An interesting property of Bulgarian solitaire

Tom Roby

May 30, 2018

Abstract

Bulgarian solitaire is a natural discrete dynamical system on the set of integer partitions of a fixed value n. It first arose as a puzzle in the early 1980s, and was popularized by Martin Gardner in one of his *Mathematical Games* columns. Here we focus on an interesting property of this action that came up in joint work with James Propp, namely the *homomesy* ("constant averages over orbits") phenomenon. Showing that Bulgarian solitaire satisfies this property can be considered an extension of the original puzzle.

Divide 15 identical tokens (in practice cards or chips will do) into any number of piles. Take one token from each pile, and make a new pile out of them. For example, if you started with piles of sizes 7,3,3,1,1, your new configuration would have piles 6,5,2,2. (The new pile has size 5 and the singleton piles disappeared.) At the next step, you get 5,4,4,1,1. Continue in this fashion until you can predict the outcome of all future iterations. This is the operation of "**Bulgarian solitaire**", so named by Henrik Eriksson, who notes that it's neither Bulgarian nor a game of solitaire. [Hop12, p. 136]. (How would one win or lose?)

Spoiler Alert: **If you haven't played with this before, I recommend that you try a few rounds first to get a feel for the operation and make your own conjecture.**

The process was presented as a puzzle in Russia in the early 1980s, and Andrei Toom published a solution in *Kvant* [Too81]. Soon after that, it was popularized by Martin Gardner in one of is *Mathematical Games* columns [Gard83]. Brian Hopkins's invaluable article [Hop12] traces the early history and discusses some more recent extensions.

Of course this definition makes sense for any number of tokens. Since we don't care about the order of our piles or the order within our piles, we can formally think of Bulgarian solitaire as a map on *integer partitions* of n, i.e., finite sequences of positive integers in weakly decreasing order $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. For example, the partitions of $n = 4$ are

$$\mathcal{P}_4 = \{(1,1,1,1), (2,1,1), (2,2), (3,1), (4)\},$$

and the partitions of $n = 5$ are

$$\mathcal{P}_5 = \{(1,1,1,1,1), (2,1,1,1), (2,2,1), (3,2), (3,1,1), (4,1), (4)\}.$$

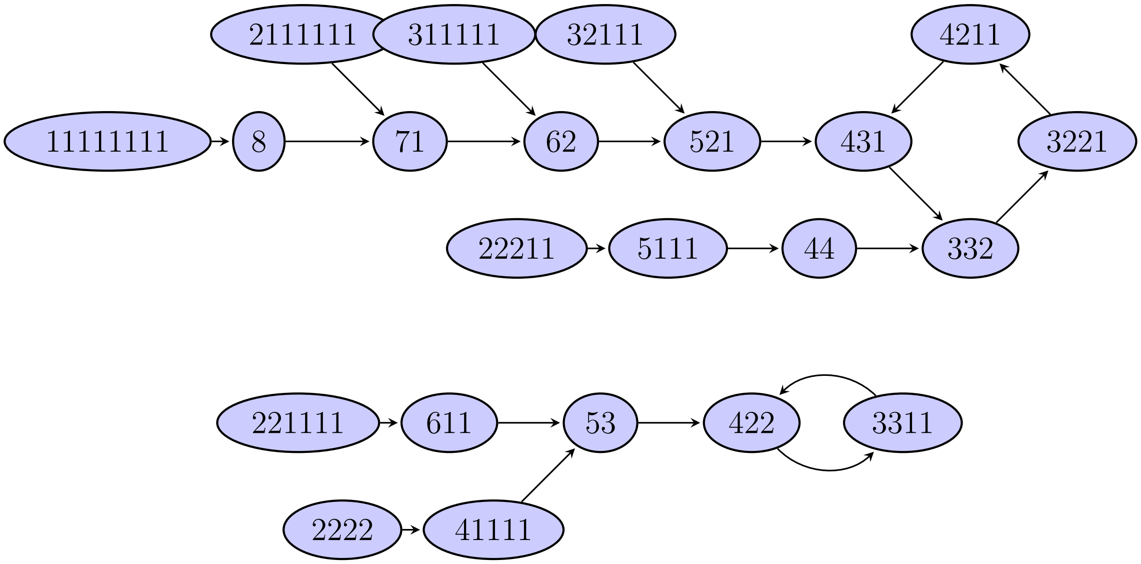
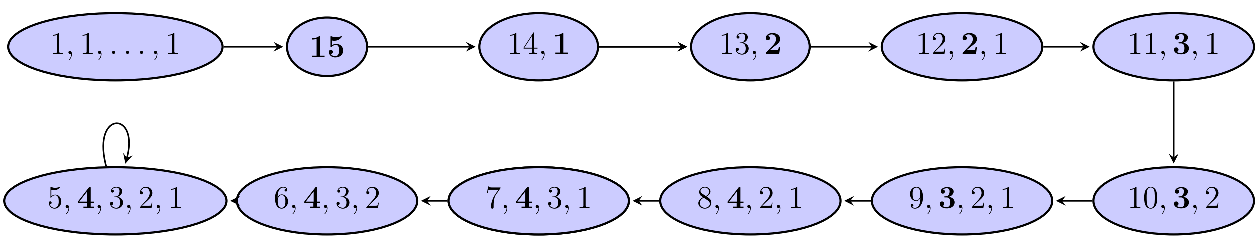


Figure 1: The action of Bulgarian solitaire on all partitions of $n = 8$

In this notation, the map takes $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ to the partition whose parts are the *nonzero* elements among $\ell, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_\ell - 1$ (which may need to be re-ordered to be weakly decreasing).

There are 176 partitions of 15. No matter where one starts, one always ends at the "staircase configuration" $(5, 4, 3, 2, 1)$, which is a "fixed point" of this action. The original puzzle was to explain why this happens.

Example 1. Bulgarian solitaire For $n = 15$, one trajectory of Bulgarian solitaire is:



We focus here on a property that came up in joint work with James Propp on the *homomesy* ("constant averages over orbits") phenomenon [PR15, Rob16].

One can show that whenever $n = 1 + 2 + \cdots + k$ is the kth *triangular number*, then any sequence of moves leads eventually to the staircase partition fixed point. A natural question is what happens for more general values of n.

Example 2. Consider Bulgarian solitaire for $n = 8$ as displayed in Figure 1. No matter where we start, we always end up in one of the two "recurrent cycles," namely $(431, 332, 3221, 4211)$ or $(422, 3311)$. The hidden structure here is that the *average* number of parts (piles) is $\dfrac{3+3+4+4}{4} = \dfrac{7}{2} = \dfrac{3+4}{2}$.

The property of a map having constant averages over recurrent cycles (or orbits when the map is interval) was dubbed "homomesy" by James Propp and me [PR15]. A precise definition follows:

Definition 3. Let $\mathcal{S}$ be a *finite* set with a (not necessarily invertible) map $\tau : \mathcal{S} \to \mathcal{S}$ (called a **self-map**). Applying the map iteratively to any $x \in \mathcal{S}$ eventually yields a *recurrent cycle*, and the *recurrent set* is the union of these cycles. (See Figure 1.) We call a statistic $f : \mathcal{S} \to \mathbb{R}$ **homomesic** if the average of f is the same over every recurrent cycle. ($\mathbb{R}$ denotes the real numbers.)

An alternate way to write this definition is to say that the average value of the statistic f as one iterates the map does not depend on the initial starting point, i.e.,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} f(\tau^k(x)) = c,$$

where c is a constant independent of the starting point $x \in \mathcal{S}$.

In the above example, Bulgarian solitaire acting on partitions of 8, the "number of parts" statistics is ℓ is homomesic with average $\frac{7}{2}$. It is not hard to show that the general situation is as follows, and the proof can be considered an extension of the original puzzle.

Proposition 4 ([Rob16, §2.3])**.** *Let $n = k(k-1)/2 + j$ with $0 \le j < k$, and consider the action of Bulgarian solitaire on the set of partitions of n. Then the length statistic ℓ which computes the number of parts of λ is homomesic with average $(k-1) + j/k$.*

Note that in Example 2, $n = 8$ corresponds to $k = 4$, $j = 2$, while in the situation that $n = k(k-1)/2$ is a triangular number (so $j = 0$), all paths lead to looping on the shape $\kappa = (k-1, k-2, \ldots, 2, 1)$.

Other statistics homomesic with respect to this action include $f_i(\lambda) := \lambda_i$, the size of the ith largest part of λ, for any $i \ge 1$. For example, when $n = 8$ one sees easily from Figure 1 that f_1, f_2, f_3, and f_4 are homomesic with respective averages $\frac{7}{2}$, $\frac{5}{2}$, $\frac{3}{2}$, and $\frac{1}{2}$.

For more information about the homomesy phenomenon, the reader can consult the original article in which the phenomenon was defined [PR15], or the expository article [Rob16].

References

[Gard83] Martin Gardner, *Mathematical Games: Tasks you cannot help finishing no matter how hard you try to block finishing them*, Scientific American **249** (1983), 12–21.

[Hop12] Brian Hopkins, *30 years of Bulgarian solitaire*, College Math J., **43**, #2, March 2012, 135–140.

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22(3)** (2015), #P3.4, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4.

[Rob16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in Andrew Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

[Too81] Andrei Toom, *Problem solution M655*, Kvant (Quantum), **7** (1981), 28–30, http://kvant.mccme.ru/1981/07/p28.htm.