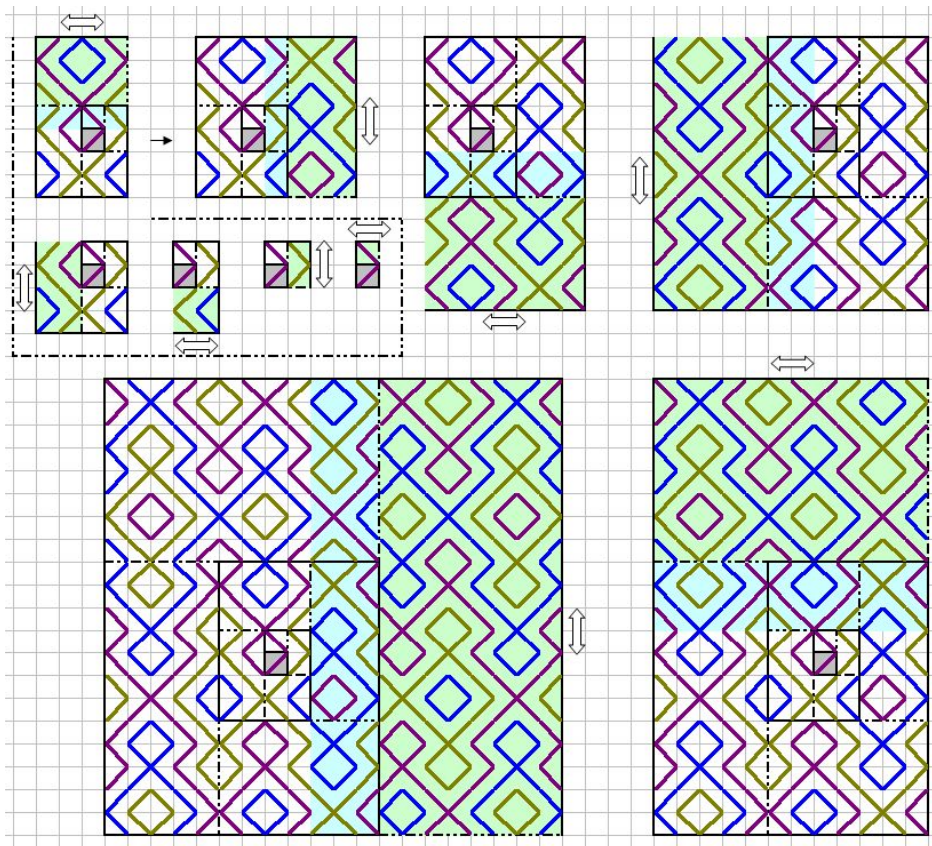


For many years I've been fascinated by a unique three-coloring of an aperiodic tiling discovered by Robert Ammann -- the golden-bee tiling, or "Ammann's A2" in Grünbaum and Shephard's Tilings and Patterns. The G4G Exchange book seems like a good place to share a few pictures. Please feel free to continue the investigation from here!

Start with a graph consisting of a unit square with two of the opposing corners connected. This will be called "stage 2" of the graph. Stage 0 is a point, stage 1 is a unit horizontal line -- it's simplest to start with stage 2. It doesn't matter at this point which initial diagonal is chosen.

Each stage  $N+1$  of the graph is composed from two copies of stage  $N$ . The second copy should be reflected in the  $Y$  axis and translated  $Fib(N/2)$  units downward in relation to the first copy. The two copies will overlap by  $Fib(N/2-1)-1$ .



$N/2$  denotes integer division by 2.  $Fib(0)=1$ ,  $Fib(1)=1$ ,  $Fib(N)=Fib(N-1)+Fib(N-2)$ .

After the copy, reflection, and translation operations, rotate the resulting larger graph 90 degrees clockwise to obtain stage  $N+1$ . For even  $N$ , the stage- $N$  graph will be a square of side  $Fib(N/2+1)-1$ . For odd  $N$ , the graph will be a rectangle  $Fib((N+1)/2)-1$  units wide, by  $Fib((N-1)/2)-1$  units tall.

The first several stages of composition of the resulting "Golden Graph" are shown above, in a spiral starting from the upper left quadrant of the figure.

In 2002 I found a way to generate the above graph, and thus the associated three-color pattern, fairly directly from two self-generating sequences related to the "Golden String" sequence:

$$S(0) = "0", S(1) = "1", S(N) = S(N-1) \& S(N-2).$$

Here "0" corresponds to a square with (say) NW and SE corners connected, and "1" is a square with NE and SW corners connected. Determining the exact recursion -- a different starting string for each of the two dimensions -- and applying three colors to the resulting graph is left as an exercise for the reader.

There is really only one color pattern in each direction. If the top of a column or the beginning of a row is a given color, any column or row starting with that color will match the colors of the original column or row, all the way to the end. Columns and rows starting with other colors can be duplicated by applying the same color shift (modulo 3) to the original column or row.

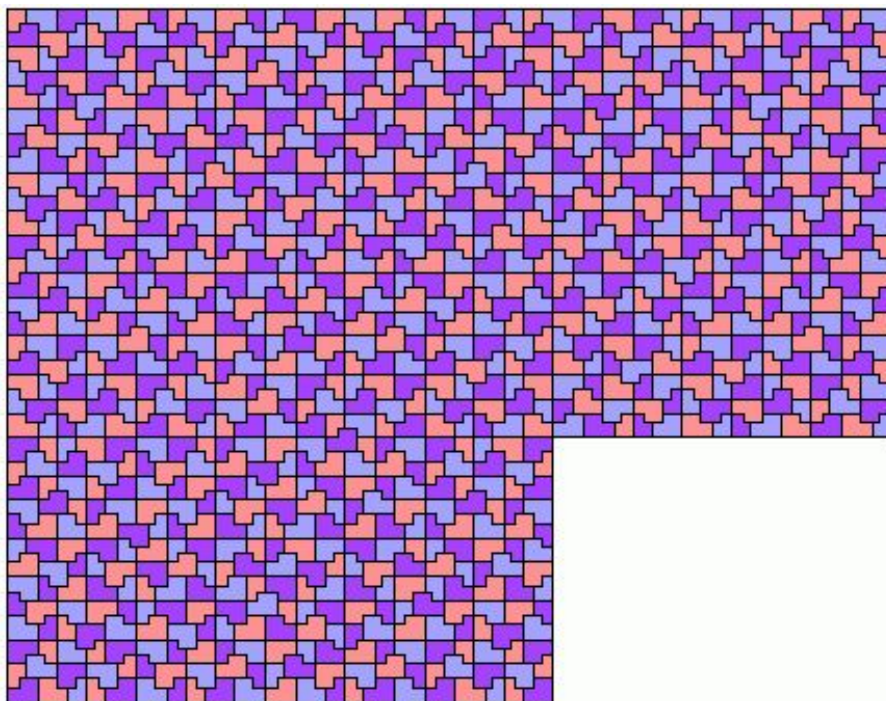
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An alternate way of obtaining this same colored graph is to start with a single Golden-B (or "Golden Bee") tile, corresponding to stage 0 of the graph. See

**<http://www.meden.demon.co.uk/Fractals/golden.html>**

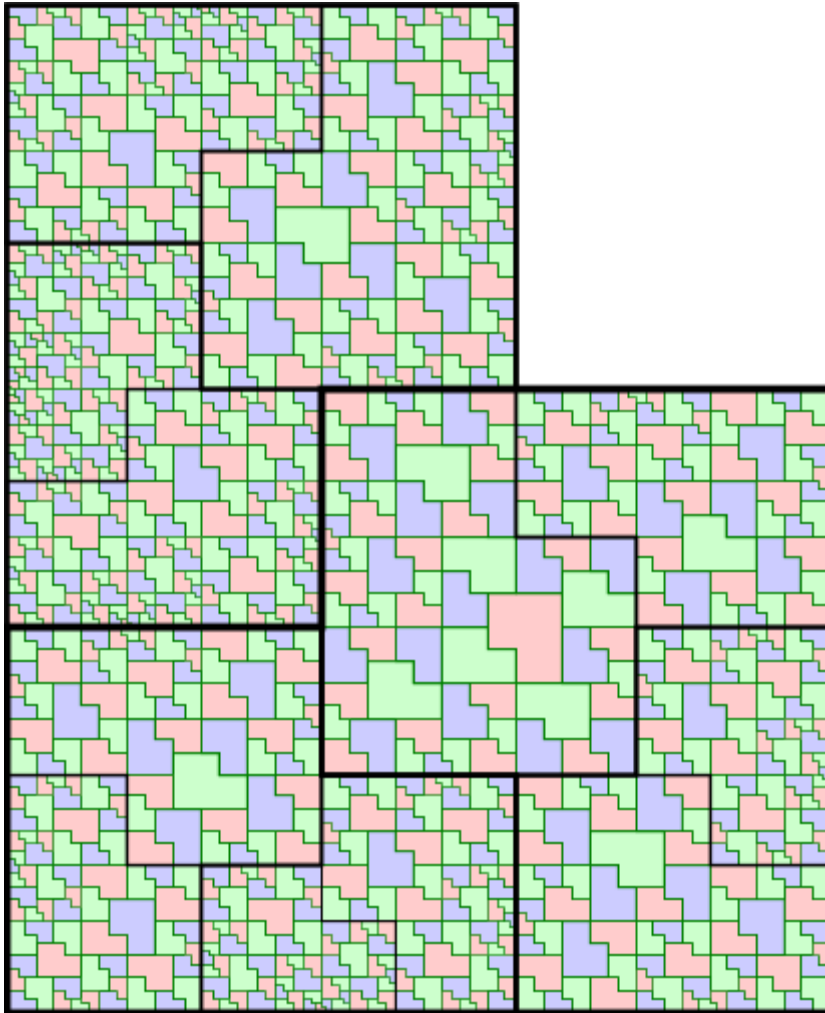
or Grünbaum and Shepard's book Tilings and Patterns for more information about this tile.

To produce each stage  $N+1$  from stage  $N$ , cut each of the  $Fib(N-1)$  \_larger\_ copies of the tile into two similar small copies. At any stage, the resulting tiling seems to be uniquely three-colorable, as long as you include the additional requirement that all three colors must be used wherever four tiles meet at a corner... and that three-coloring matches the unique three-coloring of the Golden Graph described above.



The "Golden Graph" can be produced by replacing all tiles with nodes, and drawing lines between each pair of neighboring tiles that have the same color. (Each interior tile has eight "nearest neighbors"; note that they may not be actually adjacent). Alternatively, drawing lines between pairs of neighboring tiles whose colors do not match is equivalent to rotating all the unit-cell diagonals in the graph by 90 degrees.

Another three-colorable tiling using the Golden B tile can be obtained by cutting every tile in each stage into its two similar sub-tiles, instead of just the larger tiles in each stage. Stage N of this tiling has  $2^N$  tiles instead of  $\text{Fib}(N)$ , with tiles of N different sizes.



With 0, 1, and 2 as possible colors, if the color of the *larger* sub-tile is incremented (mod 3) after every decomposition, a unique three-coloring is the result. The smaller subtile's color remains the same after each cut.

There is no need in this case for the additional restriction that all three colors must be represented at each corner. Only two colors will appear at some corners, but no two tiles of the same color will ever meet at an edge.



This kind of recursive coloring rule also works on another aperiodic tilings, apparently unrelated except for another appearance of the Golden Ratio.

Iterated dissections of Robinson triangles (half Penrose rhombs) can be three-colored in this way.

Here again, stage  $N$  of this tiling has  $2^N$  tiles instead of  $\text{Fib}(N)$ , with tiles of  $N$  different sizes.

With 0, 1, and 2 as possible colors, if the color of the *smaller* sub-tile is incremented (mod 3) after every decomposition, a unique three-coloring is the result. At each stage, the larger sub-tile's color remains the same.

A two-size Robinson triangle dissection can be created, analogous to the two-size Ammann A2 tiling in that only the larger of the two Robinson tiles is dissected at each stage of the inflation process.

However, there appears to be no unique three-coloring equivalent to the two-size Ammann A2 tiling case. A trivial two-coloring is possible, simply because such dissections can only have an even number of triangles meeting at any vertex.

Are there other similar examples of unique recursive three-colorings of aperiodic tilings?

