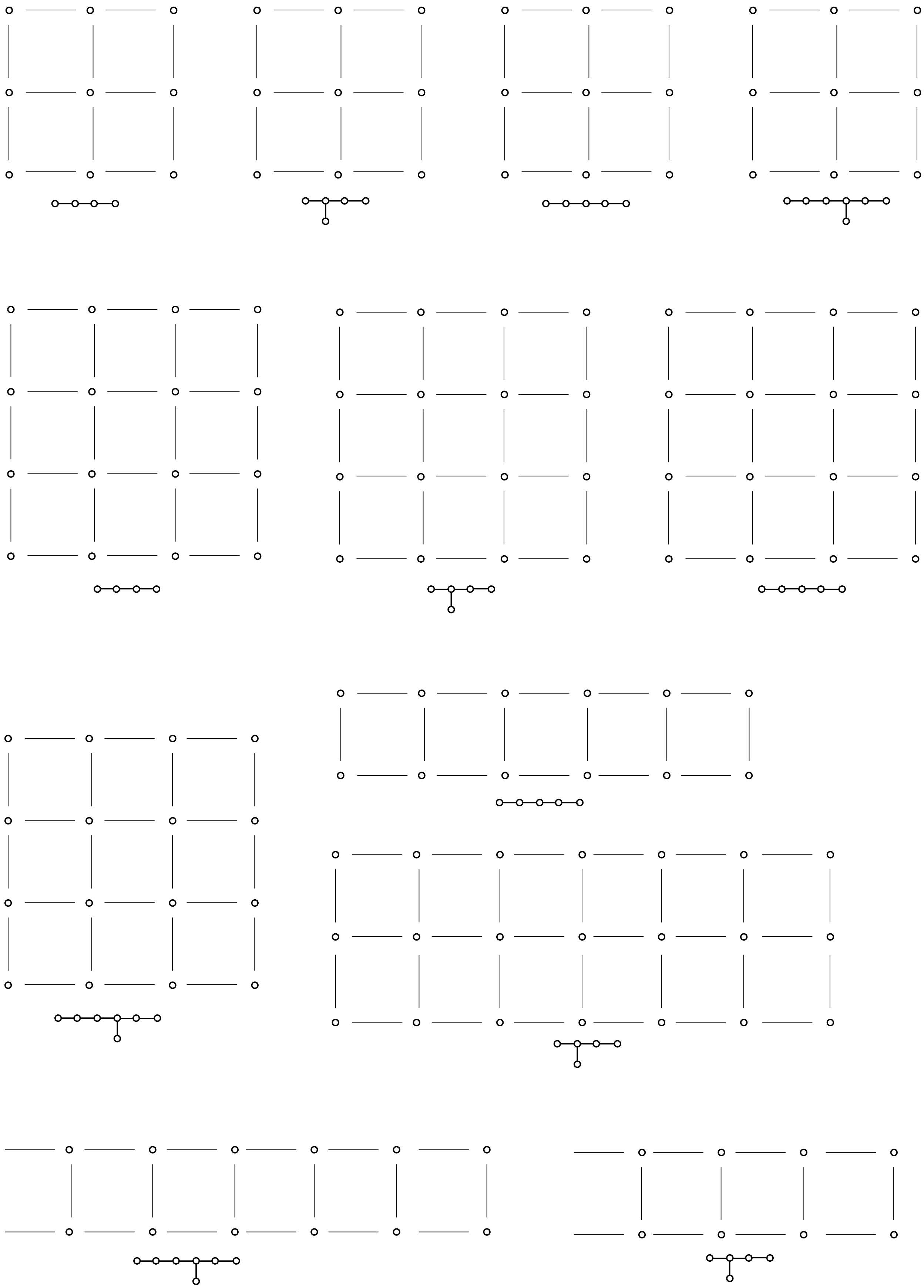


Figure 4

Many more such puzzles are easy to construct and solve. Here are a variety of extensions or generalizations of these decompositions which are all solvable and for which you might want to try to find solutions [7]. The "Examples" are small or initial cases which in some cases generalize easily, and many of which make enjoyable puzzles. Paper and pencil versions are included in Figures 4, 5, and 6.

Two-dimensional grids and cylinders.

P_n X P_{4k+n} by $T_4(1,1,2)$. Example P_3 X P_7, see Figure 6.
P_{4m} X P_{4n} by P_5. Example P_4 X P_4 on previous page.
P_{4m+2} X P_{4n+2} by P_5. Example P_6 X P_6, see Figure 6.
P_{3m} X P_{3n} by P_4. Example P_3 X P_3 on previous page.
P_{3m+1} X P_{3n+1} by P_4. Example P_4 X P_4 on previous page.
P_m X C_{4n} by P_5. Example P_3 X C_4, see Figure 6.
P_m X C_{4n} by $T_4(1,1,2)$. Example P_2 X C_4 on previous page.
P_2 X C_n by P_4. Example P_2 X C_4, the 3-cube, see Figure 6.
P_3 X C_{3n} by P_4. Example P_3 X C_3, see Figure 6.
P_3 X C_{4n} by $T_4(1,1,2)$. Example P_3 X C_4, see Figure 6.
P_4 X C_{3n} by P_4. Example P_4 X C_3, see Figure 6.
P_{3k+2} X C_n by P_4. Example P_5 X C_4, see Figure 6.
C_3 X C_{4n} by P_5. Example C_3 X C_4, see Figure 6.
C_n X C_{3n} by P_4. Example C_3 X C_4, see Figure 6.

Three-dimensional grids

P_2 X P_3 X P_{3k} by P_4, $k \geq 1$. Example P_2 X P_3 X P_3 by P_4, on next page.
P_3 X P_3 X P_n by P_4, $n \geq 1$. Example P_3 X P_3 X P_3 by P_4 on next page.
P_3 X P_3 X P_5 by P_5, see Figure 6.
P_4 X P_4 X P_{3k+1} by P_4, $k \geq 1$. Example P_4 X P_4 X P_4 by P_4.
P_2 X P_5 X P_5 by P_4, see Figure 6.

Mixed examples

P_2 X P_3 X P_3 by $6P_5 + 3P_4$, see Figure 6. (Endless similar possibilities - make some up!)
P_3 X P_4, with two colored edges decomposed by several trees, see top right of Figure 6.

In Figure 5 are a couple of three-dimensional paper and pencil grid puzzles plus a few polyhedra. For the polyhedra paper and pencil diagrams we do not insist that all edge lengths are equal, to facilitate drawing them efficiently; however, if building them out of straws and pipe cleaners the Platonic and Archimedean solids [6] are constructible with equal length straws. The "decastar," a decagon with length two paths attached to each decagon vertex, is what might be called an edge-GCD of the regular dodecahedron and icosahedron. An edge-GCD, or edge-greatest common decomposer, of two graphs G and H is a (not necessarily unique) graph with the largest number of edges that edge-decomposes both graphs. (In [6] the author showed that $T_6(1,2,3)$ is the unique edge-GCD of the five Platonic Solids.) The decastar may be easily modified to create a solution to a problem posed in Math Horizons to find a graph with the fewest number of leaves that decomposes both the dodecahedron and icosahedron [8]. In the diagram of the icosahedron on the next page, the vertices labeled *x* are " identified" and considered to be one vertex. Because the decastar easily decomposes into five copies of $T_6(1,2,3)$, the decastar's decomposition of the dodecahedron and icosahedron also provide decompositions by $T_6(1,2,3)$.

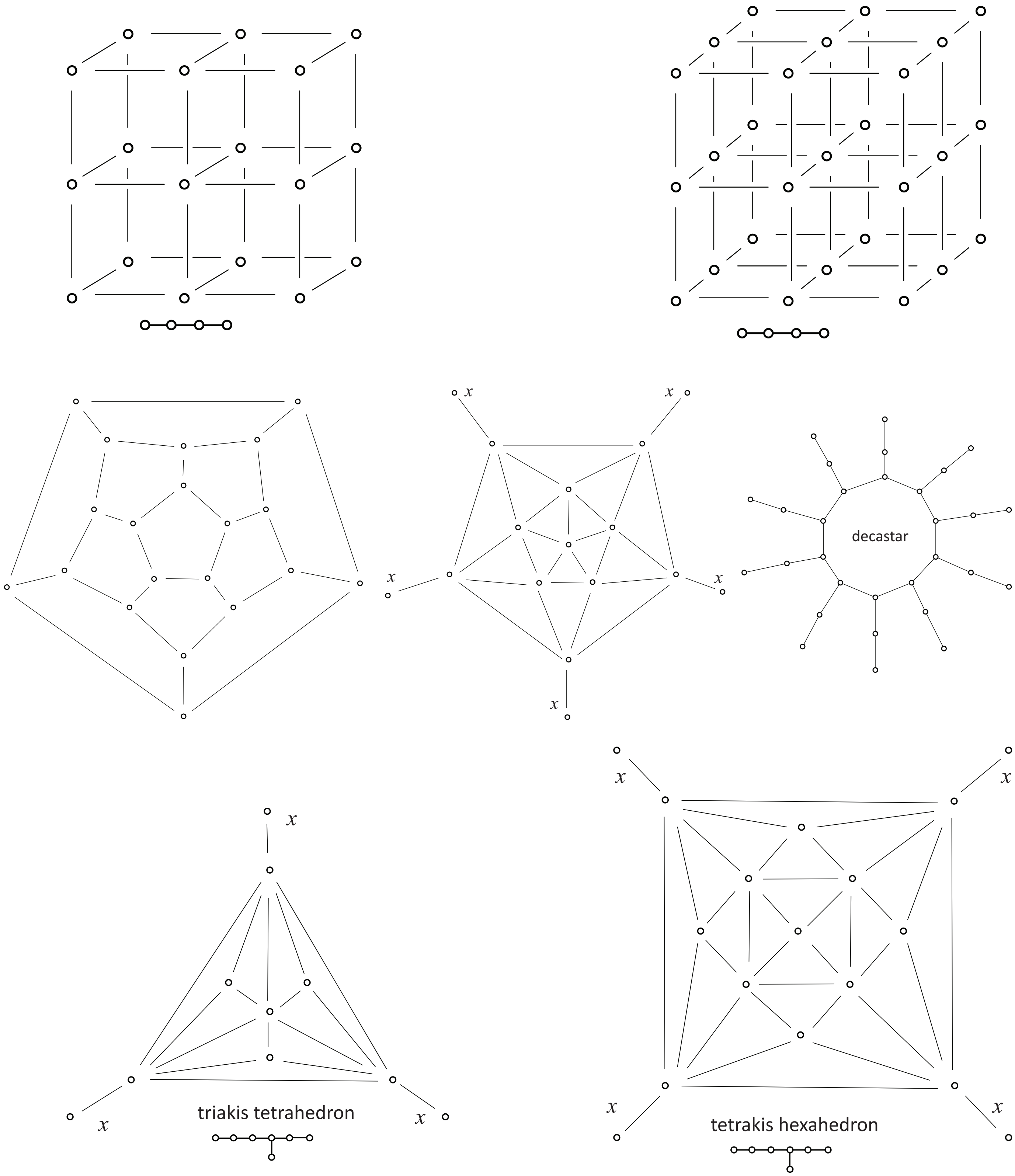


Figure 5

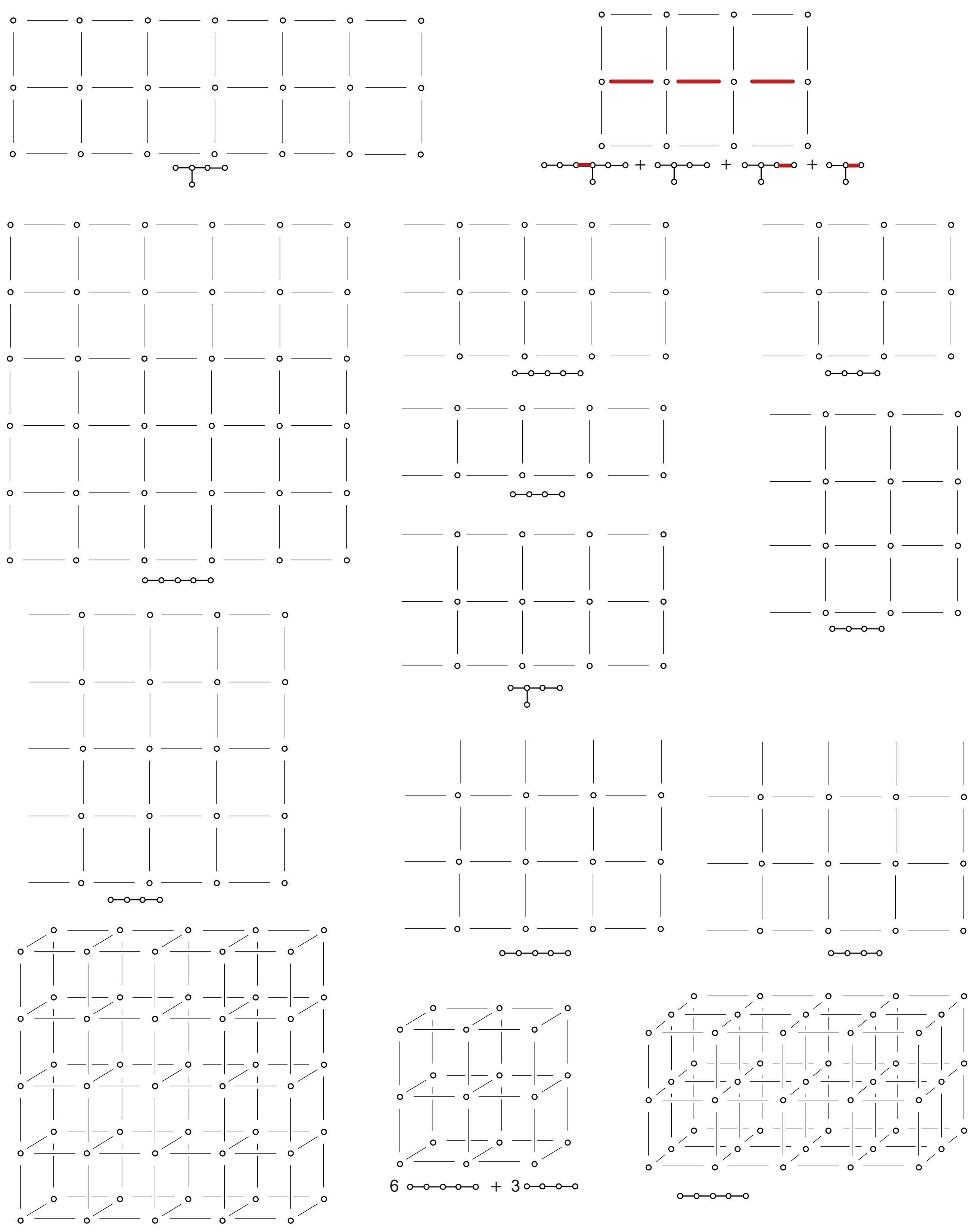


Figure 6

References

[1] Csicsery, George. Online films, at https://tinyurl.com/StringPolyhedraVideos, 2009.

[2] Devlin, Keith (direction, text), Zambra (music), and Schaffer, Karl (choreography). *Harmonious Equations*, dance concert, Santa Cruz, 2009.

[3] Schaffer, Karl . "One-dimensional Origami: Polyhedral Skeletons in Dance" in *Origami 4: Fourth International Meeting of Origami, Science, Math & Education*, ed. by Robert Lang, pp 75-83, A.K. Peters Ltd., 2009.

[4] Schaffer, Karl; Stern, Erik; Kim, Scott. *Math Dance*. MoveSpeakSpin, 2001.

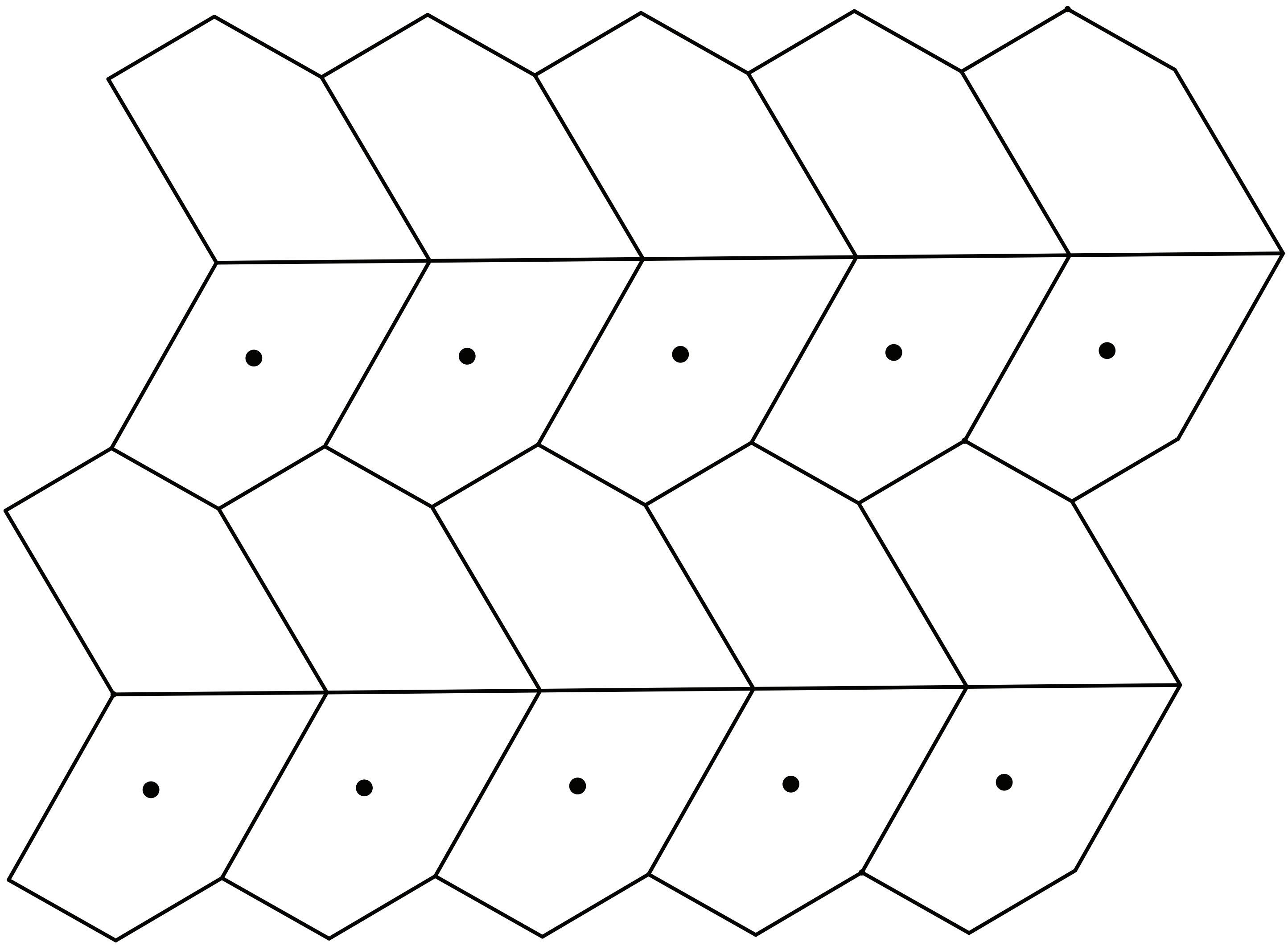
[5] Schaffer, Karl. "A Platonic Sextet for Strings," College Math Teacher, January, 2012, reprinted in *Martin Gardner in the Twenty-First Century,* ed. by Michael Henle and Brian Hopkins, pp 19-24, Math. Assoc. Amer., 2012.

[6] Schaffer, Karl. "A Skeleton Key for the Platonic Solids." In Kelly Delp, Craig S. Kaplan, Douglas McKenna, and Reza Sarhangi, editors, *Proceedings of Bridges 2015: Mathematics, Music, Art, Architecture, Education, Culture*, Phoenix, Arizona, 2015, Tessellations Publishing. Available online at http://archive.bridgesmathart.org/2015/bridges2015-199.html

[7] Schaffer, Karl. "Decompositions of basic Cartesian product graphs and other graphs by small trees." In preparation, contact author for particular solutions.

[8] Smith, Derek (2011) The Playground!, Math Horizons, 19:1, 30-33, DOI: 10.4169/mathhorizons.19.1.30. https://doi.org/10.4169/mathhorizons.19.1.30

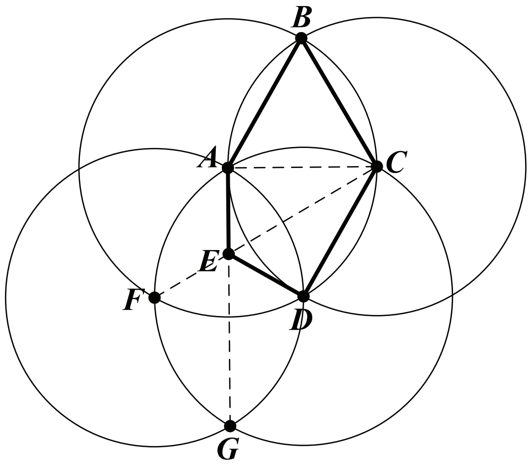
Marjorie Rice's "Versatile." Copy and cut out the tiles; assemble them into tiling patches.



Marjorie Rice's pentagonal "Versatile"

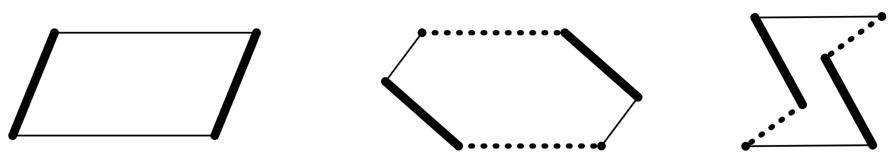
--Doris Schattschneider, Moravian College

Make your own--- with geometry software or with a compass and straightedge:



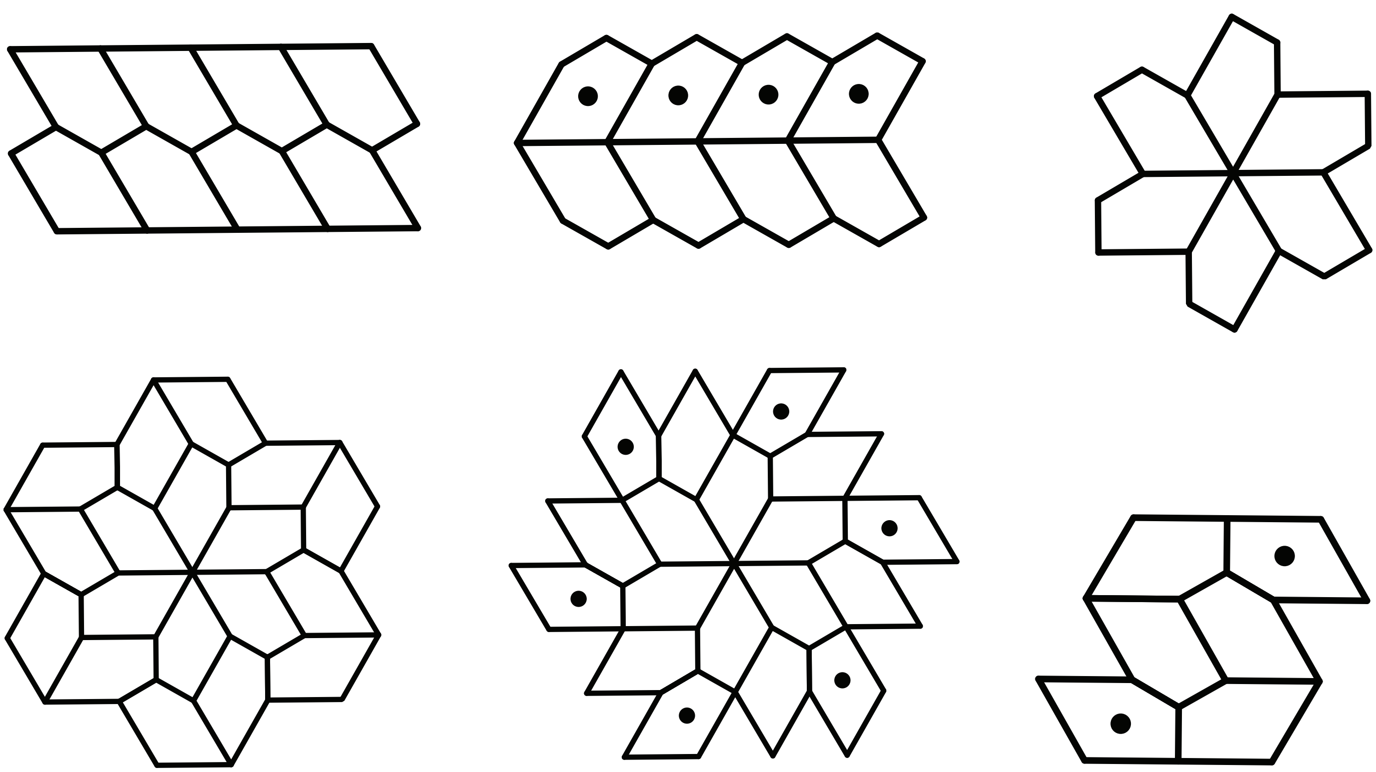
Construct two circles having radius AC, one with center A and the other with center C.
Label their points of intersection B and D as shown.
Draw AB, BC, and CD; then AB = BC = CD.
Construct a circle with center D having radius DC, and label F its intersection with the circle centered at A.
Construct a circle with center F and radius FD and label G its intersection with the circle centered at D.
Draw line segments AG and FC; let E be their intersection.
Draw AE and ED.
Since E is the center of equilateral triangle AFD, it follows that AE = ED, and $\angle E = 120°$. It is easily verified that $\angle B = 60°$, $\angle C = 120°$, $\angle D = 90°$, and $\angle A = 150°$.

Any patch of tiles that is a generalized "parallelogram" or "par-hexagon" can fill the plane using only translations. The boundary of such a patch can be partitioned into two pairs or three pairs of curves that have this property: the curves in each pair are congruent, and are translates of each other. Schematically, the boundary of such a patch looks like one of these (here "curves" are shown as straight edges; edges that match by translation have the same style of line):



Six ways that the pentagon versatile can tile the plane are shown below. Only a small patch of each tiling is shown—each patch is a generalized parallelogram or par-hexagon that can be translated to fill the plane with its copies. Tiles that have dots are reflected versions of the plain tiles.

How many other tiling patches of this versatile pentagon can you find?



Packing polyominoes into a 3-by-n box is as hard as it gets*

Tom van der Zanden

Abstract

A popular way of classifying the hardness of puzzles is by determining their membership of and completeness for the complexity class NP – essentially determining whether a certain kind of computation can be "represented" by an instance of the puzzle. The problem of determining whether a given set of polyominoes can be arranged into a given shape is NP-complete, and this is the case even if the target shape is a $2 \times n$ rectangle. From a classical viewpoint, this essentially settles the complexity. We take a more detailed look at this problem: we show that the problem of packing polyominoes into a 3-by-n rectangle is - in some sense (exact complexity) - even harder, but that moving up to $4 \times n$ or even $\sqrt{n} \times \sqrt{n}$ does not complicate things further.

1 Introduction

The term *polyomino* to describe a shape made of several connected (unit) squares was coined by Solomon Golomb [2]. In a *polyomino packing* puzzle, the goal is to take several polyominoes and arrange them into a given target shape. A simple example of such a puzzle is shown in Figure 1, where a set of 5 polyominoes can be arranged into a 3×7 rectangle.

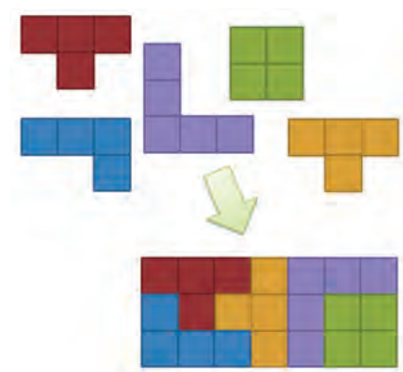


Figure 1: A simple polyomino puzzle.

At G4G6, Demaine and Demaine [3] presented a proof that established the NP-completeness of polyomino packing, even if the pieces are relatively small rectangles. This work is part of a larger framework, showing that four types of puzzles are equivalent to each other: polyomino packing, jigsaw puzzles and signed and unsigned edge-matching puzzles. This equivalence is rather interesting: given (for instance) a jigsaw puzzle, it is possible to construct an equivalent polyomino packing puzzle (equivalent in the sense that the solution to one puzzle will tell you the solution to the other), but it is also possible to do the same in the opposite direction.

Knowing the polyomino packing is NP-complete tells us that, in some sense, solving the puzzle is "hard". Informally, a problem is in NP if it is *easy to check the validity* of a solution. As an example, while solving a (partially-filled) Sudoku might be hard, given a solution (i.e., a fully-filled Sudoku puzzle) one can easily check that the solution is valid. Therefore, (generalized) Sudoku is an NP problem. As a negative example, determining whether white has a winning move in a given Chess position is (probably) not an NP problem: even if I tell you the answer there is no way to (efficiently) prove the answer is correct: I might solemnly

*This article presents results from the paper [1], but with a presentation aimed at a more general audience. The paper [1] is joint work with Hans L. Bodlaender.

swear that, yes, indeed, white does have a winning move, but you'd have to take my word for it (and if I told you what move it was, you'd still have no way of knowing whether that move is indeed a winning one).

Thus, problems in NP correspond to "puzzles" where the goal is to find some (combinatorial) object that satisfies some easy-to-check criteria. We believe that for many problems in NP there are no efficient (i.e., polynomial) algorithms, but the proof that P≠NP is still a major open problem. However, despite (essentially) not knowing whether "hard" problems exist at all, *reductions* still allow us to get good evidence that certain problems are hard: if one believed that solving jigsaw puzzles is hard, then the reduction of Demaine and Demaine [3] shows that solving polyomino packing must also be hard, because otherwise one could solve a jigsaw puzzles by constructing an equivalent polyomino packing instance and solving that instead.

There exist NP-complete problems, which have the property that they can be used to solve *any* problem in NP (and as previously mentioned, this includes polyomino packing). This is quite surprising, as to prove this one has to (essentially) show that it is possible to encode an arbitrary polynomial computation (of "checking" a candidate solution) as an instance of your problem or puzzle! Thankfully, we do not need to bother with this rather tedious task because once we know *one* NP-complete problem, we can use it to show the completeness of other problems by far simpler reductions.

2 A (very) simple proof that Polyomino Packing is hard

The following problem is well-known to be NP-complete:

3-Partition
Input: $3n$ integers $a_1, \ldots, a_{3n}$ with $\Sigma_{i=1}^{3n} a_i = M$.
Question: Can we create n groups of 3 integers a_i, a_j, a_k each, such that $a_i + a_j + a_k = M/n$ and each integer is used exactly once?

One could view this problem as having $3n$ gold bars (with differing weights) which need to be distributed among n people so that each person takes home the same amount.

There is a very simple way to model 3-partition as polyomino packing: for each integer a_i we create a $1 \times a_i$ polyomino, and we create one huge $2 \times (M + n + 1)$ polyomino, that has n "gaps" of size M/n. We then ask whether this set of polyominoes can be packed into a $2 \times (M + n + 1)$ rectangle. A solution to this problem exists only if the $1 \times a_i$ polyominoes can be partitioned into n groups of size M that fit exactly into the gaps in the large polyomino. An example of this reduction is shown in Figure 2.

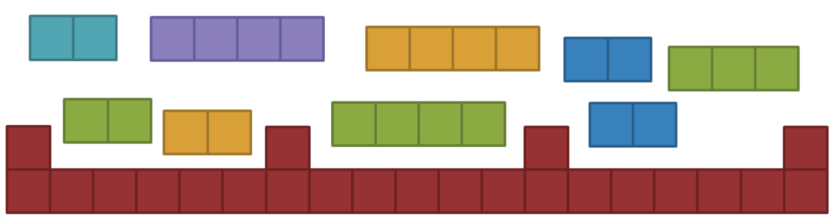


Figure 2: Figure illustrating a very simple proof showing the NP-completeness of packing polyominoes into a 2-by-n box. Note that the instance shown here does not correspond to a valid 3-partition instance, as there are too few gaps in the large piece.

Solving this polyomino packing puzzle means that each gap of size M in the large polyomino must be filled up with a_i-polyominoes summing up to the size of that gap. Thus, a solution to this polyomino packing puzzle corresponds to a solution to 3-Patition. Note that while this does not guarantee that each gap gets exactly 3 polyominoes, there is a standard technique for working around this.

One might think that if even such a simple variant of Polyomino Packing is already hard, then there is not much more to say about the problem. However, there is a surprising amount to learn about the complexity of the puzzle by studying it more closely.

3 On to Exact Complexity

The proof of NP-completeness, such as the one presented in the previous section, establishes hardness of a problem in the sense that, if P≠NP, then there is no *polynomial-time* algorithm for the problem. Any algorithm (or person) solving the puzzle must take (asymptotically) a superpolynomial number of steps. However, just knowing there is no polynomial algorithm does not give us a very precise sense of how hard a problem is: $n^{\log n}$ and 2^{2^n} are both not polynomial, but there is a world of difference between them. Instead, we are going to look at the *exact complexity* of the problem: what is the best running time we can achieve, even if it is not polynomial?

For $2 \times n$ Polyomino Packing, there exists an algorithm solving the problem in $2^{O(n^{3/4} \log n)}$ time. For the details of this algorithm we refer to [1], here we just summarize the key point: consider an algorithm that places the polyominoes in some fixed order from the first polyomino to the last. As the algorithm progresses, it has to track which squares of the $2 \times n$ target shape have already been filled up and which ones have not. As there are 2^{2n} subsets of a $2 \times n$ board, this is the number of possible solutions the algorithm would have to consider.

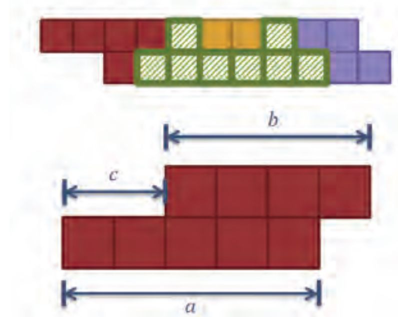


Figure 3: Top: The algorithm for packing polyominoes into a $2 \times n$ box exploits the fact that the box disconnects into Y-monotone components when placing a polyomino. Bottom: a Y-monotone polyomino can be described with three integers.

However, we can use the following trick: if we are packing polyominoes into a $2 \times n$ target shape, not all subsets of the target shape are possible and through a combinatorial argument we can show that it suffices to consider $2^{O(n^{3/4} \log n)}$ possible target shapes. For the details of this analysis we again refer to [1] but the main observation is that there are only $O(n^3)$ so-called *Y-monotone* polyominoes with n squares (and at most 2 squares high), whereas in contrast (if we allow arbitrary shapes) we can create $2^{O(n)}$ different polyominoes with n squares.

So, is this $2^{O(n^{3/4} \log n)}$-time algorithm optimal? Can we do better?

Of course, we have no hope of proving any superpolynomial lower bounds on the running time of an algorithm for any NP-complete problem, since this would mean showing that P≠NP. Instead, we have to make an *assumption* on the complexity of some base problem, and then deduce lower bounds on the running time of other problems from there. A commonly made assumption is the *Exponential Time Hypothesis* (ETH):

Assumption 1 ((Exponential Time Hypothesis) [4].) *There is no algorithm solving Satisfiability of formulas with n variables in $2^{o(n)}$-time.*

Satisfiability is a problem that asks whether a logical formula has a satisfying assignment. An example of such a formula is $(x_1 \vee x_2) \wedge (x_1 \vee \neg x_2)$, which has *variables* x_1 and x_2. To each variable we must assign either true (T) or false (F), and the formula as a whole should be satisfied. Each variable x_i has two corresponding literals, the positive literal x_i which has the same truth value as the variable, and the negation $\neg x_i$ which has the opposite truth value. For example, $x_1 = F, x_2 = T$ is *not* a satisfying assignment. The first *clause* $(x_1 \vee x_2)$ is satisfied (since x_2 is true), but the second clause $(x_1 \vee \neg x_2)$ is not since both x_1 and $\neg x_2$ (the negation of x_2) are false. Setting $x_1 = T$ will satisfy the formula (regardless of the value of x_2).

We may assume that the formula is given in *conjunctive normal form*, that is, each clause consists of taking the logical OR of several literals, and the formula consists of taking the logical AND of several such clauses. We may furthermore assume that each clause consists of at most 3 literals, and that each variable occurs in at most 3 clauses.

Note that the Exponential Time Hypothesis states that there is no $2^{o(n)}$-time algorithm. This means that there could be, for instance, a 1.0001^n-time algorithm, but not a $1000^{n^{0.99}}$-time one. Essentially, the function appearing in the exponent must be linear.

Suppose that we had a hypothetical reduction from Satisfiability to Polynomio Packing, that maps a n-variable formula to a Polyomino Packing instance with a target shape of area n^2. Supposing we also had an algorithm, solving Polyomino Packing for target shapes of area A in time $2^{A^{0.25}}$. If we applied this algorithm to the instance created by the reduction, we would obtain a $2^{(n^2)^{0.25}} = 2^{\sqrt{n}}$-time algorithm for Satisfiability! Thus, if we believe the Exponential Time Hypothesis (and this hypothetical reduction existed), we would conclude that no such algorithm can exist. In fact, the reduction tells us that (hypothetically) Polyomino Packing would not have a $2^{o(\sqrt{A})}$-time algorithm.

One good piece of evidence in favor of the Exponential Time Hypothesis is that, overwhelmingly often, the best known reduction and best known algorithm match up perfectly. E.g., for almost all problems for which we know $2^{O(\sqrt{n})}$-time algorithms we have reductions that turn n-variable Satisfiability formulas into $O(n^2)$-size problem instances [5].

Unfortunately, the chain of reductions from Satisfiability to 3-Partition is quite complicated, and does not give a tight lower bound for Polyomino Packing in a $2 \times n$ box (and therefore, we do not know the answer to the question we stated a few paragraphs earlier). In the following section, we will instead derive a tight lower bound for polyomino packing into a $3 \times n$ box.

4 Lower bound for $3 \times n$ Polyomino Packing

In the previous section, we showed that there exists an algorithm solving Polyomino Packing with a $2 \times n$ box as target shape in time $2^{O(n^{3/4} \log n)}$. In this section, we will discuss the following contrasting result:

Theorem 1 *Assuming the Exponential Time Hypothesis, there is no algorithm solving Polyomino Packing where the target shape is a $3 \times n$ rectangle in $2^{o(n/\log n)}$ time.*

This truly is a big "jump" in difficulty between the difficulty of the two problems (of packing into a $2 \times n$ rectangle v.s. a $3 \times n$ one). The main reason for this gap is that we can use binary encoding of integers to construct polyominoes. Given an binary integer, say, 10110111, we can create a 2×8 polyomino consisting of a single solid top row, and then a bottom row that has a square whenever the bitstring has a 1. This gives us, using $2n$ squares, 2^n distinct polyominoes.

So why does this explain the difference in hardness between the $2 \times n$ case and the $3 \times n$ case? Certainly, the polyominoes we just described are only two squares high so they could also appear in a $2 \times n$ Polyomino Packing instance. However, the big difference is, that in the $2 \times n$ case, the way two such polyominoes can *interact* is very limited. However, as illustrated in Figure 4, in the $3 \times n$ case, we can create a second polyomino (the *complementary polyomino*) that fits together only with that specific other polyomino into a rectangle 3 squares high.

Figure 4: Building two polyominoes that interlock in a specific way using bitstrings.

Suppose that we have some satisfiability formula with $O(n)$ variables and clauses. Using the method sketched above, we can construct for each variable and clause a unique *corresponding polyomino* and *complementary polyomino* which have the property that the corresponding polyomino for a specific variable (or clause) only fits together with the complementary polyomino for that specific variable (or clause). Figure 5 illustrates this, showing corresponding and complementary polyominoes for variables x_1, x_2 and clause c_3. Note that, thanks to the property previously discussed, these polyominoes need only be $O(\log n)$ squares wide to be able to distinguish $O(n)$ distinct clauses and variables.

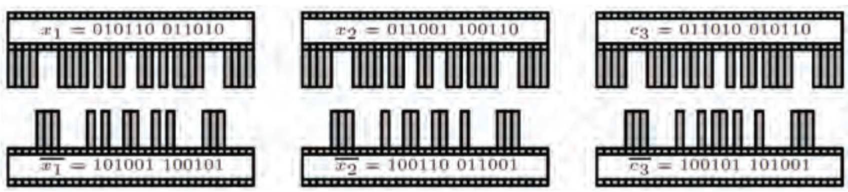


Figure 5: Complementary and corresponding polyominoes. Note that the picture is shown compressed in the X-axis.

We can further create two other polyominoes, the blocking polyomino and the wildcard polyomino - shown in Figure 6. The wildcard fits together with *any* corresponding polyomino, whereas the blocking polyomino only fits together with the wildcard. These two polyominoes are important building blocks in the reduction.

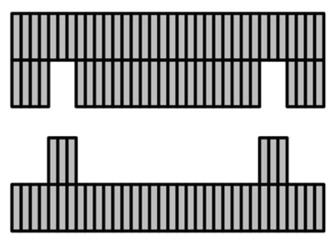


Figure 6: Blocking polyomino (top) and wildcard (bottom).

In the following, as an example, we will use the following formula: $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$. Note that for simplicity this is a 2-CNF formula (the solving of which is not even NP-hard) but the reduction easily generalizes to 3-CNF and above. We number the variables x_1, x_2 and the clauses c_3, c_4, c_5 (so $(\neg x_1 \vee \neg x_2)$ is c_5).

Let us look at a single variable (say x_1). If we make x_1 true, we would satisfy c_3. If instead we make x_1 false, we would satisfy c_4 and c_5. We encode this information in a *formula-encoding* polyomino, shown in Figure 7.

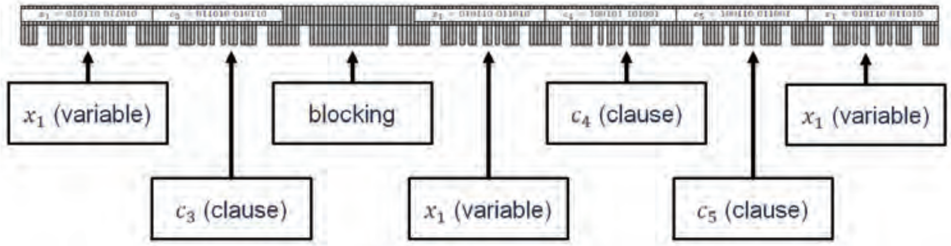


Figure 7: Polyomino encoding the clauses satisfied by assignments to x_1.

The formula-encoding polyomino is constructed by taking (the corresponding polyomino for) x_1, c_3, the blocking polyomino (for padding, to make the two parts of equal length), again x_1, c_4, c_5 and finally another copy of the corresponding polyomino for x_1.

Note that we thus end up with a single polyomino with two *parts*: both parts are delineated by two copies of the corresponding polyomino for variable x_1 (and they share the middle copy), and each part contains polyominoes corresponding to clauses that would be satisfied by a true or false assignment respectively.

Next, as shown in Figure 8, we create the *variable-setting* polyomino for x_1: we simply take a copy of (the complementary polyomino for) x_1, two wildcards, and another copy of x_1.

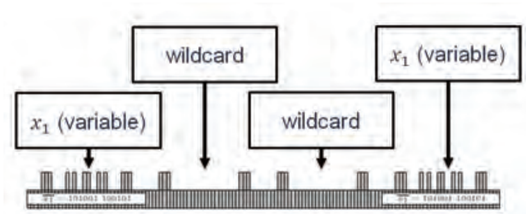


Figure 8: Variable-setting polyomino for x_1.

There are exactly two possible placements for this variable-setting polyomino: either it is packed together with the first "part" of the formula-encoding polyomino, or packed together with the second "part". The former placement corresponds to a false assignment to x_1 (since the clauses that would be satisfied by a true assignment are covered by the polyomino, leaving us free to pack clause polyominoes into the places created for clauses satisfied by a false assignment), the latter placement to a true assignment.

We repeat this process for every variable, creating one formula-encoding and one variable-setting polyomino for each. Finally, we create *clause-checking* polyominoes, which are just copies of the complementary polyominoes for each clause (one copy for each clause). The entire set of polyominoes created by the reduction (when applied to the example formula) is shown in Figure 9.

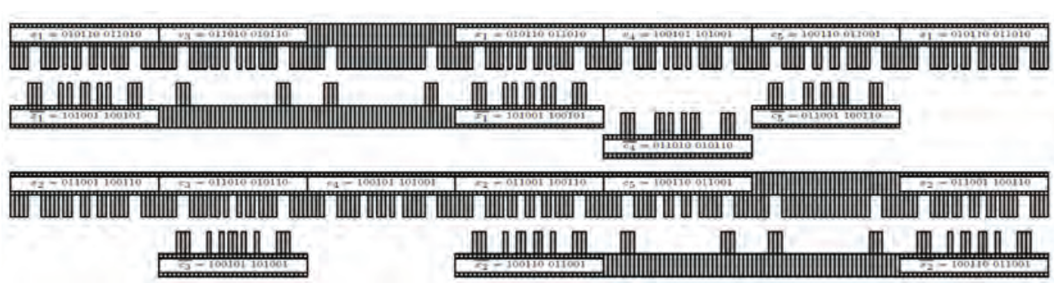


Figure 9: Overview of the construction created in the reduction for the example formula $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$.

To pack the thus created polyominoes into a rectangle 3 squares high, we must place the variable-setting polyominoes together with the formula-encoding polyominoes, and their relative placement gives the truth assignments for each variable. We will then have space to pack all the clause-checking polyominoes only if the formula is satisfiable. More technical details are given in [1].

Note that if we start with a formula with $O(n)$ clauses/variables, we end up creating a Polyomino Packing instance with a $3 \times O(n \log n)$ target shape (each building block is

$O(\log n)$ polyominoes wide, and we use $O(n)$ of them). This thus (under the Exponential Time Hypothesis) rules out a $2^{o(n/\log n)}$-time algorithm.

5 Conclusions

The lower bound proof, presented in the previous section, is tight: even for an arbitrary target shape with area n, we can solve Polyomino Packing in $2^{O(n/\log n)}$ time. For the details, we again refer to [1]. This means that, unless the Exponential Time Hypothesis is violated, both the algorithm and the reduction are optimal.

Packing Polyominoes into a $1 \times n$ box is trivial, packing them into a $2 \times n$ box is moderately hard, and packing them into a $3 \times n$ box is harder, and actually as hard as it gets: the lower bound for $3 \times n$ Polyomino Packing is tight against the algorithm we have for solving general Polyomino Packing – and thus, $3 \times n$ Polyomino Packing is rightfully "as hard as it gets", since it requires as much time to solve as solving any other polyomino puzzle, while solving $2 \times n$ puzzles can be done faster.

These results give us some insight into not only the fact that Polyomino Packing is hard, but also *why* it is hard. The proof of hardness for $3 \times n$ polyomino packing exploits that polyominoes inside a $3 \times n$ box can have complex interactions, and we only need polyominoes of area $O(\log n)$ to identify n distinct pieces. For $2 \times n$ Polyomino Packing, the algorithm exploits precisely the fact that the pieces can not interact in very complex ways. Furthermore, the algorithm for general Polyomino Packing exploits the following symmetry: in a Polyomino Packing instance with total area n, at most $n/\log n$ polyominoes can consist of more than $\log n$ squares, and the remaining (at most n) polyominoes have at most $\log n$ squares. The worst case running time is achieved exactly when the polyominoes have area exactly $\log n$, and the instance created in our reduction has precisely this property.

References

[1] Hans L. Bodlaender and Tom C. van der Zanden: *On the Exact Complexity of Polyomino Packing*. Fun with Algorithms (FUN) 2018.

[2] Solomon W. Golomb: *Checker boards and polyominoes*. The American Mathematical Monthly, 61(10), 675-682, 1954.

[3] Erik D. Demaine and Martin L. Demaine: *Jigsaw Puzzles, Edge Matching, and Polyomino Packing: Connections and Complexity*. Graph. Comb. 23, 1, 2007.

[4] Russell Impagliazzo and Ramamohan Paturi: *The Complexity of k-SAT*. 14th IEEE Conference on Computational Complexity (1999).

[5] Dániel Marx: *The Square Root Phenomenon in Planar Graphs*. Frontiers in Algorithmics and Algorithmic Aspects in Information and Management (2013).

Puzzles that Solve Themselves

Peter Winkler *

February 23, 2018

Abstract

Roughly speaking, we say that a puzzle "solves itself" if the stupidest way you can think of to get an answer works. Often, that means guessing an answer, and then fixing it in the obvious way until it becomes a solution. But how do you know when this happy state of affairs exists?

1 Problems

Let's start with a simple example.

Flipping the Bulbs

In front of you is a 9×9 array of light bulbs, some on, some off. At the left end of each row, and at the top of each column, is a switch that will reverse the state of every bulb in that row or column.

Is it possible to flip switches in such a way that every row and every column has most of its bulbs on?

The obvious thing to do here is to find some line (row or column) that has most of its bulbs off, then flip its switch. Trouble is, that might cause some intersecting lines to go from mostly on to mostly off; thus, you might *increase* the number of bad lines. Then, after more corrections, you might find yourself back at the original configuration without having found a solution.

But a little thought will convince you that this process can never cycle back to any previous configuration, and in fact will solve the problem rather quickly. The key observation is that when you flip a line that has more bulbs off than on, *you increase the total number of lit bulbs.* This can't go on forever and only reaching a solution can stop you.

*Department of Mathematics, Dartmouth College, Hanover NH 03755-3551, USA. Research supported by NSF Grant DMS-1600116.

Sometimes you don't even have to be clever enough to find *any* possible route to a solution.

Breaking a Chocolate Bar

You have a rectangular chocolate bar marked into $m \times n$ squares, and you wish to break up the bar into its constituent squares. At each step, you may pick up one piece and break it along any of its marked vertical or horizontal lines.

How should you break up the bar so as to minimize the number of breaks needed?

This puzzle, which I first heard from the late, great mathematician Paul Halmos, looks geometrical but isn't. The fact is, breaking the $m \times n$ bar into its constituent square takes exactly $mn - 1$ breaks, no matter how you do it, simply because every break increases the number of pieces by one.

Obvious when you know it, but many smart folks have been led astray by the grid lines and failed to count pieces.

The next puzzle really does have some geometric content.

Red Points and Blue

Given n red points and n blue points on the plane, no three on a line, can you find a "heterosexual" pairing of red and blue points so that if you connect each red point to its blue mate with a line segment, no two line segments cross?

Let's be dumb and match up the points any old way, then draw in the corresponding line segments. Maybe they never cross!

If they do, pick two segments that cross, and switch partners to that they don't cross any more. Great. But, of course, that action might create many more crossings. Ugh.

Ah, but uncrossing two segments always *reduces the total length of the crossings.* Why? Because crossing segments are the diagonals of a convex quadrilateral, and replacing them with opposite sides reduces length since they no longer have to meet in the middle of the quadrilateral. (Technically, we are employing the "triangle inequality" here.)

There are only a finite number of ways (namely, n factorial) to match up the red and blue points, so eventually you must reach a matching with no crossings.

Conceptually speaking, you can prove non-algorithmically that such a matching exists by just choosing, from the start, the matching that minimizes the sum of the pairwise distances between matched pairs of points. But if you really need to *find* the matching, the above untangling scheme typically works quite fast.

We continue with a familiar campus entity–the Athletic Committee.

Picking the Athletic Committee

The Athletic Committee is a popular service option among the faculty of Quincunx University, because while you are on it, you get free tickets to the university's sports events. In an effort to keep the committee from becoming cliquish, the university specifies that no one with three or more friends on the committee may serve on the committee—but, in compensation, if you have three or more friends on the committee you can get free tickets to any athletic event of your choice.

To keep everyone happy, it is therefore desirable to construct the committee in such a way that even though no one on it has three or more friends on it, everyone *not* on the committee *does* have three or more friends on it.

Can this always be arranged?

This problem (in an abstract form, with the number 3 replaced by an arbitrary integer k) arose in the work of my computer science colleague Deeparnab Chakrabarty. What's the dumbest way to try to solve it? How about this: start with an arbitrary set S of faculty members, as a prospective Athletic Committee. Oops, Fred is on the committee and already has three friends on the committee? Throw Fred out. Mona is not on the committee, but has fewer than three friends on it? Put Mona on. Continue fixing in this haphazard manner.

Now, why in the world would you expect this to work? Clearly, the above actions could make things worse; for example, throwing Fred off the committee might create many more Monas; maybe we should have thrown off one of Fred's on-committee friends instead. So there doesn't seem to be anything to prevent cycling back to the same bad committee. Moreover, even if you don't cycle back, there are exponentially many possible committees and you can't afford to consider every one. Suppose there are 100 faculty members in all; then the number of possible committees is $2^{100} > 10^{30}$ which, even if you spent only a nanosecond considering each committee, would take a thousand times longer than all the time that has passed since the Big Bang.

But if you try it—and if there's one idea that you take from this paper, it's *try it!*—you will find that after shockingly few corrections, you end up with a valid committee. And this happens whether in situations where there is only one valid committee, as well as when there are many.

How can this be? Well, as in Flipping the Bulbs, perhaps there is something that is improving each time you throw someone off or add someone to the current prospective committee. Let's see: when you throw someone off, you destroy at least three on-committee friendships; when you put someone on, you add at most two. Let $F(t)$ be the number of friendships on the committee *minus* $2\frac{1}{2}$ *times the number of people on the committee* at time t. Then when Fred is thrown off, $F(t)$ goes down by at least $\frac{1}{2}$. When Mona is put on, $F(t)$ *again* goes down by at least $\frac{1}{2}$. But $F(0)$ can't be more than $(100 \times 99)/2 - 250 = 245$ and $F(t)$ can never dip below -250, so there can't be more than $2 \times (245 - (-250)) = 990$ steps total. (A computer scientist would say that the number of steps in the process is at worst quadratic in the number of faculty members.)

In practice, the number of steps is so small that if there are 100 faculty members and you start with (say) the empty committee, you will reach a solution easily by hand. Of course, you'll need access to the friendship graph, so you might need to do some advance polling. It'll be interesting to see who claims friendship with whom that isn't reciprocated!

Sometimes the correction process is continuous.

Squaring the Mountain State

Can West Virginia be inscribed in a square?

It must be tough living in a state with two panhandles, but that doesn't mean you can't make a square map of your state in which the state outline exactly reaches all four edges. Certainly you can make a *rectangular* map with this property, just by orienting the state in the familiar way—north equals up—and drawing horizontal and vertical lines through the northernmost, southernmost, easternmost and westernmost points in the state. You won't have a square; West Virginia is slightly wider than it is tall.

Now rotate the state slowly clockwise (say), moving the horizontal lines smoothly up and down and the vertical ones left and right so as to stay tangent to the state boundary. When you've got the state rotated 90 degrees, so that it's northern panhandle is pointing to the right, the rectangle in which it is inscribed will be too tall to be a square instead of too short. It follows (by the *intermediate value theorem*, if you must know) that at least once during the rotation, the horizontal and vertical sides of the rectangle were the same length. And at that moment, you had WV where you wanted it—inscribed in a square.

We wind up with a marvelous puzzle devised by ace probabilist and puzzle-maker Ander Holroyd, who as you read this is visiting Cambridge University.

Self-Referential Number

The first digit of a certain 8-digit integer N is the number of zeroes in the (ordinary, decimal) representation of N. The second digit is the number of ones; the third, the number of twos; the fourth, the number of threes; the fifth, the number of fours; the sixth, the number of fives; the seventh, the number of sixes; and, finally, the eighth is the number of distinct digits that appear in N. What is N?

If you try to work out this number by intelligent reflection, it ain't easy. Instead, pick any 8-digit number, say M, and write out a new 8-digit number M' as follows: the first digit of M' is the number of zeroes in M, the second is the number of ones in M, etc., and the last digit is the number of different digits in M. Now repeat, starting with M'. In short order you will find that you have converged to a number that doesn't change, and that's the unique answer; I leave it to you to discover it.

There's one catch. In all the previous problems, we could determine exactly why the obvious procedure works so well. But neither Ander nor I knows why this particular puzzle is self-solving; some similar ones are not. If you figure it out, let us know!

SCIENCE

Introducing the PiTOP | Kenneth Brecher | Page 197

Sketching a Projectile on a Ramp

Gary Antonick
Stanford H-STAR
antonick@stanford.edu

Abstract
This informal paper presents a circle-and-line method for constructing the trajectory of a projectile bouncing up and down a ramp. The method is presented as a series of collaborative discoveries.

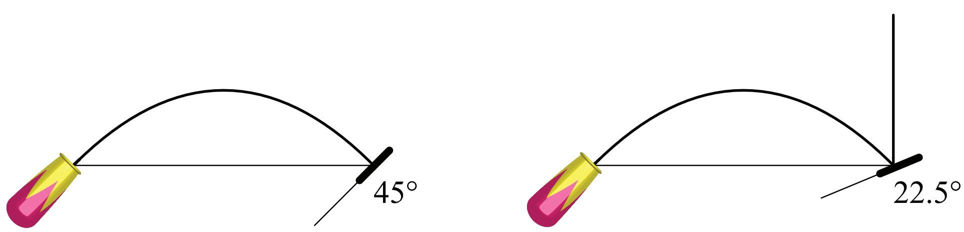
Latent structure is master of obvious structure. —Heraclitus

1 A Second Reflection

Raise a cannon halfway to vertical and fire. The cannonball flies over to a small trampoline, bounces, and retraces its path back to the cannon.

1) What is the angle of the trampoline?
2) What other angle will work?

Solution: A trampoline at 45° will return the ball to the cannon, clearly, but half that angle will also work: the ball will fly over to the trampoline, bounce vertically, then retrace its path back to the cannon.



Nick McKeown (Stanford CS) shared this problem with me back in 2010, and I immediately featured it in The Times Numberplay column. Readers readily identified these two angles. But why stop there?

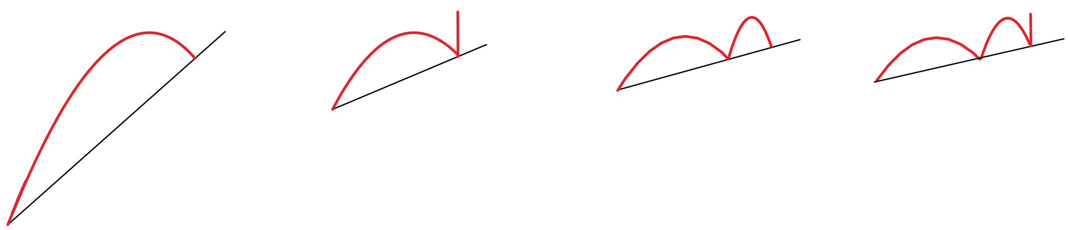
https://wordplay.blogs.nytimes.com/2010/09/06/numberplay-a-little-reflection/
https://wordplay.blogs.nytimes.com/2010/10/11/numberplay-the-ultimate-answer/

2 Seeking Structure

It was clear that 45° and 22.5° were not the only solutions—with a long enough trampoline, infinitely many angles would work. But what was the underlying pattern?

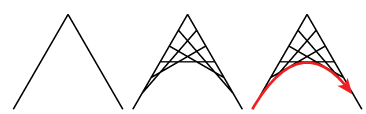
In an ongoing Numberplay discussion, the team of Dr. W, Marco Moriconi, Tudor, Hans Chen, Pummy Kalsi, and Pradeep Mutalik settled on the fire-from-ramp approach and computed a number of angles and distances between bounces. Nick Baxter created the corresponding images of the flight paths. Peter Norvig, director of research at Google, created a useful projectile/ramp simulation tool.

http://norvig.com/inclined_plane.html



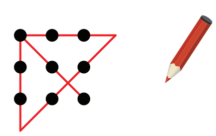
The Method of Apollonius

A parabola's envelope of tangent lines can be created by taking a sloping line segment, mirroring it, dividing each by any number of equal segments, then connecting as follows.

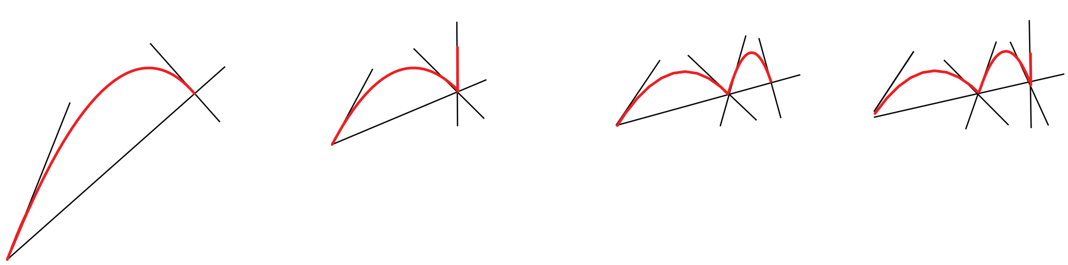


The Nine-Dot Problem

Nine dots arranged in a grid can be connected with a series of connected straight line segments as follows—the iconic outside-box solution.

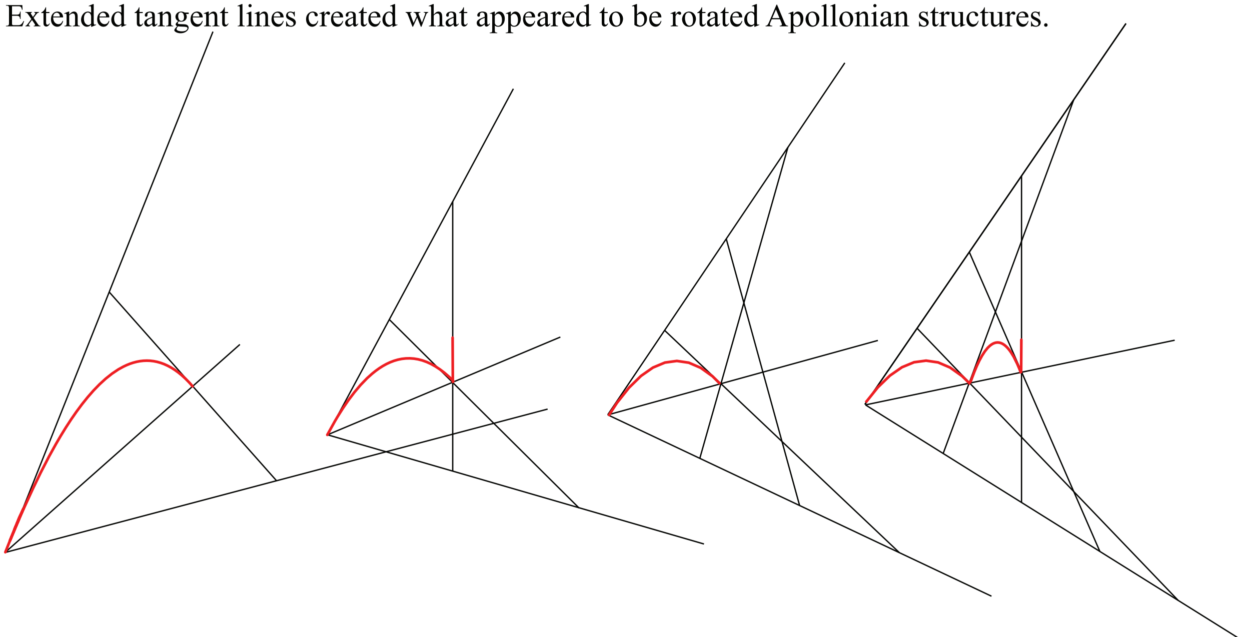


It would seem that the underlying structure would have to include the following tangent lines. What form would these lines take if extended?



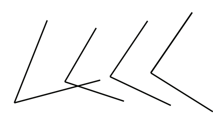
3 Apollonius Sideways

Extended tangent lines created what appeared to be rotated Apollonian structures.



If this approach actually worked—this remained to be proven—it would be a new way to think about the path of a projectile on an inclined plane. What the logic behind these structures?

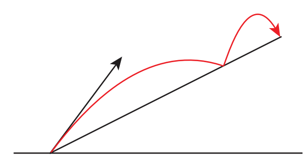
Key to the structures were the sideways V shapes, which could be divided into 2, 3, 4, 5 segments to create tangents for 2, 3, 4, 5 bounces. How could these be derived?



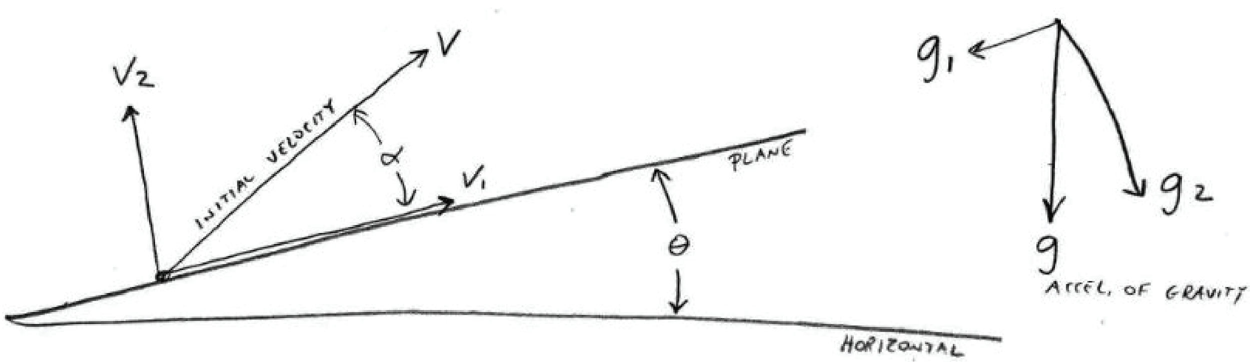
For ideas I turned to Nick Baxter, who suggested throwing the problem to geometers. George Hart came to mind. We phrased the challenge this way:

What Angles?

Place a cannon directly on a ramp. What launch and ramp angle will cause a ball to bounce up the ramp n times before reversing direction?



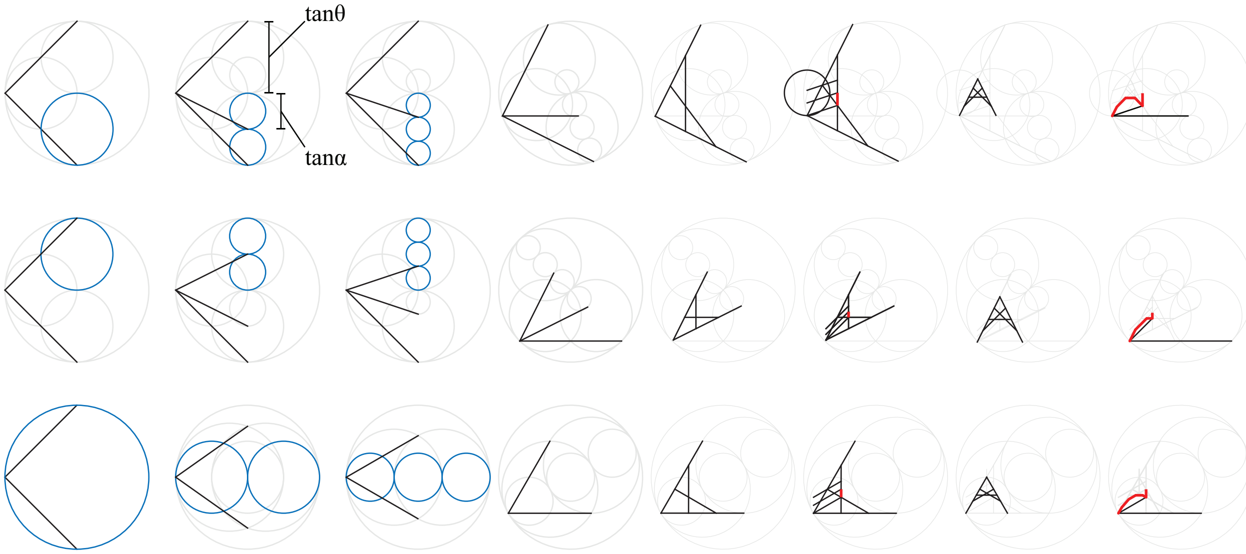
George's conclusion: For a ramp angle θ and incremental launch angle α, the number of bounces up a ramp is 1/2tanθtanα. (Full solution in appendix). This was exactly what I was looking for.



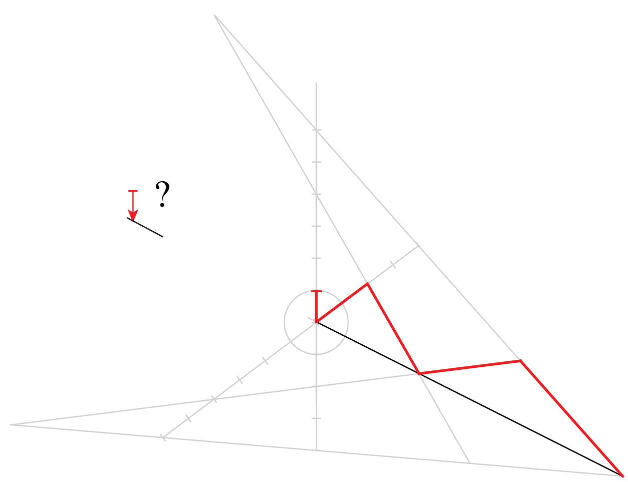
3 A Deeper Structure

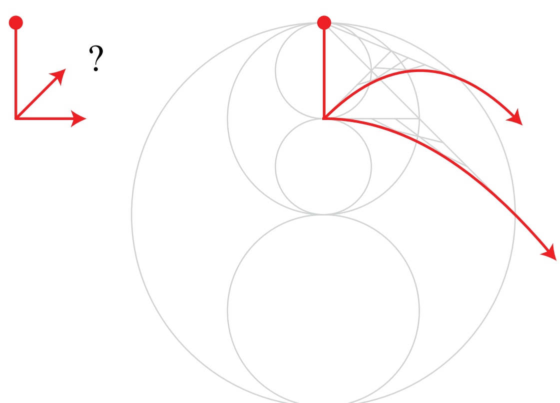
The structure was now clear. The V shapes had openings of 2θ and were tilted at angle α from horizontal, where tanθtanα = 1/bounces. (There are several ways to add bounce detail.)

Below are three possible θ and α combinations for 1, 2, and 3 bounces, with the 3-bounce scenarios fully worked out (there are several ways to construct bounce-level detail). I find circle constructions practical, precise, and aesthetically pleasing.



This was the deeper structure I had been seeking. The basic idea could be used to easily and accurately construct not only the launch/ramp angles to generate any number of bounces, but also estimate the number of bounces for any launch/ramp configuration, accurately determine the path of a ball bouncing down a ramp, and determine the trajectory of a ball fired in any direction based on a single vertical launch. The reflection was complete.

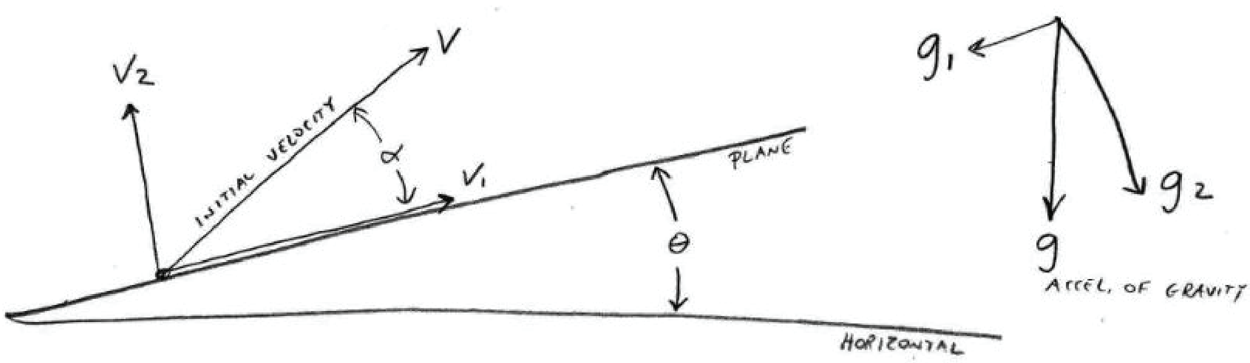




Acknowledgements

In exploring this problem I also received suggestions and encouragement from Ravi Vakil, Susan Holmes, Peter Winkler, and Dana Mackenzie. I'd like to thank Tadashi Tokeida for verifying the originality of this construction method, and Andrey Sushko for writing up a proof (see appendix).

Appendix 1: George Hart's Solution



[KNOWING $\Delta V = a\tau$, and separating into components 1 & 2 IN direction of plane surface and normal to it.]

V = initial velocity with components $V_1 = V\cos\theta$, $V_2 = V\sin\theta$

g = accel. of gravity with components $g_1 = g\sin\alpha$, $g_2 = g\cos\alpha$

τ_R (RISE TIME) SATISFIES $g_2 t_R = V_2$ so $\tau_R = V_2/g_2$

τ_B (BOUNCE TIME) $= 2\tau_R = 2V_2/g_2$

τ_S (time until V_1 motion STOPS) satisfies $g_1\tau_S = V_1$ so $\tau_S = V_1/g_1$

n (number of bounces while contact point move rightward)

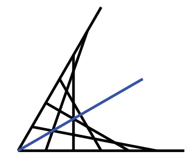
$$n = \tau_S/\tau_B = \frac{V_1/g_1}{2V_2/g_2} = \frac{V\cos\theta\ g\cos\alpha}{2V\sin\theta\ g\sin\alpha} = \frac{1}{2\tan\theta\ \tan\alpha}$$

[THIS ASSUMES ALL THE USUAL THINGS ABOUT ZERO AIR FRICTION, PERFECTLY ELASTIC COLLISIONS, PARALLEL CONSTANT GRAVITY, ETC.]

G. HART
NOV 2012

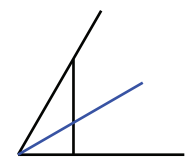
Appendix 2: Proof

We seek to prove that the intersection points between the construction shown in black and a line bisecting the angle at the bottom left vertex (in blue) are spaced quadratically with the zero point at the single perpendicular intersection.



Let N be number of subdivisions of the bounding lines such that the subdivision points are equidistant and N is one more than the number of lines between these subdivision vertices. In the above case, there are 5 lines and N = 6. Labelling the subdivision vertices by their distance from the vertex (down to arbitrary scale factor) such that the bottom left vertex is 0 and the end of each bounding line is N, the lines will connect a vertex X on one bounding line to N − X on the other.

Consider, now, a triangle bounded by the two bounding lines and one of the lines from X to N − X (drawn for X = 4, N − X = 2). The diagonal divides this into 2 triangles. Let us label the distance from the vertex to the intersection with the diagonal as Yx. If the angle at the vertex between the two bounding lines is θ, we can find the areas of the two triangles using the sine angle formula as



$A_1 = XY_x sin(\frac{\theta}{2})$

$A_2 = (N-X)Y_x sin(\frac{\theta}{2})$. We can find the total area in the same way as

$A_1 + A_2 = X(N-X)sin(\theta)$. So

$$XY_x sin(\tfrac{\theta}{2}) + (N-X)Y_x sin(\tfrac{\theta}{2}) = X(N-X)sin(\theta)$$
$$\Rightarrow Y_x = \frac{X(N-X)}{X+(N-X)} \frac{sin(\theta)}{sin(\frac{\theta}{2})}$$
$$= X(N-X) \frac{sin(\theta)}{Nsin(\frac{\theta}{2})}$$

Since we care only about the relative ratios of Yi we can ignore the constant multiplicative factor and state that Yx=X(N-X). We now want to re-express those lengths as distances relative to the perpendicular intersection point. We can easily see that this intersection occurs for a line with

$X = N - X \Rightarrow X = \frac{N}{2}$ so let

$$Z_i = Y_{N/2} - Y_{N/2+i}$$
$$= \frac{N^2}{4} - \left(\frac{N}{2}+i\right)\left(N-\frac{N}{2}-i\right)$$
$$= \frac{N^2}{4} - \left(\left(\frac{N}{2}\right)^2 - i^2\right)$$
$$= i^2$$

exactly as desired.

The Dynamics of Spinning Polyominoes

George Bell   Mar 21st 2018   gibell@comcast.net

Q: Which two pentominoes are indistinguishable as rigid, rotating bodies?

A: Two rigid bodies have the same rotational dynamics if they have the same **principal moments of inertia**. Therefore, to answer this question, we need to calculate the **moment of inertia tensor** for each of the 12 pentominoes (the principal moments of inertia are the eigenvalues of this matrix). The pentominoes are 2D objects obtained by joining 5 squares along their edges in all possible ways. As spinning objects, we consider that each pentomino is composed of five 1x1 squares of mass 1 and thickness *h* as shown in Figure 1.

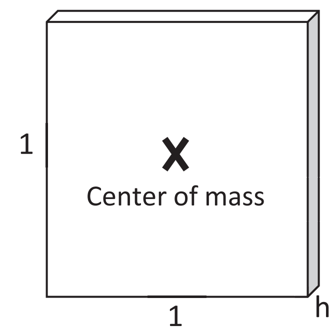


Figure 1: Polyomino building blocks.

As we will see, we can use any h between 0 and 1 and our results do not change qualitatively. We do not even have to build our pentominoes from squares of height *h*, we can use any object with square symmetry, or we could use circles or spheres (connected at points).

All moment of inertia tensors we will calculate are taken about the center of mass. The moment of inertia tensor J for an object composed of *n* squares is given by the sum of the moment of inertia tensors of the component squares, nJ_1, plus the moment of inertia tensor J_2 of the squares as point masses displaced from the center of gravity [1]. The moment of inertia tensor for the rectangular solid in Figure 1 is given by

$$J_1 = \frac{1}{12}\begin{pmatrix} 1+h^2 & 0 & 0 \\ 0 & 1+h^2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The limiting cases $h \to 0$ (thin plate) and $h = 1$ (cubes) are interesting special cases. In the cube case J_1 is 1/6 times the identity matrix.

Now we consider the contribution from the squares as point masses displaced from the center of gravity. As an example take the “P” pentomino (Figure 2).

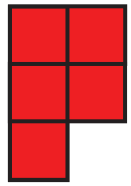


Figure 2: The P-pentomino.

The 5 squares have their centers at coordinates $(0,0), (0,1), (0,2), (1,1), (1,2)$. The center of mass of these 5 point masses is at $(0.4,1.2)$, so subtracting this from each coordinate we obtain a set of 5 point masses with center of mass at the origin:
$(-0.4,-1.2), (-0.4,-0.2), (-0.4, 0.8), (0.6,-0.2), (0.6,0.8)$. For a set of unit point masses in 2D at coordinates $(x_i, y_i, z_i = 0)$ the moment of inertia tensor about the origin is given by

$$J_2 = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

where

$$I_{xx} = \sum_i y_i^2 + z_i^2 = \sum_i y_i^2$$
$$I_{yy} = \sum_i x_i^2 + z_i^2 = \sum_i x_i^2$$
$$I_{zz} = \sum_i x_i^2 + y_i^2 = I_{xx} + I_{yy}$$
$$I_{xy} = -\sum_i x_i \, y_i$$

Calculating this for the P-pentomino, we get the total moment of inertia tensor about the center of gravity (0,0)

$$J = 5J_1 + J_2 = 5J_1 + \begin{pmatrix} 2.8 & -0.6 & 0 \\ -0.6 & 1.2 & 0 \\ 0 & 0 & 4.0 \end{pmatrix}$$

The principle moments of inertia are the eigenvalues of J, these are always real and non-negative. Because J_1 is diagonal, it only affects the magnitude of the eigenvalues. We calculate the eigenvalues of J_2 as 4, 3 and 1 with corresponding unit eigenvectors $(0,0,1)$, $(3,-1,0)/\sqrt{10}$ and $(1,3,0)/\sqrt{10}$. We now adopt the convention of displaying the pentomino with the two principal axes beginning at the center of mass, with length proportional to the magnitude of the eigenvector. The largest eigenvalue is always aligned with the z-axis and is not shown in these (2D) diagrams.

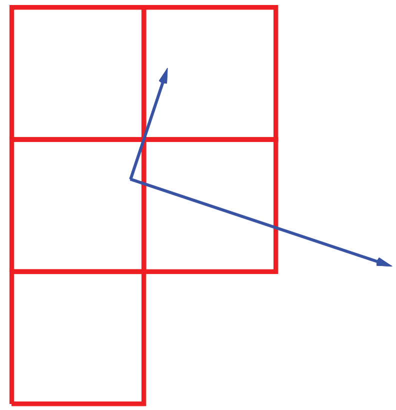


Figure 3: The P-pentomino, with principle axes of inertia shown at the center of mass.

We now repeat these calculations for all 12 pentominoes, with results shown in Table 1. This table shows the polyominoes sorted by decreasing principal moments of inertia. These eigenvalues are those of the matrix J_2, to obtain the eigenvalues of J we add $n/6$ to the largest eigenvalue and $n(1+h^2)/12$ to the other two. From here on out we assume $h=0$ (2D polyominoes).

Name	λ_1	λ_2	λ_3
I	10	10	0
L	7.6	$(19+3\sqrt{29})/5 \cong 7.031099$	$(19-3\sqrt{29})/5 \cong 0.568901$
N	6.4	$(16+\sqrt{181})/5 \cong 5.890725$	$(16-\sqrt{181})/5 \cong 0.509275$
V	6.4	5	1.4
Z	6	$3+\sqrt{5} \cong 5.236068$	$3-\sqrt{5} \cong 0.763932$
Y	6	$3+\sqrt{5} \cong 5.236068$	$3-\sqrt{5} \cong 0.763932$
W	5.6	5	0.6
U	5.2	4	1.2
T	5.2	3.2	2
F	4.8	$(12+\sqrt{29})/5 \cong 3.477033$	$(12-\sqrt{29})/5 \cong 1.322967$
P	4	3	1
X	4	2	2

Table 1: The principal moments of inertia of the 12 pentominoes (same order as in Figure 4).

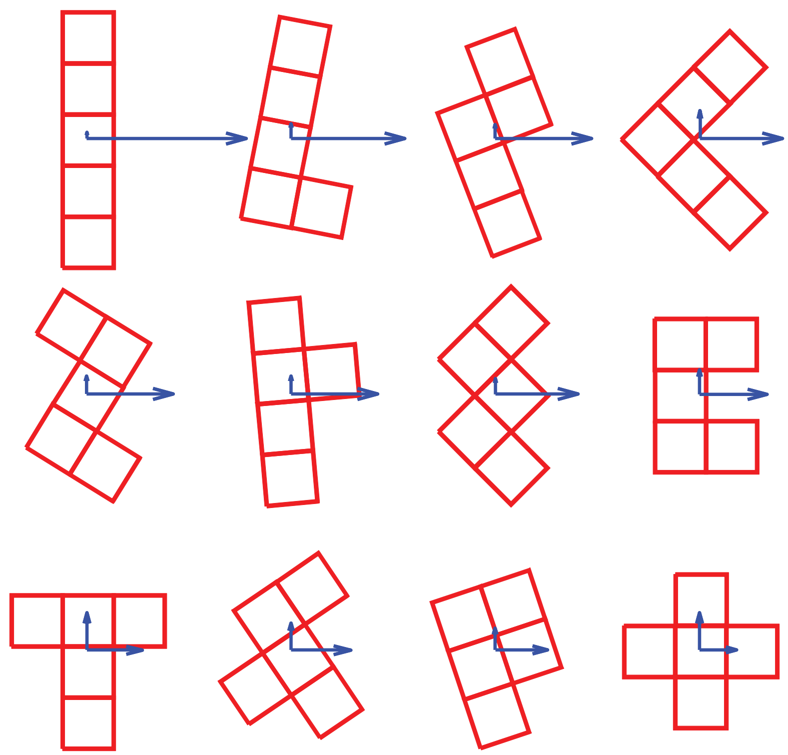


Figure 4: The 12 pentominoes oriented with principal axes aligned with the coordinate axes.

We see in Table 1 that there are exactly two pentominoes, Y and Z, which have identical principal moments of inertia. These will rotate exactly the same as freely spinning objects. If we rotate each so that their principal axes correspond, we obtain Figure 5a. We note that the Z pentomino has rotational symmetry, while the Y pentomino has no symmetry.

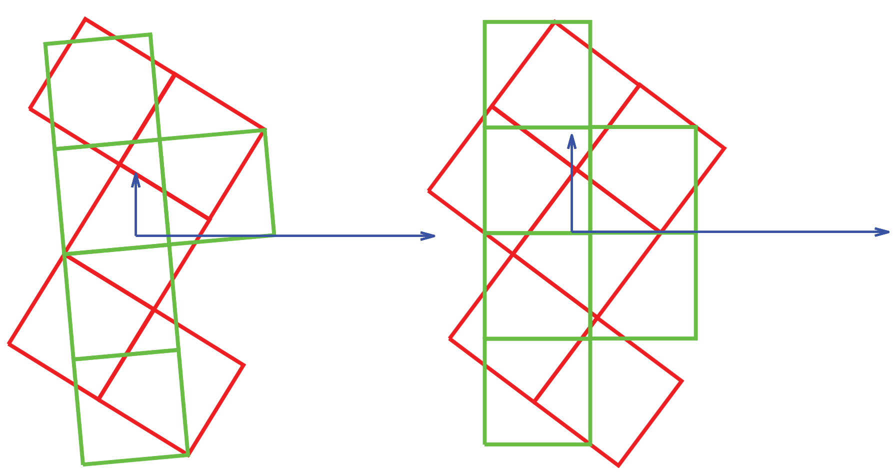


Figure 5: The Z and Y pentominoes with principle axes aligned and identical (left). The two hexominoes with identical moments of inertia.

What about larger polyominoes?

We have repeated these calculations through octominoes (n=8). There are exactly two hexominoes with the same principal moments of inertia (Figure 5b). These can be obtained from the Z and Y pentominoes by adding a square to each.

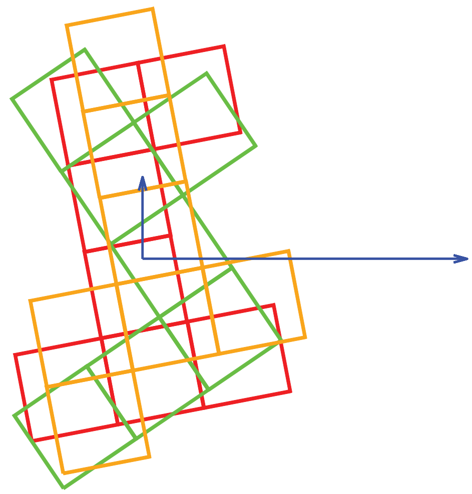


Figure 6: Three heptominoes (n=7) all sharing the same three principal moments of inertia.

Beyond n=6, polyominoes sharing the same principal moments of inertia are common. Figure 6 shows **three** septominoes (n=7) with the same principal moments of inertia. Figure 7 shows **six** octominoes which all share the same three principal moments of inertia: $(19.5,16,3.5)$!

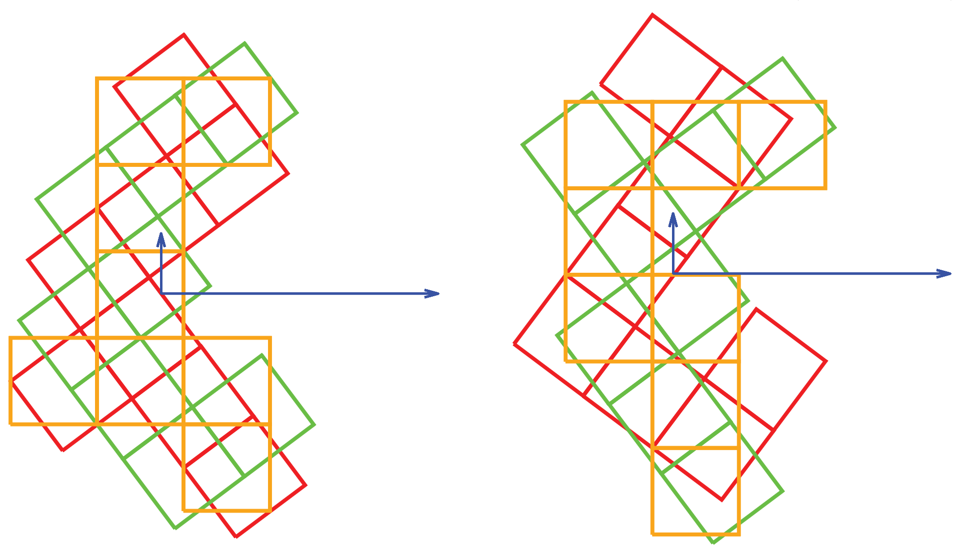


Figure 7: Six octominoes (n=8) all sharing the same principal moments of inertia.

[1] H. Goldstein, **Classical Mechanics**, Chapter 5, 1980 Addison-Wesley

Introducing The πTOP® or PiTOP®

Kenneth Brecher
Departments of Astronomy and Physics
Boston University
Boston, MA 02215, U.S.A.
Email: brecher@bu.edu
PiTOP® Website: http://www.thepitop.com

Why Pi at G4G13?

(Martin Gardner caricature by Ken Fallin, 2010)

$$e^{\pi\sqrt{163}} \equiv 262{,}537{,}412{,}640{,}768{,}744.0$$

Martin Gardner demonstrated a playful interest in Pi. His April 1975 column in *Scientific American* entitled "Six Sensational Discoveries" reported that in 1974, Ramanajun's 1913 conjecture shown above had been proven to be an exact result!!!

What is the PiTOP®?

It is a physical embodiment of the mathematical constant π. This disk, has a radius of r = 1" and thickness t = 1/π" ~ .32". When made in brass, it weighs ~ 4.8 ounces. It displays the first 109 digits of Pi in a spiral pattern on one side. (The pattern was designed in collaboration with Kaz Brecher.)

What is the point of the PiTOP®?

It is a tactile hand sized stress reliever.

It is an elegant paperweight.

It is a beautiful March 14 Pi Day gift.

It is a personal fidget device.

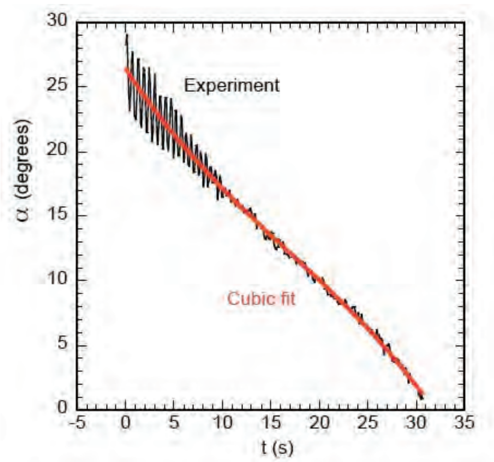
And it also symbolizes profit in economics!

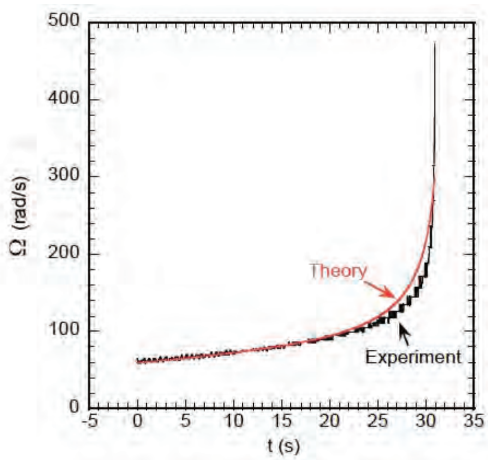
Sound and Light Effects

The PiTOP® was designed to optimize its dynamical properties based on a variety of experiments that I carried out with many prototypes. As the PiTOP® spins and precesses, it produces a hypnotic sound and light display.

PiTOP® Dynamics

After spinning it on its edge like a coin, the PiTOP® loses rotational energy due to friction. As the angle α that it makes with the horizontal decreases with time, its precession frequency Ω increases, tending toward a "finite time singularity".





The above data was collected from time-lapse photographic measurements of the spin of a PiTOP prototype that I sent for analysis to Professor Rod Cross at the University of Sydney, (cf. "Effects of Rolling Friction on a Spinning Coin or Disk", *European Journal of Physics*, 39, #3, 5, 2018).

Cubing the PiTOP®

Although one cannot square the circle in a finite number of steps using only a compass and a straightedge, the PiTOP® automatically cubes a right circular cylinder of radius r since it has volume $V_{PiTOP} = \pi r^2 t = \pi r^2 r/\pi = r^3 = V_{cube}$.

The PiTOP® and The PhiTOP®

The PhiTOP was previously introduced at G4G12. They can both be found at:

https://www.etsy.com/shop/SiriusEnigmas

What the Liar Taught Achilles[1]

Gary Mar
GROUP FOR LOGIC AND FORMAL SEMANTICS
SUNY at Stony Brook, Stony Brook, New York 11794-3750

ABSTRACT. Zeno's paradoxes of motion and the semantic paradoxes of the Liar have long been thought to have metaphorical affinities. There are, in fact, isomorphisms between variations of Zeno's paradoxes and variations of the Liar paradox in infinite-valued logic. Representing these paradoxes in dynamical systems theory reveals fractal images and provides other geometric ways of visualizing and conceptualizing the paradoxes.

KEYWORDS: Zeno's paradox, Liar paradox, semantic paradox, dynamical systems, chaos, fractal, infinite regress, Lewis Carroll, Łukasiewicz, Sierpinski, Tarski.

In his classic "What the Tortoise Taught Achilles" [1895], Lewis Carroll borrowed characters from Zeno's paradoxes of motion and transported them into a dialogue about a paradox he had discovered in attempting to justify fundamental laws of logic. Carroll did not claim that there were any *formal* similarities between the infinite regresses in Zeno's paradoxes of motion and the infinite regress of logical justification. Instead, it is likely that Carroll's parable represents an attempt to express some difficulties that he intuitively felt but could not adequately explain.

In this paper we will show that there are, in fact, mathematically demonstrable isomorphisms between variations of Zeno's paradoxes and intriguing new variations of the paradox of the Liar (see Mar and Grim [1991]). These similarities can be visualized using the computer graphic tools of dynamical systems theory. The results of this paper support Wesley Salmon's observation that our current resolutions of Zeno's paradoxes often go hand in hand with our current mathematical tools.[2]

I.

Zeno's paradox of motion known as the DICHOTOMY PARADOX comes in two forms. In the PROGRESSIVE FORM, Achilles is never able to complete the racecourse. If it is possible for Achilles to complete the racecourse, then he must first reach the halfway point. But before he can complete the racecourse, he must reach the halfway point of the remaining distance, and so on *ad*

[1] This article was published in *The Journal of Philosophical Logic*, 1992, vol. 28, pp. 29-46. In this revised version, some explanatory remarks have been added to the footnotes, including an intuitive account of fractal dimensions in footnote 4 and an update of Devaney's mathematical definition of chaos in footnote 9.
[2] The introduction to Wesley Salmon [1970].

infinitum. Achilles will never reach his final destination, so the argument goes, for to do so would require Achilles to traverse an infinite number of points in a finite amount of time. The REGRESSIVE FORM of the paradox shows, by a similar argument, that Achilles is never able to get started. Before he can reach the halfway point, he must reach the point halfway to the halfway point, and so on *ad infinitum*. Even for Achilles to get started at all would require him to traverse an infinite number of points in a finite amount of time.

Let's combine the Progressive and Regressive Dichotomies in a natural way. In this variation of Zeno's paradox, Achilles always runs *half* the distance to either the starting point or the ending point of the racecourse, whichever is *farther*. We call this the AMBIVALENT ACHILLES.

> AMBIVALENT ACHILLES: I run *half* the distance to the beginning of racecourse or the ending of the racecourse, whichever is *farther*.

The path of the AMBIVALENT ACHILLES is given by

$$x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n > 1/2 \\ (1+x_n)/2 & \text{if } x_n \leq 1/2 \end{cases}$$

and the path is attracted to two points on the racecourse, namely, the points 1/3 and 2/3. Achilles quickly oscillates between these two points in the limit. If Achilles were to reach one of these points, he would then be forever trapped in a cycle of period 2, oscillating back and forth between the two fixed points.

The path of the AMBIVALENT ACHILLES can be visualized graphically. Consider a time series diagram with x_n representing the successive positions of Achilles on the unit interval. Given an initial starting value of $x_n = 0$ we have convergence toward a cycle of period two of the fixed-point attractors 1/3 and 2/3. An alternative way of visualizing this dynamical behavior in dynamical systems theory is in terms of a web diagram. A *web diagram* is a method of graphing the iterated values of a function f(x). Beginning by drawing a line vertically from (x_0, 0) to (x_n, x_{n+1}), the web diagram next draws a line horizontally from (x_n, x_{n+1}) to (x_{n+1}, x_{n+1}) and then iterates the process by using x_{n+1} for x_n. Figure 1 compares a time series graph with its corresponding web diagram. The fact that the AMBIVALENT ACHILLES approaches a cycle of period 2 is visually evident as the lines of the web diagram converge on a simple box whose corners intersect the $x = y$ line at the attractor points 1/3 and 2/3.

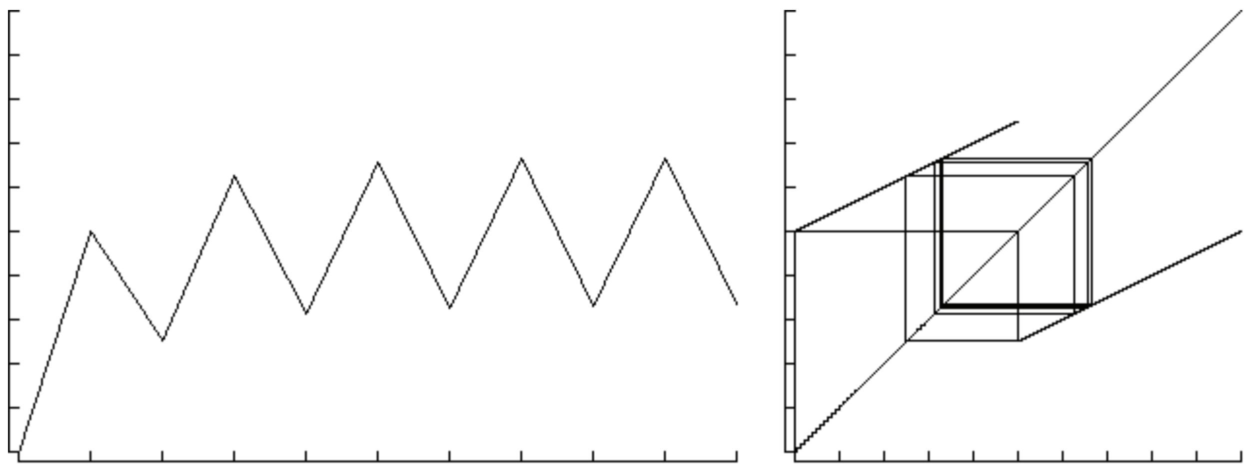


Figure 1. A times series graph and web diagram for the AMBIVALENT ACHILLES.

The behavior of the AMBIVALENT ACHILLES can be conveniently analyzed in binary arithmetic. We can represent running half way to the starting point by prefacing a '0' and right-shifting the binary string that represented Achilles previous position on the [0,1] interval. Similarly, we can represent running half way to the end point by prefacing the binary string with a '1' and right-shifting. It is now easy to see why the AMBIVALENT ACHILLES is drawn inexorably to the attractor points 1/3 and 2/3, which in binary notation are the strings 0.010101..... and 0.101010......, respectively. The successive positions of the AMBIVALENT ACHILLES can be seen as right-shifting and alternately prefacing of '0' and '1' to the binary representation of the initial starting point. The difference between the initial starting point and the attractor points, therefore, quickly diminishes as the binary string for the initial position is prefaced by increasingly long strings of alternating '0's and '1's.[3]

[3] It is intriguing to note that Conway's "surreal" numbers can be represented as a sequence of 1's and 0's and modeled on Zeno's paradoxes. Each number is represented by a finite or transfinite sequence of 1's and 0's. Each repeated initial sequence of 1's advance one unit forward. Hence, 1 := 1; 2 := 11, 3 := 111, etc. However, once there is a change from 1 to 0 or from 0 to 1, you go half your current unit in the opposite direction. Thus, for example, 111 is 3, but 110 is 3/2, 11 is 2, but 10 is 1/2. See Conway [1976], Knuth [1974], and Shulman [1995].

That Zeno's runners rehearse self-similar patterns at decreasing scales suggests the *fractal character* of Zeno's paradoxes. This fractal character becomes visually evident when we generalize the one-dimensional AMBIVALENT ACHILLES to two dimensions. We call this two-dimensional Achilles the TRIVALENT ACHILLES. In this new variation Achilles will have the *three* end points of a triangle, rather than just *two* end points of a line, toward which to run. We can imagine Achilles beginning somewhere in the middle of a triangular field. Achilles trifurcates, and each of his three counterparts runs halfway toward the three goal points. Then each of these counterparts trifurcates, and this process is repeated.

The limit points of the TRIVALENT ACHILLES form the famous SIERPINSKI FRACTAL. The Sierpinski fractal is generated by the iterative procedure of dividing an equilateral triangle in four equal triangles and removing the middle fourths. Figure 2 shows a variation of the Sierpinski fractal generated from an isosceles right triangle. Fractals derive their name from the fact that they can have fractional dimensions. The *Hausdorff dimension* for the Sierpinski fractal, for example, is log3/log2, which is approximately 1.58.[4]

To generate a computer image of the Sierpinski fractal, we plot a random sampling of all such paths in a process which has been dubbed by Michael Barnsley [1988] as the *chaos game*.

TRIVALENT ACHILLES: I run *halfway* to one of the three goal points chosen at random.

A *deterministic* way of obtaining the Sierpinski fractal (intuitively obtained by playing backwards a "movie" of one of the TRIVALENT ACHILLES runners) is discussed by Manfred Schroeder [1991]. It proceeds as follows:

ESCAPIST ACHILLES: I run *twice* the distance away from the nearest point (along a straight line from that point).

---

[4] Fractal dimensions are a generalization of integer dimensions. Suppose we divide the sides of a square in half to obtain four smaller copies. So when the reduction factor $r = 2$, the number of similar members $m = 4$. Intuitively, the dimension of the square is the power d such that $m = r^d$ so the dimension d of the square is 2 since $4 = 2^2$. Notice if we subdivide the side of the square into thirds, then the reduction factor $r = 3$, $m = 9$, and again the dimension $d = 2$ since $9 = 3^2$. Dividing the side of a cube by 2 results in 8 smaller cubes so $m = 8$, $r = 2$, and so the dimension d of the cube is 3 since $8 = 2^3$. Now when the side of the Sierpinski triangle is divided by $r = 2$, we obtain only $m = 3$ copies of the Sierpinski triangle, and so, generalizing the above idea, the *fractal* or *Hausdorff* dimension d satisfies the equation $3 = 2^d$. Thus, the *fractal* or *Hausdorff dimension* of the Sierpinski trangle $d = \log 3/\log 2 \approx 1.58$. For discussion see Schroeder [1991], pp. 16-17.

The set of all points that do *not* eventually escape from the triangle forms the Sierpinski triangle.

The reason why the ESCAPIST ACHILLES generates the Sierpinski triangle is easily seen when analyzed in binary arithmetic. Consider a square whose corners are (0,0), (0,1), (1,0), and (1,1). Let's say that the first three points are the goal posts for the TRIVALENT ACHILLES. Now doubling the distance from the nearest point is equivalent to left-shifting the binary strings. Consider, for example, (x_0, y_0), where x_0 = .0011 and y_0 = .0101. This point will *not* be in the Sierpinski fractal because there is a '1' in both strings in the fourth position. We can verify this by tracing the path of the ESCAPIST ACHILLES. We begin at the point (.0011, .0101) or (3/16, 5/16). We then double the distance from (0,0), the closest of the goal posts, to arrive at (.011, .101) or (3/8, 5/8). We then double the distance from (0,1) (which are the leading values of the binary expansion) to arrive at (.11, .01) or (3/4, 1/4). Doubling the distance from (1,0), the nearest goal point, we arrive at (.1, .1) or (1/2, 1/2). Doubling the distance from (0,0) once again, the ESCAPIST ACHILLES finally escapes the Sierpinski triangle arriving at the point (1,1).

The pairs of binary strings that are points in the Sierpinski triangle are those where both values cannot simultaneously be greater than or equal to 1/2. In binary notation, these will be precisely those pairs of strings that do not have '1's in the same position in their respective strings.

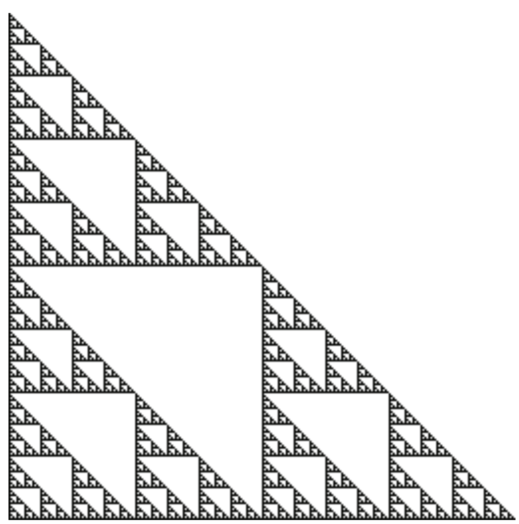


Figure 2. The TRIVALENT and ESCAPIST ACHILLES generate the Sierpinski triangle.

The above characterization of the points in the Sierpinski fractal makes clear an intriguing connection between Zeno's paradoxes of motion and propositional logic. The points *not* in the Sierpinski fractal are precisely those that have the constant value 0 for a binary bit-wise conjunction. The following simple QBasic program for plotting points not in the Sierpinski triangle can verify this:

```
Screen 9
For p = 0 to 255
For q = 0 to 255 – p
If p and q then Pset (p, q)
Next p
Next q
```

Suppose the propositional letters p and q have the following final columns in a truth table: $/p/ = \langle 1, 1, 0, 0\rangle$ and $/q/ = \langle 1, 0, 1, 0\rangle$. Then $/(p \wedge q)/ = \langle 1, 0, 0, 0\rangle$. Given x = $/p/$ and y $=/q/$, the plot at the point (x, y) is $/(p \wedge q)/$. The point (x,y) is *not* plotted only if $/(p \wedge q)/ = \langle 0,0,0,0\rangle$. This will be true if the bit-wise conjunction for /p/ and /q/ is *not* true, i.e., if the Sheffer Stroke of p and q, $(p|q)$, is true. If we now consider not merely finite truth assignments, but *infinitary* ones, we generate the Sierpinski fractal. The successive approximations of the Sierpinski fractal therefore represent the non-tautologies. In the limit, therefore, the Sierpinski triangle created by the negative space among the plotted points can be thought of as a picture of the set of tautologies of propositional logic.[5]

II.

To make the structural identity of a variation of Zeno's paradox and paradoxes of logic precise, we will set forth an infinite-valued Łukasiewiczian logic with a self-reference operator based on ideas discussed by Rescher [1969] and van Fraassen [1972]. We call the language

[5] See St. Denis and Grim [1997].

DIALOGUE (for Dynamical Iterative Algorithmic Language Offering Genuinely Unstable Evaluations). The syntax for DIALOGUE is given by first specifying an *alphabet* of symbols:

p, *q*, *r* (with or without subscripts)	propositional variables
~ , →	propositional connectives
) , (	parentheses
/ /	the propositional value operator
t , ***f***	truth-value constants
V	2-placed multi-valued truth predicate
θ	the self-referential propositional operator
+ , ÷	the addition sign and the division sign

We complete the specification of the syntax by giving a set of *grammatical rules* defining the set of *well-formed formulas* (wffs) and the set of *value terms*.

(G1) Any propositional variable is a wff.

(G2) θ is a wff.

(G3) If φ and ψ are wffs, then so are ~φ and (φ → ψ).

(G4) The truth-values ***t*** and ***f*** are value terms.

(G5) If α and β are value terms so are α+ β and α ÷ β.

(G6) If φ is a wff, then /φ/, the *value* of the wff φ, is a value term.

(G7) If φ is a wff and α is a value term, then Vαφ is a wff.

We shall say that a wff in which θ occurs is *self-referential*; otherwise, we shall call the wff *normal*.

The semantics for DIALOGUE is given by assigning to all normal wffs a real value in the [0,1] interval and assigning to self-referential sentences an iterative evaluative algorithm.

(S1) For each propositional variable *p*, we assign *p* a value /*p*/ ∈ [0,1].

(S2) /θ/ is assigned a value x_0 ∈ [0,1].

(S3) For normal wffs φ and ψ, where /φ/ and /ψ/ are the values of φ and ψ, respectively, we have:

(A) /~ φ/ = 1 – /φ/.

(B) /(φ → ψ)/ = MIN{1, 1 – /φ/ + /ψ/}.

(C) /Vαφ/ = 1 – ABS(/α/ – /φ/).

(S4) The truth-values ***t*** and ***f*** are assigned the values /***t***/ = 1 and /***f***/ = 0, respectively.

(S5) If α and β are value terms, where $/\alpha/$ and $/\beta/$, then $/\alpha + \beta/ = /\alpha/ + /\beta/$, and $/\alpha \div \beta/ = /\alpha/ \div /\beta/$, where $/\beta/ \neq 0$ in which case the expression is undefined.

(S6) If χ is a self-referential sentence, then $/\chi/$ is an iterative evaluative algorithm defined as follows:

(A) If $\chi = \sim\theta$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - x_n$, where $x_0 = /\theta/$.

(B) If $\chi = (\theta \rightarrow \psi)$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - \mathrm{ABS}(\mathrm{MIN}\{1, 1 - x_n + /\psi/\} - x_n)$;
if $\chi = (\varphi \rightarrow \theta)$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - \mathrm{ABS}(\mathrm{MIN}\{1, 1 - /\varphi/ + x_n\} - x_n)$;
if $\chi = (\theta \rightarrow \theta)$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - \mathrm{ABS}(\mathrm{MIN}\{1, 1 - x_n + x_n\} - x_n)$;
where $x_0 = /\theta/$.

(C) If $\varphi = \theta$, then $/V\alpha\theta/$ is the algorithm $x_{n+1} = 1 - \mathrm{ABS}(/\alpha/ - x_n)$, where $x_0 = /\theta/$.

This completes the semantics for DIALOGUE. A few comments are in order.

First, the infinite-valued rules for normal sentences in (S3) are *faithful* to classical logic: when the values of the sentences are restricted to the classical truth-values, we obtain the classical truth tables. Intuitively, the *negation* of *p* is true to the extent that *p* is *un*true, i.e., to the extent the value of *p* *differs* from the value of 1 or complete truth. The rule for the conditional is what makes the system characteristically Łukasiewiczian. Given the definitions:

$$(\varphi \vee \psi) := ((\varphi \rightarrow \psi) \rightarrow \psi)$$

$$(\varphi \wedge \psi) := \sim(\sim\varphi \vee \sim\psi)$$

we obtain Łukasiewicz's *Boolean* evaluation rules:

$$/(\varphi \wedge \psi)/ = \mathrm{MIN}\{/\varphi/, /\psi/\} \ ,$$

and

$$/(\varphi \vee \psi)/ = \mathrm{MAX}\{/\varphi/, /\psi/\} \ .$$

Given the classical equivalence $(\varphi \leftrightarrow \psi) := ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$, we derive the *biconditional rule*:

$$/(\varphi \leftrightarrow \psi)/ = 1 - \mathrm{ABS}(/\varphi/ - /\psi/) \ .$$

This rule states that the *biconditional* is *true* to the extent that its constituents do *not differ* in truth-value.

Secondly, the value of the proposition asserting that the proposition *p* has the value α, $V\alpha p$, is given by the schema $/V\alpha p/ = 1 - \mathrm{ABS}(\alpha - /p/)$. This schema is a generalization of the Tarskian (T) schema. Using the biconditional rule, we can state *Tarskian* (T) *schema* by $/T'p'/ = 1 - \mathrm{ABS}(\boldsymbol{t} - /p/)$, where $\boldsymbol{t}$ is the value of 1 or complete truth. Replacing the constant $\boldsymbol{t}$ with a parameter α (which ranges over the [0,1] interval) and replacing the bivalent truth predicate 'T' with a multi-valued relation '$V\alpha p$' (which is to be read 'α is the truth-value of the sentence *p*'), we obtain Rescher's [1969] schema for his parametric-operator development of many-valued logics.[6] Intuitively, Rescher's schema states that the sentence $V\alpha p$ is true to the extent to which the value of *p* does *not differ* from α.[7]

Thirdly, DIALOGUE contains its own truth predicate and a form of self-reference but avoids the inconsistency of semantically closed languages by assigning iterative semantic algorithms to self-referential sentences rather than univocal truth-values. The semantic paradoxes have, in a way, been a trap for logicians who, in their attempts to *solve* the paradoxes, have tended to view the patterns of paradox as simpler and more predictable than they actually are. Even in the sophisticated work of Barwise and Etchemendy on the Liar [1987], the cyclical regularity of the semantical paradoxes has been obvious but their incalculable complexity has remained hidden. Here, instead of searching for simple patterns of semantic *stability* (as in Gupta [1982] and Herzberger [1982]), in *DIALOGUE* we will exhibit infinitely complex and chaotic patterns of semantic *in*stability, which have gone virtually unexplored.

III.

The above infinite-valued Łukasiewiczian logic can be used to obtain generalizations and variations on the classical paradox of the Liar. Recall that the CLASSICAL LIAR is a sentence that asserts it own falsity:

---

[6] Rescher [1969], p. 81.

[7] The Tarskian (T) schema was stated above in terms of sentences; Rescher states his V**v***p* schema in terms of propositions. For present purposes, we set aside the philosophical controversy as to what should properly be regarded as the bearers of truth. See Church [1956], p. 27, footnote 72.

The boxed sentence is false.

According to the Tarskian (T) schema, a sentence stating that a sentence *p* is true has the same truth-value as *p* itself. Hence,

(1) 'The boxed sentence is false' is true if and only if the boxed sentence is false .

But since it is empirically true that

(2) 'The boxed sentence is false' is identical to the boxed sentence ,

we may infer from (1) and (2) by Leibniz's Law that

(3) The boxed sentence is true if and only if the boxed sentence is false .

The assumption that the boxed sentence is true leads to the conclusion that it is false, and the assumption that it is false leads to the conclusion that it is true. The semantic behavior of the CLASSICAL LIAR can therefore be represented as an infinite oscillation between the classical truth-values true and false.

Intuitively, the CHAOTIC LIAR is a sentence that self-referentially states that it has the value of falsehood. In *DIALOGUE* we can represent this sentence by $\mathrm{V}f\,\theta$. The semantic algorithm for this sentence of *DIALOGUE* is given by the iterative algorithm $x_{n+1} = 1 - \mathrm{ABS}(0 - x_n)$. Given an initial estimated value of x_0, the successive estimated values for the continuous valued CLASSICAL LIAR will be an alternating cycle of period 2 between the values x_0 and $1 - x_0$. The single exception is the fixed-point value of 1/2. Figure 3 shows a web diagram for the CLASSICAL LIAR within an infinite-valued logic with the periodic values 1/3 and 2/3, reminiscent of the AMBIVALENT ACHILLES.

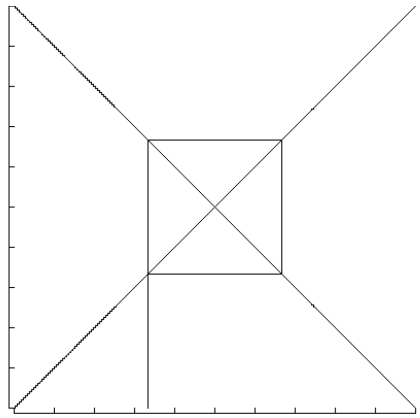


Figure 3. Web diagram for the CLASSICAL LIAR in an infinite-valued context.

Other infinite-valued variations of Liar-like sentences are possible. *DIALOGUE* allows us to evaluate sentences that do not attribute to themselves a *particular* truth-value α but rather attribute to themselves a truth-value expressed as a *function* of its self-referential values. To obtain the algorithm for evaluating such sentences, we successively replace the α in the Vαθ schema with the sequentially estimated values of the self-referential wff. Consider, for example, the following pair of Liar-like sentences:

CAUTIOUS TRUTH-TELLER: This sentence is *half* as true as it is estimated to be true.

CONTRADICTORY LIAR: This sentence is *as true as the contradiction* consisting of the conjunction of itself and its negation.

This CAUTIOUS TRUTH-TELLER is represented by V/θ/÷2 θ, where the numeral '2' abbreviates '/***t***/ + /***t***/'. The semantic algorithm for the CAUTIOUS TRUTH-TELLER is therefore:

$$x_{n+1} = 1 - \text{ABS}(x_n \div 2 - x_n) \ .$$

The CONTRADICTORY LIAR, on the other hand, can be represented by V/(*p*∧~*p*)/θ. Using Łukasiewicz's rule for conjunction, we obtain the following semantic algorithm:

$$x_{n+1} = 1 - \text{ABS}(\text{MIN}\{x_n, 1 - x_n\} - x_n) \;.$$

The semantic behaviors of these Liar-like sentences can be made visually perspicuous using web diagrams. The semantics for the continuous-valued CLASSICAL LIAR appears in a web diagram as a nested series of simple boxes. Given an initial value 1/3, for example, the CLASSICAL LIAR will oscillate monotonously in a cycle of period 2 between 1/3 and 2/3. The behaviors of the CAUTIOUS TRUTH-TELLER and the CONTRADICTORY LIAR, on the other hand, are diametrically opposed. The CAUTIOUS TRUTH-TELLER yields a *fixed-point attractor*: no matter what the initial value, the successively estimated values are inevitably drawn toward the fixed point of 2/3. The CONTRADICTORY LIAR, in contrast, yields a *fixed-point repellor*: for any values other than the fixed point 2/3, the successively revised estimates for the CONTRADICTORY LIAR are repelled away from 2/3 until the values settle on the oscillation between 1 and 0, characteristic of the CLASSICAL LIAR.

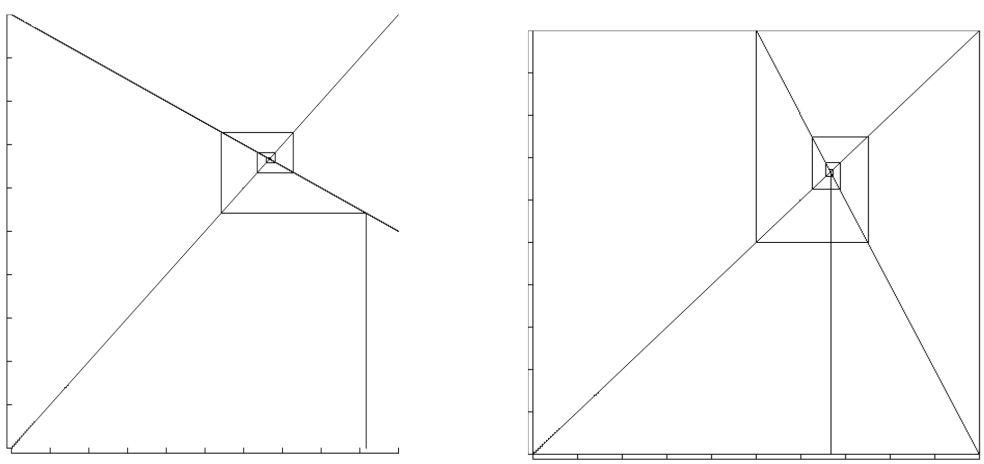


Figure 4. The CAUTIOUS TRUTH-TELLER and the CONTRADICTORY LIAR exhibit opposite semantic behaviors in terms of fixed-point attractors and repellors.

The semantic behavior of the CLASSICAL LIAR, though identical to the CONTRADICTORY LIAR on classical values, diverges from the CLASSICAL LIAR on the range of values between 0 and 1. Sentences like the CAUTIOUS TRUTH-TELLER and the CONTRADICTORY LIAR, on the other hand, have the opposite semantic behavior in terms of being attractors and repellors around the same fixed point. Infinite-valued logic can, therefore, reveal intriguing new patterns of paradox that have remained hidden in a classical bivalent setting.[8]

IV.

Consider next an intriguing variation of Zeno's paradox, which we call the SISYPHUS. Imagine Sisyphus is pushing a boulder up a hill. If Sisyphus is less that halfway up the hill, he is able to push the boulder to a point *twice* as far up the hill as he is from the bottom. Once Sisyphus passes the halfway point, however, his fortunes reverse. Sisyphus slips down the hill until he is at a point that is *twice* as far from the bottom as he had left to reach the top of the hill. This Sisyphean task continues on *ad infinitum*.

> SISYPHEAN ACHILLES: If I'm less than halfway up the hill, I'll double my progress. However, if I'm more than halfway up the hill, I will slip to a point that is twice that distance I had left.

The successive points in SISYPHUS's journey is given by the algorithm:

$$x_{n+1} = \begin{cases} 2^*x_n & \text{if } x_n < 1/2 \\ 2^*(1 - x_n) & \text{if } x_n \geq ½ \end{cases}$$

In binary notation the successive values of the Sisyphean algorithm for an initial value x_o can be obtained by the following operation on binary strings:

(1) if the leading digit is '0', shift all the digits to the left,

and

(2) if the leading digit is '1', take the complement and then shift all the digits to the left.

---

[8] Consider the CONTINGENT LIAR based an infinite-valued generalization of the paradox due to Kleene and Rosser [1935], discussed by Curry [1941], and also known as Löb's paradox (see Barwise and Etchemendy [1987], footnote 14). The CONTINGENT LIAR is the sentence: *This sentence is as true as the conditional: if this sentence* (i.e., the CONTINGENT LIAR itself) *is true then q*. Here *q* is some contingent sentence. Using the infinite-valued rule for the Łukasiewiczian conditional, the sequence of estimated values for the Contingent Liar is $x_{n+1} = 1 - \mathrm{ABS}(\ \mathrm{MIN}\{1 - x_n + /q/,\ 1\} - x_n\ )$. The CONTINGENT LIAR exhibits the fixed-point semantic behavior of the TRUTH-TELLER (*This sentence is true*) on the interval [0, /q/] and exhibits chaotic semantic behavior on the interval [/*q*/,1].

If you start with any irrational initial value, then the orbits of the algorithm will *appear* to be completely *random* even though the rule for generating the value is completely *deterministic*. Practically speaking, unless we know the initial value with infinite precision, the successive values of this algorithm are unpredictable. The algorithm for the Sisyphean Achilles is a paradigmatic example of what is known as *deterministic chaos* (see Devaney [1989]).

`Consider next an intriguing, infinite-valued generalization of the CLASSICAL LIAR, which asserts of itself, not that it is simply false, but that it is *true* to the *extent* that it is *false*:

The boxed sentence is as true as it is false.

This is perhaps the most natural generalization of the CLASSICAL LIAR into an infinite-valued context. This sentence has been dubbed the CHAOTIC LIAR for reasons that will soon become apparent.

What the CHAOTIC LIAR asserts of itself is that it is false; hence, $S(x_n) = 1 - \mathrm{ABS}(0 - x_n)$. Since $x_n \geq 0$, we have that $S(x_n) = 1 - x_n$. Hence, the CHAOTIC LIAR can be expressed by V/~*p*/*p* and its semantic algorithm will be given by:

$$x_{n+1} = 1 - \mathrm{ABS}((1 - x_n) - x_n) .$$

The CHAOTIC LIAR derives it name from the fact that its algorithm is *chaotic* in a precise mathematical sense.[9]

As a chaotic function, the algorithm for the CHAOTIC LIAR, will exhibit:

---

[9] There are stronger and weaker definitions of chaos. Devaney's [1989] definition of chaos is as follows.
A function $f: I \rightarrow I$ is *chaotic* on a set I if all three of the following conditions hold:
(i) f has *sensitive dependence* on initial conditions: there exists points arbitrarily close to x which eventually separate from x by at least δ under iteration of f, i.e., $\exists\delta > 0\ \forall x \in I\ \forall$ neighborhood N of $x\ \exists y \in N\ \exists n \geq 0\ |f^n(x) - f^n(y)| > \delta$ (here '$f^n(x)$' represents the nth iteration of the function f);
(ii) f is *topologically transitive*: f has points which eventually move under iteration from one arbitrarily small neighborhood to any other, i.e., $\forall$ opens sets $U, V \subset I\ \exists k > 0\ f^k(U) \cap V \neq \phi$;
(iii) the *periodic points are dense* on I: there is a periodic point between any two periodic points in the interval I, where a point x is *periodic* if $\exists n\ f^n(x) = x$.
Devaney [1992], p. 119 notes that J. Banks, *et. al.* in "On Devaney's Definition of Chaos," *Amer. Math. Monthly*, **99**, (1992), 332 – 334 show that sensitive dependence follows from the density of periodic points and topological transitivity.

- *unpredictability*: the sensitive dependence on initial conditions of chaotic functions entails that prediction will fail if the initial conditions are not known with infinite accuracy;
- *infinitely many periodic cycles*: since the CHAOTIC LIAR has a cycle of period three it will have cycles of all other periods according to Sarkovskii's ordering (see Devaney [1989], pp. 60-62);
- *fractal images of the semantics of paradox*: *fractals* or sets with a fractional Hausdorff dimension (see Peitgen and Saupe [1988], pp. 28-29) are characterized by self-affinity at increasing powers of magnification. Within the patterns of paradox for DUALIST forms of the Liar, for example, there are infinitely complex fractal patterns (see Mar and Grim [1991]).

It happens that this semantic algorithm for the CHAOTIC LIAR is mathematically identical to the algorithm for the successive positions of the Sisyphean Achilles given above. The Sisyphean Achilles therefore inherits all the mathematical properties of the CHAOTIC LIAR. The web diagram for the CHAOTIC LIAR makes it clear that the complexity of its semantic behavior far surpasses the monotonous regularity of the CLASSICAL LIAR:

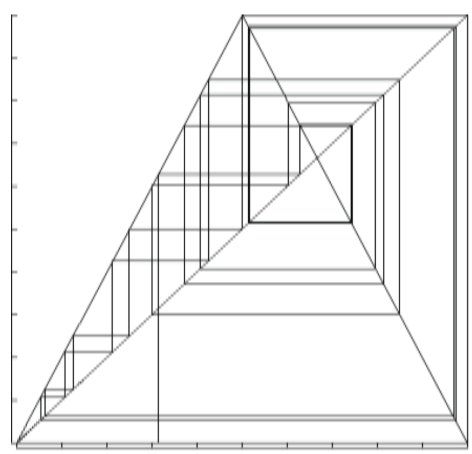


Figure 5. The web diagram for the CHAOTIC LIAR and the SISYPHEAN ACHILLES.

The variation of Zeno's paradox that generated the Sierpinski fractal can be analyzed separately into the behaviors of its x and y components. The function for the iterated values of each coordinate is given by the *Baker function*:

$$x_{n+1} = \begin{cases} 2^* x_n & \text{if } x_n < 1/2 \\ 2^* x_n - 1 & \text{if } x_n \geq 1/2 \end{cases}$$

We can obtain the algorithm for the coordinates of the CHAOTIC LIAR or Sisyphean Achilles by simply reversing the slope of the graph for values greater than 1/2. When we do so, we again obtain the algorithm for the SISYPHEAN ACHILLES or CHAOTIC LIAR expressed as the chaotic tent function.[10]

The Baker function was used to generate the Sierpinski fractal: given an ordered pair (x_0, y_0), we check to see if the successive x_n and y_n values as computed by the Baker function, both exceed the threshold of 1/2. If so, then (x_0, y_0) is *not* in the Sierpinski fractal. Similarly, the algorithm for the Chaotic Liar can be used to generate a new fractal, which we call the Sisyphean fractal. Given (x_0, y_0), we check to see if the successive x_n and y_n values as computed by the Chaotic Tent function, both exceed the threshold of 1/2. If so, then (x_0, y_0) is *not* in the Sisyphean fractal.

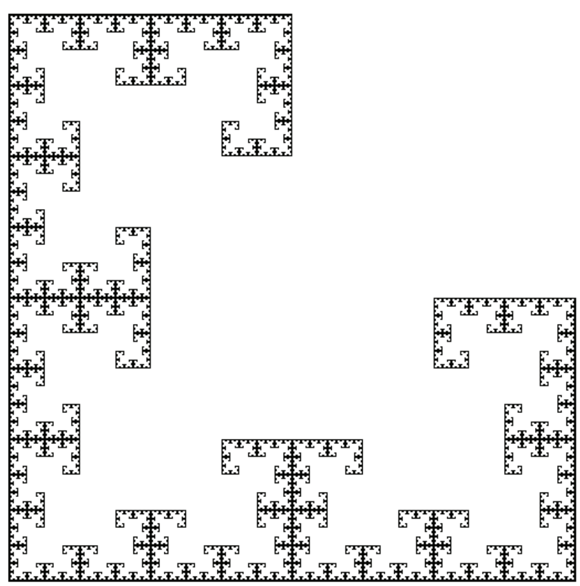


Figure 6. The Sisyphean Fractal is generated by the SISYPHEAN ACHILLES and the CHAOTIC LIAR.

[10] The tent function is mentioned in Robert May's [1976] groundbreaking paper as a "mathematical curiosity." Here, however, the tent function appears as perhaps the simplest generalization of the CLASSICAL LIAR that yields chaotic semantic behavior.

The Sisyphean fractal can be obtained, like the Sierpinski fractal, by an iterative geometric construction. In this construction we remove the quadrant where both x and y are greater than 1/2. The resulting L-shaped figure is folded on top of the square quadrant where x and y are both less than 1/2, which is now considered to be the new unit square. We again remove the quadrant where both x and y are greater than 1/2. The L-shaped figure is folded on top of the new unit square and this procedure is repeated.

An analysis of the binary arithmetic of the algorithm for the CHAOTIC LIAR reveals its intimate connection with *Gray codes*. Gray codes (invented by Frank Gray in 1872) is a way of symbolizing numbers in a positional notation so that when the numbers are in counting order, any adjacent pair will differ in their digits in at most one position. There are different Gray coding schemes, but the most familiar is *binary reflected* Gray codes. It is generated by reflection in the following way: beginning with 0, 1, the next numbers are obtained by taking the mirror image of the digits and prefixing 1. This procedure is iterated, to obtain the values:

0	0	7	100	14	1001
1	1	8	1100	15	1000
2	11	9	1101	16	11000
3	10	10	1111	17	11001
4	110	11	1110	18	11011
5	111	12	1010	19	11010
6	101	13	1011	20	11110

To convert a binary number to its reflected Gray equivalent, we start with the digit at the right and consider each digit in turn:

(1) if the next digit to the left is 0, let the former digit alone;

(2) if the next digit to the left is 1, change the former digit. (The digit at the extreme left is assumed to be 0.)

For example, 41 in binary is 101001 and is assigned the Gray code of 111101. This procedure is equivalent to performing the following operations in binary strings: (1) right-shifting the binary string, and (2) taking the bit-wise *exclusive disjunction* of the two binary strings.

Given the second procedure above, we can obtain a simple QBasic program for the Sisyphean fractal modeled on the previous program for the Sierpinski fractal. Simply replace the binary strings with their binary reflected Gray Codes equivalents and alter the range of q:

Screen 9

For p = 0 to 255

For q = 0 to 255

Gp = p XOR INT(p/2): REM The Gray code for p

Gq = q XOR INT(q/2): REM The Gray code for q

If Gp and Gq then Pset (p, q)

Next p

Next q

For our last example, consider a TRIPLIST version of the Liar paradox (such variations were discussed by the medieval master of paradox John Buridan (see Scott [1966] and more recently by Tyler Burge [1982] and Brian Skyrms [1982]):

SOCRATES: What Plato says is true.

PLATO: What Socrates says is false.

CHRYSIPPUS: It is not the case that both Socrates and Plato speak truly.

What Socrates says is that what Plato says is *true*, but Plato says that what Socrates says is *false*. Hence, Socrates' statement is true to the extent that it is false. Similarly, what Plato says is true to the extent that it is false. Hence, both Socrates' and Plato's statements are modeled by the evaluative algorithmic sequences of the CHAOTIC LIAR. Chrysippus statement is true to the extent that not both Socrates and Plato speak truly.

We can represent the history of bivalent semantic evaluations of the TRIPLIST LIAR as an expanding binary expansion. Each place in the expansion represents a triple of values for a bivalent semantic evaluation. The entire binary string can be seen as a way of encoding the *semantic history* of one of the speakers. Now let each ordered pair (x, y) of binary strings expressed in Gray codes represent a possible semantic history of Socrates' and Plato's statements. We associate with each ordered pair (x, y) a value z, the value of the *bit-wise binary conjunction* of the pair of binary strings. The value z represents the value of Chrysippus' evaluation of Socrates' and Plato's statements considered as semantic histories.

When the value z is expressed in terms of Gray codes and is mapped as the height above the point (x, y), we obtain a tetrahedral Sisyphean fractal. Figure 6 is an approximation of the Sisyphean fractal for the first four iterations.[11] We have rotated the cube so that the origin (0,0,0) is located in the upper right-hand corner of the square base.

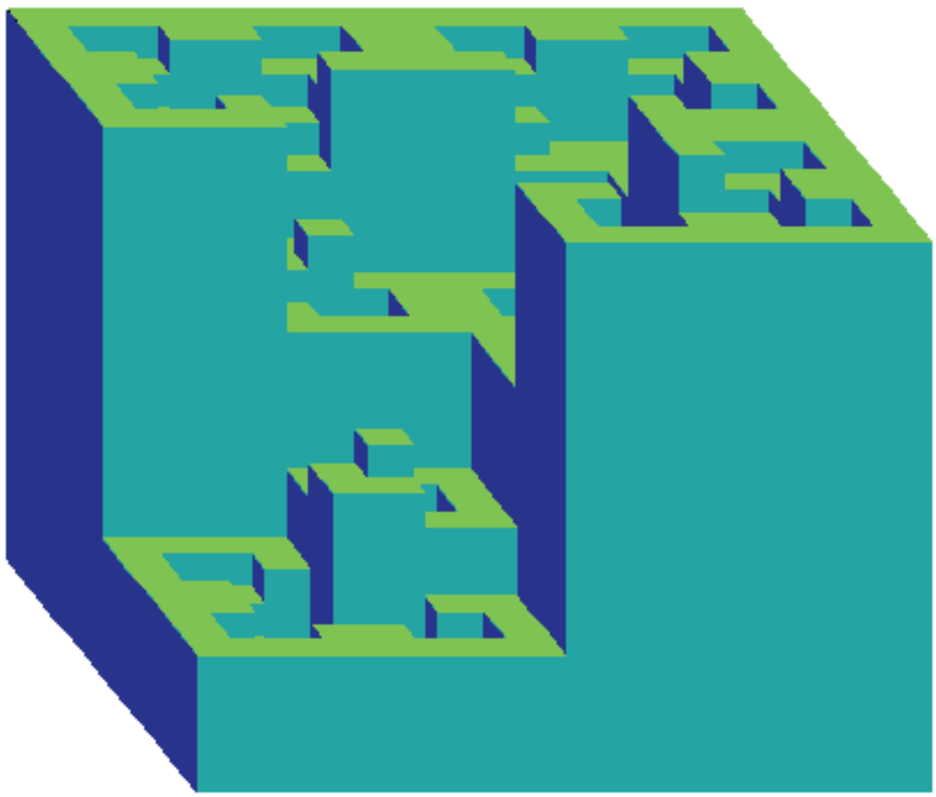


Figure 7. A TRIPLIST LIAR represented as a three-dimensional Sisyphean fractal.

V.

The paradoxes of Zeno and the paradoxes of logic have long been thought to have some metaphorical affinities. We have shown that the affinity between the two variations of Zeno's paradoxes of motion and the paradox of the Liar can be strengthened to mathematical identity. In retrospect, the intuitive equivalence between Zeno's paradoxes of motion and the semantical paradoxes of the Liar may now seem obvious. The isomorphism is due to the fact that both the

[11] We are indebted to Robert Rothenberg for programming assistance.

paradoxes of motion and the paradoxes of semantics involve infinite-regresses that have a fractal character. This fractal character can now be precisely characterized using the mathematics of dynamical systems theory. Commenting on the Protean power of Zeno's paradoxes, Wesley Salmon observed

> Each age, from Aristotle on down, seems to find in the paradoxes difficulties that are roughly commensurate with the mathematical, logical, and philosophical resources then available. When more powerful tools emerge, philosophers seem willing to acknowledge deeper difficulties that would have proved insurmountable for more primitive methods. We may have resolutions which are appropriate to our present level of understanding, but they may appear quite inadequate when we have advanced further. The paradoxes do, after all, go to the very heart of space, time, and motion, and these are profoundly difficult concepts.

This paper can be seen as a confirmation of Salmon's observation. Using the mathematics of dynamical systems theory, the intuitively felt but unproved underlying structural similarities between the paradoxes of Zeno and the paradoxes of the logic can now be mathematically proven.

REFERENCES

Barnsley, Michael
[1988] *Fractals Everywhere*, Academic Press: San Diego, CA.
Barwise, Jon, and John Etchemendy
[1987] *The Liar*, Oxford University Press: New York.
Burge, Tyler
[1982] The Liar Paradox: Tangles and Chains, *Philosophical Studies* **41**, 353-366. Reprinted in Martin [1984], 83-117.
Carroll, Lewis (Charles Dodgson)
[1895] What the Tortoise Said to Achilles, *Mind*, New Series **4**, 278-280.
Church, Alonzo
[1956] *Introduction to Mathematical Logic, vol. 1*, Princeton University Press: Princeton, NJ.
Conway, John Horton
[1976] *On Numbers and Games*, Academic Press: San Diego, CA.
Curry, Haskell B.
[1941] The paradox of Kleene and Rosser, *Trans. Am. Math. Soc.* **50**, 454-516.
Devaney, Robert L.
[1989] *An Introduction to Chaotic Dynamical Systems, 2nd edition,* Addison-Wesley: Menlo Park, CA.
[1992] *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley: Menlo Park, CA.

van Fraassen, Bas
[1972] Inference and Self-Reference, in *Semantics of Natural Language*, ed. by Davidson and Harman, D. Reidel: Dordrecht-Holland, 695-708.
Gardner, Martin
[1972], "The Binary Gray Code," *Scientific American*, **277** (August), 106 ff.
Gupta, Anil
[1982] Truth and paradox, *J. of Phil. Logic* **11**, 1-60.
Herzberger, Hans
[1982] Notes on naive semantics, *J. of Phil. Logic* **11**, 61-102.
Kleene, Stephen C.
[1936] λ-definability and recursiveness, *Duke Math. J.* **2**, 340-353.
Kleene, Stephen C. and Barkley J. Rosser
[1935] The inconsistency of certain formal logics, *Ann. Math*. **36**, 630-636.
Knuth, Donald E.
[1974] *Surreal Numbers*, Addison-Wesley: Menlo Park, CA.
Mar, Gary and Patrick Grim
[1991] Pattern and chaos: new images in the semantics of paradox, *Noûs* **25**, 659-693.
Martin, Robert L.
[1984] *Recent Essays on Truth and the Liar Paradox*, Oxford University Press: New York.
May, Robert
[1976] Simple mathematical models with very complicated dynamics, *Nature* **261**, 459-467.
Peitgen, Heiz-Otto and Dietmar Saupe (eds.)
[1988] *The Science of Fractal Images*, Springer-Verlag: New York.
Rescher, Nicholas
[1969] *Multi-valued Logic*, McGraw-Hill: New York.
Salmon, Wesley
[1970] *Zeno's Paradoxes*, Bobbs-Merrill: Indianapolis, IN.
Schroeder, Manfred
[1991] *Fractals, Chaos, Power Laws*, W. H. Freeman: New York.
Scott, Kermit
[1966] *John Buridan: Sophisms on Meaning and Truth*, Appleton-Century-Crofts: New York.
Shulman, Polly
[1995] Infinity Plus One, and the Surreal Numbers, *Discover* **16**, 96-105.
Skryms, Brian
[1982] Intensional Aspects of Semantical Self-Reference, in Martin [1984], 119-131.
St. Denis, Paul and Patrick Grim
[1997] Fractal Images of Formal Systems, *J. of Phil. Logic* **26**, 181-222.

GARY MAR
GROUP FOR LOGIC AND FORMAL SEMANTICS
Stony Brook University
Stony Brook, New York, USA 11974-3750
email: Gary.Mar@stonybrook.edu

List of Authors | VOLUME 2

G4G13 Exchange Book

Volume 2

Legacy, Puzzles, & Science

Atlanta, Georgia

APRIL 11 - APRIL 15, 2018

Welcome to this glimpse of some of the fun and excitement of the 13th Gathering for Gardner (G4G13) in Atlanta, Georgia, April 11-15, 2018. Here you will find the program of events, and 78 papers that are write-ups by many of the presenters who made this event so vibrant. The subjects are far-ranging, all touching on subjects that fascinated Martin Gardner. Placed into sections on Art, Games, Math, and Magic... these papers describe puzzles, games, illusions, magic, and curiosities both mathematical and otherwise.

- excerpt from the Preface by Doris Schattschneider

Published by:
Gathering 4 Gardner™