Crocheting Hyperbolic Regular Octagon and Pair of Pants

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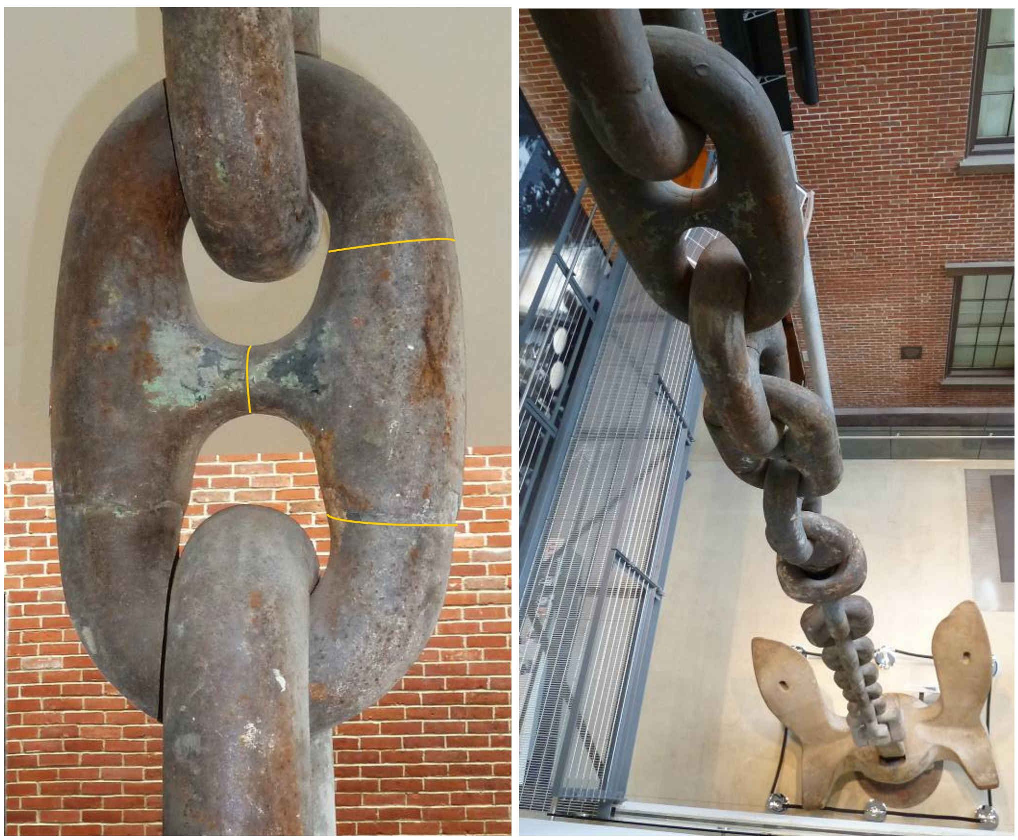
No interesting mathematical topic is self-contained or complete: rather, it is full of "holes", or natural questions and ideas not readily answered by techniques native to the topic. These holes often give rise to connections between the given topic and other topics that seem at first unrelated. – W.P. Thurston[1]

For me, such unrelated topics seemed to be the hyperbolic plane and quantum computing. When in 1997 I first crocheted a hyperbolic plane, the next thing my husband asked me to crochet was a hyperbolic regular octagon with 45-degree interior angles which can be edge-identified (see end of this paper) to form a two-holed torus or *an anchor ring*. Not being a topologist, I did not know about the importance of this figure, but it was interesting to figure out how to crochet it.



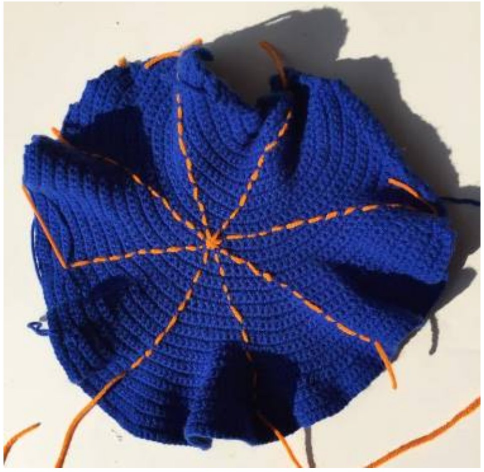
A two-holed torus is sometimes called an "anchor ring"
for reasons that are clear in this photo from the Brooklyn Navy Yard.

The 45-degree regular octagon tiles the hyperbolic plane and can also be arranged as a *pair of pants*. Topologically a *pair of pants* is a sphere minus three open disks (one disk removed as a waist and two other disks removed as cuffs for the pants). A pair of pants is indicated by the yellow boundaries in the photo. In hyperbolic geometry, a pair of pants is the smallest building block used for the decomposition of closed surfaces. Recently, I learned that the pair of pants has become of interest in topological quantum field theory[2] and topological quantum computing[3].

However, it was interesting for me to create a pair of pants (and two-holed torus) from a regular hyperbolic octagon with 45-degree interior angles:

Crochet a hyperbolic plane using acrylic yarn. Start with 15 chain stitches and then use single crochet stitches (increase ratio 5 to 6). After eight rows start to shorten rows as shown in a picture. (That is necessary to eliminate unnecessary extra areas on both sides.)

Shortened rows to produce a hyperbolic plane for constructing regular octagon with approximate radius 22 cm.



Central angle divided in eight equal parts.

Constructing octagon. Fold the plane in half and mark the straight line with a thread. Then construct a perpendicular straight line to the first one approximately at the center of the plane. Mark it. Now do two more folds and mark them, so that central angle is divided in eight equal parts.

Make a paper wedge with 45-degree angle and check that central angle is divided equally. Fold the wedge in half to mark its angle bisector.

Checking wedge and central angle.

There is really no precise procedure for constructing the 45° octagon needed. After you construct eight 45° angles around the center, mark equal distances in all eight directions (in the model shown in the picture, the distance is approximately 17 cm). Construct one of the sides of the octagon by folding the line between the marks on two adjacent lines and marking it with stitches.

Notice that marking any distance from the center will give you a regular octagon, but not every regular octagon will give the required model: we need one with 45° interior angles. Therefore, after you construct two sides of the octagon, you have to check whether the angle between these two sides is 45°. This can be done more precisely if we instead use our wedge to check that the angle between the side and radius (the line from the center) is 22.5°. Lay one edge of the paper angle along a ray and see how the other edge lines up with the side of the octagon. Working with just two adjacent rays, adjust the distance from the center until the side forms the necessary angles with the rays. Now mark the equal distances from the center along all eight rays accordingly.

Octagon with 45 degree angles

Cut two strips of Velcro closure length of the side of your octagon. In this example the side of the octagon is 20 cm. Attach Velcro at the four sides of the octagon as you can see in the picture.

Fold in and lightly stich excess fabric.

Fasten Velcro strips. Your hyperbolic pants are done!

Hyperbolic pair of pants

More examples can be found in my book:

> *Crocheting Adventures with Hyperbolic Planes: tactile mathematics, art and craft for all to explore*, CRC Press, 2018, 2nd edition.

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[1] William P. Thurston, *Three-Dimensional Geometry and Topology*, Princeton University Press, 1997, p. ix.

[2] Michael Atiyah, "An Introduction to Topological Quantum Field Theories", *Tr. J. of Mathematics* 21 (1997), p. 1-7.

[2] Slavomir Klimek, Matt McBride, Sumedha Rathnayake, Kaoru Sakai, "The Quantum Pair of pants, Symmetry, Integrability and Geometry: Methods and Applications" *SIGMA 11* (2015), p. 012.

[3] Eric C. Rowell, Zhenghan Wang, "Mathematics of Topological Quantum Computing", 5 Dec 2017, https://arxiv.org/pdf/1705.06206.pdf, p8.