# 2184: AN ABSURD (AND ADSURD) TALE

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*Received: , Revised: , Accepted: , Published:*

**Abstract**

An old problem of De Morgan leads to the observation that the number 2184 is 3 less than a power of 3 and 13 less than a power of 13. Such a number is called "doubly absurd." Doubly absurd numbers relate in subtle ways to the Ramanujan-Nagell equation, Catalan's equation and Pillai's equation. The author conjectures that there are only seven other doubly absurd numbers, and that 2184 is the only one where both powers are cubes or higher exponents. Some partial progress is made toward proving the conjecture.

## 1. Introduction

Augustus De Morgan, the nineteenth century's closest analogue to Martin Gardner, once posed this puzzle: "At one point in my life, the square of my age was the same as the year." What year was he born in? It seems as if there is not enough information, until you realize that he wrote this in 1864. The only year he could have been born in (given a normal life span) was $43^2 - 43 = 1806$, so he was 43 in the year 1849.

The next birth years that could solve De Morgan's puzzle (allowing higher powers as well as squares) are 1892, 1980, 2046, 2070, 2162, 2184, 2184, ... . Notice that the year 2184 appears twice! People born in the year 2184 will first be able to celebrate in 2187, when their age will be the seventh root of the year. And in case they miss this great occasion, they will get another chance ten years later: in 2197, their age will be the cube root of the year. (See [1] and [8] for popular expositions based on this idea.)

Is there some explanation for this curious fact? Is 2184 the only number that appears twice? These are the questions we will explore, and partially answer, in this paper.

First we need some terminology. If a number $n$ can be written in the form $n = x^a - x$, we will call the number *absurd* (literally, "without the surd") because it is equal to an integer $x^a$ minus a perfect root, or surd, of that integer. Likewise, if $n$

can be written as $n = x^a + x$, we will call it an *adsurd* number. If $a \geq 3$ in the above definitions, we will say that $n$ is *strictly absurd* or *strictly adsurd*, respectively.

Furthermore, if a number $n$ is absurd in two different ways, i.e.,

$$n = x^a - x = y^b - y, \tag{1}$$

we will call it *doubly absurd*. We can define doubly strictly absurd, doubly adsurd and doubly strictly adsurd numbers analogously. We are now prepared to say what is (probably) unique about the number 2184.

**Conjecture 1.** *(2184 Conjecture.) The only doubly strictly absurd number is 2184.*

Likewise, it appears that there is only one doubly strictly adsurd number.

**Conjecture 2.** *(130 Conjecture.) The only doubly strictly adsurd number is 130.*

We have not found a previous occurrence of the 130 Conjecture in the literature. However, the question of finding doubly absurd numbers has come up several times. The 2184 Conjecture is in fact a special case of the following more general conjecture, which is apparently due to Mike Bennett ([2], also see [9]).

**Conjecture 3.** *(Bennett) The eight numbers listed below are the only doubly absurd numbers.*

*(i)* $6 = 2^3 - 2 = 3^2 - 3$

*(ii)* $30 = 2^5 - 2 = 6^2 - 6$

*(iii)* $210 = 6^3 - 6 = 15^2 - 15$

*(iv)* $240 = 3^5 - 3 = 16^2 - 16$

*(v)* $2184 = 3^7 - 3 = 13^3 - 13$

*(vi)* $8190 = 2^{13} - 2 = 91^2 - 91$

*(vii)* $78120 = 5^7 - 5 = 280^2 - 280$

*(viii)* $24299970 = 30^5 - 30 = 4930^2 - 4930.$

We will call these "Bennett's eight solutions" and refer to them by number, so 2184 is Bennett's solution (v). Note that we will always write solutions to (1) with the smaller variable first, so throughout the paper we will assume $x < y$ and $a > b \geq 2$.

## 2. History and Heuristics

Considering that equation (1) and Conjectures 1 and 3 are not "famous," it is surprising to see what a rich history they have. In fact, there are at least three plausible routes leading to the problem: the Moret-Blanc-Mordell thread, the Ramanujan-Nagell-Skinner thread, and the Catalan-Pillai-Bennett-Mihailescu thread.

### 2.1. The Moret-Blanc Thread.

Is there an integer $n$ that can be expressed both as a product of two consecutive numbers and as a product of three consecutive numbers? If we let the two numbers be $y-1$ and $y$, and the three numbers be $x-1, x$, and $x+1$, then we arrive at equation (1) in the special case where $a=3$ and $b=2$.

The earliest reference to this problem that we have found is [3] from 1881. Eugene Lionnet posed the above problem and Claude Seraphin Moret-Blanc, a high-school teacher in Le Havre, derived the two solutions, $n=6$ and $n=210$ (Bennett solutions (i) and (iii)).

Moret-Blanc's proof was not complete; he makes an assumption of convenience that lets him get the two stated solutions. A complete proof that 6 and 210 are the *only* solutions can be found in Mordell [4]. The proof is not elementary, as it uses the fact that a certain cubic number field has unique factorization. Thus the case $a=3, b=2$ of equation (1) has been completely solved, the only substantive case that has.

### 2.2. The Ramanujan Thread.

In 1913, Srinivasa Ramanujan conjectured that there are only five integer solutions to the equation

$$2^{a+2} - 7 = z^2. \tag{2}$$

This equation is easily reduced to a special case of (1). Clearly $z$ must be odd, so we can write it as $z = 2y-1$. Substituting this into (2) and dividing by 4, we get $2^a - 2 = y^2 - y$, which is equation (1) with $x=2$ and $b=2$. Two of the five solutions, $(a,y)=(1,1)$ and $(2,2)$, are trivial, but the other three are not and they lead to Bennett's solutions (i), (ii) and (vi).

In 1948, Trygve Nagell proved Ramanujan's conjecture. In 1988, Chris Skinner (who at that time was a high-school student) replaced 2 with an arbitrary prime $q$ and thus considered the equation $4q^a - 4q + 1 = z^2$. In a remarkable paper for a teenager, or indeed a mathematician of any age, Skinner [5] showed there are only two other solutions. Combining Nagell's and Skinner's results, we conclude that Bennett's (i), (ii), (iv), (vi) and (vii) are the complete list of solutions to (1) where $x$ is prime and $b=2$.

The main result of this paper (Theorem 1) is quite analogous to Skinner's Theorem. We will show that Bennett's (v) is the complete list of solutions to (1) where $y$ is prime and $b = 3$. (In addition, we will have to assume $a$ is odd).

### 2.3. The Catalan Thread.

In 1842, Eugene Catalan conjectured that the only consecutive numbers that are perfect powers are 8 ($= 2^3$) and 9 ($= 3^2$). That is, *Catalan's equation* $x^a - y^b = 1$ has only one positive integer solution. This was finally proved by Preda Mihailescu [6] in 2002, and is one of the landmark results in number theory so far this century.

Meanwhile, in the 1930s and 1940s, S.S. Pillai framed a generalization of Catalan's conjecture: for any constant $c$ there are only finitely many solutions to the equation $y^b - x^a = c$. His conjecture remains open. Bennett's paper [2] proves the much more modest statement that for any two *fixed* values of $x$ and $y$ there are at most two solutions to Pillai's equation. Note that he allows exponents of 1, so one possibility is that the two solutions are

$$y^b - x^a = y - x = c, \tag{3}$$

which is simply another version of equation (1). In this way he arrived at his conjectured list of eight solutions to (1).

For any readers who might wish to compare this paper with Bennett's, it is somewhat tricky. Although the problems we consider are very similar, the point of view is quite different. In Bennett's paper, $x$ and $y$ are thought of as *parameters* and written as $b$ and $a$ (respectively), while $a$ and $b$ are thought of as *variables* and written as $y$ and $x$ (respectively). Also, note that his target of interest is $c$ (in equation (3)), while ours is $n$ (in equation (1)). For example, his Theorem 1.4 appears at first glance to say that $c$ cannot be too small compared to $y^b$. In fact, though, it says that $c$ cannot be too small compared to the *smaller* of $y^b$ or $y$, which is $y$. Specifically, his result implies that $y < 6001c = 6001(y - x)$, or in other words $y/x > 6001/6000$.

Bennett's results are excellent in their context, but virtually orthogonal to the problems treated in this paper. Nevertheless, the Catalan thread is extremely important for our approach to equation (1). The fundamental idea is to reduce (1) to Catalan's equation. As we shall see, for $b > 3$ this reduction is not always completely successful, and it will be very convenient to call on the extensive computer work that has been done [7] to find "small" solutions of Pillai's equation.

By contrast, we have not been able to find any prior literature on Conjecture 2, doubly adsurd numbers, or the equation analogous to (1) with the minuses replaced by pluses. We will merely point out here that the only two doubly adsurd numbers less than 2 billion are $30 = 5^2 + 5 = 3^3 + 3$ and $130 = 5^3 + 5 = 2^7 + 2$. We hope that some readers will be motivated to pursue this problem further.

The main purpose of this article is to demonstrate the following theorem.

**Theorem 1.** *The only solution to $n = x^a - x = y^3 - y$ for which $x$ is a positive integer, $y$ is a prime, $a > 3$, and $a$ is odd, is $n = 2184$.*

In fact, we will prove a generalization of Theorem 1 to all $b \leq 14$ (Theorem 2). The generalization, however, requires additional assumptions on $a$ and $y$, so equation (1) is far from being completely solved even for these small exponents.

Although the proof of Theorem 1 looks technical, the main idea is quite simple. Suppose we are looking for solutions to the equation

$$x^7 - x = y^3 - y, \tag{4}$$

i.e., equation (1) with $a = 7$ and $b = 3$. We start by multiplying by $x^2$ to get $x^2(y^3 - y) = x^9 - x^3 = m^3 - m$, where we have defined a new variable $m = x^3$. Because $m$ is "small" compared to $m^3$ and $y$ is "small" compared to $y^3$, we conclude heuristically that $x^2y^3 \approx m^3$. On the other hand, if we define $j$ to be the integer closest to $m/y$, so that $jy \approx m$, then $m^3 \approx j^3y^3$. Comparing these two approximate equations, we conclude that $x^2 \approx j^3$.

But what does "approximately equal" mean when the variables in question are integers? Ideally, it means the two integers differ by at most one. That is, $x^2 - j^3 = \pm 1$. But this is exactly Catalan's equation! By Mihailescu's theorem, it has the unique solution $x = 3, j = 2$. Since $jy \approx m = x^3 = 27$, it's easily seen that $y$ must equal 13, and thus equation (4) has the unique solution $x = 3, y = 13$.

The proofs of Theorems 1 and 2 simply formalize the above argument and generalize it to other exponents $a$ and $b$. The argument does not work at all if $b = 2$, because $m$ is not sufficiently small compared to $m^2$. It works extremely well if $b = 3$. If $b \geq 4$ the argument works pretty well but with some complications that force us to look at Pillai's equation rather than Catalan's.

In Section 3 we will collect all the inequalities we need; this section does not involve any number theory. In Section 4 we move to the context of integers and prove Theorem 1 and its generalizations. Section 5 will offer some directions for future research.

## 3. Through the Eye of the Needle.

The main new tool involved in the proof of Theorem 1 is the following set of inequalities, which for the most part do not require $x$, $y$, $a$, and $b$ to be integers. Only when we get to Lemma 3 will we assume that $b$ (but not the others) is an integer; otherwise, all the variables in this section are merely assumed to be real numbers.

**Lemma 1.** *Given that $x^a - x = y^b - y$, $x > 1$, $y > 1$, and $a > b \geq 3$. Let $t = (a - b)/(b - 1)$ and let $m = x^{1+t}$. Finally, let $j = m/y$. Then*

$$1/x > x^t - j^b > 0. \tag{5}$$

*Proof.* Multiply both sides of equation (1) by $x^t$, to obtain

$$x^t(y^b - y) = x^{a+t} - x^{1+t} = m^b - m. \tag{6}$$

Equation (6) will be the starting point for all of our Lemmas as well as Theorem 1 and its generalizations. We start by establishing a few basic inequalities. First, because $x^t > 1$, we have $m^b - m > y^b - y$. Because $f(x) = x^b - x$ is an increasing function on $[1, \infty)$, it follows that $m > y$.

Likewise, note that $y^b - y = (x^{a/b})^b - x > (x^{a/b})^b - x^{a/b}$. By the same argument as above, $y > x^{a/b}$.

Next, we point out an important dichotomy. If $b^2 \geq a$, then

$$x^t < y^{tb/a} = y^{(ab-b^2)/(ab-a)} \leq y. \tag{7}$$

If on the other hand $b^2 < a$, then $y > x^{a/b} > x^b$. These are the possibilities referred to as Case 1 (i.e., $b^2 \geq a$ and $x^t < y$) and Case 2 (i.e., $b^2 < a$ and $x^b < y$) in Lemma 4, and I will continue to refer to them by those names throughout the paper. Notice that in either case, we can say from the first part of equation (7) that $x^t < y^{b/(b-1)}$.

Finally, note that (6) can be rewritten as follows: $y^b - y = x(m^{b-1} - 1)$. Consequently, we get the following inequality that will be used in Lemmas 3 and 4:

$$(y^b - y) < xm^{b-1} < y^b. \tag{8}$$

With the preliminaries finished, we turn to the proof of (5). We plug $m = jy$ into equation (6) to obtain

$$(x^t - j^b)y^b = (x^t - j)y = x^t y - m = x^t y - x^{1+t} = x^t(y - x) > 0.$$

This proves the right-hand side of (5). For the left-hand side, in Case 1 we have

$$(x^t - j^b)y^b < x^t y < y^2.$$

Thus

$$x^t - j^b < 1/y^{b-2} < 1/x^{b-2} \leq 1/x.$$

In Case 2 we have $x^b < y$ and $x^t < y^{b/(b-1)} \leq y^{3/2}$, since $b \geq 3$. Proceeding as above we conclude that $x^t - j^b < 1/y^{1/2} < 1/x^{b/2} < 1/x$. □

**Remark.** Lemma 1 formalizes the idea, stated in section 1, that $x^t \approx j^b$. Here we had to assume that $m = jy$. Lemmas 2 and 3 relax this assumption to $m \approx jy$ (in the specific sense of $\approx$ that was mentioned earlier).

**Lemma 2.** *Given $x$, $y$, $a$, $b$, $t$, and $m$ as defined in Lemma 1, let $j' = (m-1)/y$. Then*

$$(b+1)/x > x^t - (j')^b > 0. \tag{9}$$

*In fact, the right-hand side of (9) can be strengthened to:*

$$x^t - (j')^b > \frac{b}{x}\left(1 - \frac{1}{y^{b-1}}\right)\left(\frac{1}{1+b/m}\right). \tag{10}$$

*Proof.* By Lemma 1, if $j = m/y$ then $0 < x^t - j^b < 1/x$. We note that $j' = j - 1/y$. Obviously, $0 < j^b - (j')^b$. From the Mean Value Theorem, applied to the function $f(x) = x^b$ with endpoints $j$ and $j'$, we have $j^b - (j')^b < bm^{b-1}/y^b < b/x$ (using (8)). Adding these inequalities, we get (9).

To improve on the lower bound, note that $(j - 1/y)^b(j + 1/y)^b < j^{2b}$, so

$$(j - 1/y)^b < \frac{j^{2b}}{(j + 1/y)^b} < \frac{j^b}{(1 + 1/m)^b} < \frac{j^b}{(1 + b/m)}.$$

We combine this with the left-hand side of (8) and do a little bit of algebra (left to the reader) to obtain inequality (10). □

**Lemma 3.** *Given $x$, $y$, $a$, $b$, $t$, and $m$ as defined in Lemma 1, let $j'' = (m+1)/y$. Furthermore, assume that $b$ is an integer. Then the following inequalities hold:*

*(a) If $b \geq 4$ and $a \leq b^2$ (we will call this "Case 1"), then*

$$\frac{1}{x}\left[b + \frac{2^b}{(\pi b)^{1/4}\sqrt{m^2-1}}\right] > (j'')^b - x^t > \frac{b-1}{x}. \tag{11}$$

*(b) If $b \geq 4$ and $a > b^2$ (we will call this "Case 2"), then*

$$(b+1)/x > (j'')^b - x^t > (b-1)/x. \tag{12}$$

*(c) If $b = 3$ and $m \geq 4$, then*

$$4/x > (j'')^b - x^t > 0. \tag{13}$$

*Proof.* We note that $j'' = j + 1/y$, where $j = m/y$. As in the proof of Lemma 2, we can apply the Mean Value Theorem, with $f(x) = x^b$ and endpoints $j$ and $j''$, to show the right-hand side of inequality (11), (12) or (13).

To get the left-hand side, we note that

$$(j'')^b - j^b = \sum_{k=1}^{b} \tbinom{b}{k}\, j^{b-k}(1/y)^k.$$

(Here is where we use the assumption that $b$ is an integer.) In many cases the leading term is the largest, so we separate it out and try to bound the rest.

$$(j'')^b - j^b = \frac{bj^{b-1}}{y} + \frac{j^{b-1}}{y}\sum_{k=2}^{b} \tbinom{b}{k}\left(\frac{1}{m}\right)^{k-1}. \tag{14}$$

By Schwarz's Lemma, the sum in equation (14) is bounded above by

$$\left(\sum_{k=0}^{b}\tbinom{b}{k}^2\right)^{1/2}\left(\sum_{k=1}^{\infty}\left(\frac{1}{m}\right)^{2k}\right)^{1/2} = \tbinom{2b}{b}^{1/2}\left(\frac{1}{m^2-1}\right)^{1/2}.$$

By a well-known inequality (which follows from Stirling's formula), $\binom{2b}{b} < 4^b/\sqrt{\pi b}$, and the left-hand side of inequality (11) follows.

In Case 2, $2^b \le x^b < y < \sqrt{m^2-1}$, so inequality (12) follows. Finally, if $b = 3$, then (14) reduces to $(j'')^3 - j^3 = (3 + 3/m + 1/m^2)j^2/y$, and the expression in parentheses is less than 4 when $m \ge 4$. Part (c) follows. □

**Remark.** While all three of these lemmas say that, in some sense, $x^t - j^b \approx 0$, they actually provide very narrow windows or "eyes of the needle" that $x^t - j^b$ must lie in. We put this information to good use in the next section.

## 4. From Reals to Integers.

In this section we will assume that $x$, $y$, $a$ and $b$ are all integers and start investigating the consequences of the "eye of the needle" inequalities. We begin with the proof of Theorem 1, in which we are assuming that $y$ is prime and that $b = 3$.

*Proof.* First note that $x = 2$ cannot be a solution because $x^a - x \equiv 2 \pmod 4$ but $y^3 - y \equiv 0 \pmod 4$. Thus we can assume $x \ge 3$. We define the integers $t$ and $m$ as in Lemma 1 (noticing that $t$ is an integer because $a$ is odd and $b - 1 = 2$). By equation (6) we see that $y|(m^3 - m)$, and therefore either $y|m$, $y|(m + 1)$, or $y|(m - 1)$. We will take the three cases in that order.

Suppose then that $m = jy$ for some integer $j > 1$. Then from Lemma 1,

$$1 > 1/x > x^t - j^3 > 0.$$

Because $x$, $t$, and $j$ are integers, this case leads to a contradiction.

Next, suppose that $m = j'y + 1$ for some integer $j'$. By Lemma 2,

$$4/x > x^t - (j')^3 > 0.$$

Because all quantities in the equation are integers, this forces $x = 3$ and $x^t - (j')^3 = 1$. If $t > 1$, Mihailescu's theorem says that there is only one solution: $x = 3$, $t = 2$, and $j' = 2$. Then we have $a = 3b - 2 = 7$, $m = x^{1+t} = 27$ and $y = (m - 1)/2 = 13$, and we recover the solution to equation (1), $3^7 - 3 = 13^3 - 13$. If $t = 1$, then we have $x = (j')^3 + 1$ and $x = 3$, which is a contradiction.

Finally, suppose that $m = j''y - 1$ for some integer $j'' > 1$. Because $x \ge 2$ and $y > x$, we have $y \ge 3$ and therefore $m \ge 5$. Thus by Lemma 3,

$$4/x > (j'')^b - x^t > 0.$$

This is only solvable in integers if $x = 3$ and $(j'')^3 - x^t = 1$. But by Mihailescu's theorem the second equation has no solution if $t > 1$. If $t = 1$ then $x = 3$ and $x = (j'')^3 - 1$, a contradiction.

Thus we conclude that $y = 13$, $x = 3$, $a = 7$ is the only integral solution to equation (1) under the conditions of Theorem 1. □

It is natural to wonder whether we can use the same approach to solve equation (1), or rule out solutions, for other values of $b$. The answer is yes. First, we start with a strengthening of Lemma 2 that holds in the integer case. The basic idea of the proof, like everything in this section, is that the discreteness of the integers lets us turn inequalities into equalities.

**Lemma 4.** *Suppose that $x$, $y$, $a$, $b$, $t$, $m$ and $j'$ are defined as in Lemma 2, and are all integers. Assume $b \geq 4$ and $j' \geq 2$. Let $c = x^t - (j')^b$. Then $cx = b$.*

*Proof.* We already know from Lemma 2 that $cx < b + 1$. Therefore it remains only to show that $cx > b - 1$. Assume, for the sake of deriving a contradiction, that $cx \leq b - 1$.

Lemma 2 also says that

$$(1 + b/m)cx > (1 - 1/y^{b-1})b \geq (1 - 1/3^{b-1})b.$$

(The latter inequality holds because $y > x$ and $x \geq 2$.) We substitute $cx \leq b - 1$ and conclude that $3^{b-1}(1 + b/m)(b - 1) > (3^{b-1} - 1)b$. This simplifies to

$$m < \frac{3^{b-1}b(b-1)}{3^{b-1} - b},$$

from which it easily follows that $m < b^2$. Because $m = x^{1+t}$, it follows that $x^t < b^2$. Then $c = x^t - (j')^b < b^2 - 2^b \leq 0$, where the first inequality uses the assumption that $j' \geq 2$ and the second uses the assumption that $b \geq 4$. This contradicts the fact proven in Lemma 3 that $c > 0$. □

**Theorem 2.** *There are no integer solutions of $x^a - x = y^b - y$ such that $a > b$, $y$ is prime, $(b-1)|(a-1)$, $(y-1, b-1) \leq 2$, and $4 \leq b \leq 14$.*

For example, if $b = 5$, Theorem 2 says that there are no solutions with $y$ prime, $y \equiv 3 \pmod 4$, and $a \equiv 1 \pmod 4$.

*Proof.* Suppose, for the sake of deriving a contradiction, that all the conditions in Theorem 2 are true. Define $t$ and $m$ as in Lemma 1. Just as in Theorem 1, $y|(m^b - m)$ and $m$ is not a multiple of $y$. Therefore $o(m)|(b-1)$ (where $o(m)$ is the order of $m \pmod y$). But also $o(m)|(y-1)$ by Fermat's little theorem. Because $(b-1, y-1) \leq 2$, it follows that $o(m) = 1$ or 2. Because $y$ is prime, the only residues with order 1 or 2 are $\pm 1$. Thus $m \equiv 1 \pmod{y}$ or $m \equiv -1 \pmod{y}$. Also, note that if $b$ is even then $(b-1, y-1) = 1$, so in that case $o(m) = 1$ and $m \equiv 1$.

Accordingly we consider two cases, first where $m \equiv 1 \pmod{y}$. In this case, there is an integer $j' \geq 1$ such that $m = j'y + 1$. If $j' = 1$, then $y = m - 1 = x^{1+t} - 1$. By hypothesis, $y$ is prime, and this is only possible if $x = 2$. By Lemma 2, $2c = cx < b + 1 \leq 15$, so $c \leq 7$. In addition, $2^t - 1 = c$, from which we conclude that $c$ must equal 1, 3, or 7. Then $(a-b)/(b-1) = t = 1, 2,$ or $4$. Thus, equation (1) reduces to one of the following three possibilities: $2^{2b-1} - 2 = 3^b - 3$, $2^{3b-2} - 2 = 7^b - 7$, or $2^{5b-4} - 2 = 31^b - 31$. We leave it to the reader to show that none of these equations has an integer solution $b \geq 4$.

Thus we can assume henceforth that $j' \geq 2$, and apply Lemma 4. As in that lemma, let $c = x^t - (j')^b$. Then, by Lemma 5, $cx = b$, so we get the remarkable equation

$$x^t - (j')^{cx} = c. \tag{15}$$

If $t = 1$, then $14 \geq b \geq x = c + (j')^b \geq 1 + 2^4 = 17$, a contradiction. Thus we may assume that $t > 1$. In that case, by Mihailescu's theorem, $c = x^t - j^b > 1$. Thus $2 \leq c, x \leq 7$. It is easy to show that $c$ and $x$ are relatively prime, so there are only eight possibilities: $(x, c) = (2,3), (3,2), (2,5), (5,2), (2,7), (7,2), (3,4)$ or $(4,3)$.

Although we could go through the eight cases one by one, it is more interesting to give an approach with greater generality.

Suppose $x = p$ and $c = 2$, where $p$ is an odd prime. Then $(j')^{2p} \equiv -2 \pmod{p^2}$. Thus $2^{p-1} \equiv (-2)^{p-1} \equiv (j')^{2p(p-1)} \equiv 1 \pmod{p^2}$, where the last step follows because the order of the multiplicative group (mod $p^2$) is $p(p-1)$. This means that $p$ is a Wieferich prime! Only two such primes are known: $p = 1093$ and $p = 3511$, and any further Wieferich primes are at least $4.9 \times 10^{17}$. Of course, for our proof it is sufficient to note that 3, 5, and 7 are not Wieferich primes.

Similarly, suppose $x = 2$ and $c = p$, where $p$ is an odd prime. If $t$ is even, the left side of equation (15) factors, and we easily get a contradiction. Thus $t$ must be odd. Then, reducing (15) modulo $p$ and using Fermat's little theorem, we have $(j')^2 \equiv (j')^{2p} \equiv 2^t \pmod{p}$. Thus $2^t$ is a quadratic residue (mod $p$); hence 2 is a quadratic residue (mod $p$), and by the Quadratic Reciprocity Theorem, $p \equiv 1$ or $7 \pmod{8}$. In particular, $p$ cannot be equal to 3 or 5.

If $x = 2$ and $c = 7$, the above congruence argument does not help. However, we have a delightful surprise: equation (15) reduces to $2^t - [(j')^7]^2 = 7$. This is just the Ramanujan-Nagell equation, discussed in section 1! In particular, we conclude that $(j')^7 = 1, 3, 5, 11,$ or $181$, the five solutions to the Ramanujan-Nagell equation. The possibility $j' = 1$ was ruled out earlier, and the other four possibilities are not seventh powers. Thus we have a contradiction.

The cases $x = 4, c = 3$ and $x = 3, c = 4$ lead to contradictions by similar congruence arguments (with no need to call upon advanced theorems). This completes the proof of Theorem 2 in the case where $m \equiv 1 \pmod{y}$.

Now we consider the other possibility, which is that $m \equiv -1 \pmod{y}$. In this case, there is an integer $j'' > 1$ such that $m = j''y - 1$. Recall from the first paragraph of the proof that $b$ must be odd.

Here we can apply Lemma 3. First, if $a > b^2$, we set as usual

$$c = (j'')^b - x^t \tag{16}$$

and note that $b + 1 > cx > b - 1$. Thus $cx = b \leq 14$. By Mihailescu's theorem $c > 1$. Thus $b$ is an odd composite number less than 14, which means that $b = 9$ and $c = x = 3$. But then $(j'')^b = 3^t + 3$, which is impossible because the right-hand side is divisible only once by 3.

Hence we can assume that $a \leq b^2$, and we note that this also means that $t \leq b$. It is easy to rule out $t = b$ because we can factor the right-hand side of (16) (details again left to the reader). Thus we can assume that $t \leq b - 1$.

At this point, the argument gets a little bit messy because we have to use the upper bound in Lemma 3(a), which is much worse than the one in Lemma 3(b). However, we also have the benefit of massive computer calculations that have been done to identify small solutions (in $c$) to the equation $(j'')^b - x^t = c$, Pillai's equation. Specifically, sequence A076427 in the Online Encyclopedia of Integer Sequences, and the linked table at [7], lists all of the solutions to this equation for which $c \leq 100$ and for which the two powers, $(j'')^b$ and $x^t$ are less than $10^{18}$. As it turns out, the table is sufficient to solve our problem for $b \leq 13$.

We will leave the easier cases, $b = 5$, 7, and 9, to the reader and give the proof in the two most difficult cases, $b = 11$ and 13.

Suppose that $b = 11$. Note that $x = 2$ can never be a solution to (1) when $b$ is odd, by a congruence argument (mod 4). Thus $x \geq 3$ and $y \geq 5$ (remembering that $y$ is a prime greater than $x$). We also know that $m \geq 2y - 1 \geq 9$. Applying equation (11), we conclude that $(j'')^b - x^t = c$, where $cx < 105.5$. Because $x \geq 3$, we have $c \leq 35$, which puts it within the range of table [7]. Also because $c \geq 2$, we have $x \leq 52$, so $x^t \leq 52^{10} < 1.4 \times 10^{17}$, so $x^t$ and $(j'')^b$ are also within the range of the table. Thus equation (16) must appear among the 274 known solutions of Pillai's equation in table [7]. But none of those solutions involves an eleventh power, so we have a contradiction.

The case $b = 13$ requires a little extra work. First, we can rule out $x = 3$ by reducing equation (1) modulo 9, and we can rule out $x = 4$ by reducing equation (1) modulo 8. Hence any solution to (1) must have $x \geq 5$, $y \geq 7$ and $m \geq 13$. Now Lemma 3 says that $cx \leq 263.1$. Because $x \geq 5$, $c \leq 52$, which puts $c$ within the range of table [7]. Also, if $x \leq 31$, then $x^t < 31^{12} < 10^{18}$, which puts $x^t$ and $(j'')^b$ within the required range as well.

Now if $x \geq 32$, then $y$ is a prime greater than $x$, so $y \geq 37$ and $m \geq 73$. We can recompute the upper bound (9) with the new value of $m$ and we find that $cx < 44.4$. This is already a contradiction, because $c \geq 2$ and $x \geq 32$.

Thus $x \leq 31$ and so $c$, $x$, and $j''$ must appear among the 274 known solutions of Pillai's equation in table [7]. However, none of those solutions involve a thirteenth power, and so we arrive at our final contradiction. $\square$

While the last part of the proof of Theorem 2 is inelegant, the main point is that, when $m \equiv -1 \pmod{p}$, we were able to place an upper bound on $t$, depending on $b$. Thus the search for solutions was reduced to a finite, albeit large, calculation. Fortunately, that calculation was already done for us!

## 5. Ideas for Future Study.

In this final section, I will leave the reader with two unsolved problems.

1) Prove Conjecture 2. That is almost certainly too hard, but it would be interesting to see if analogues of Theorems 1 and 2 can be proven for doubly adsurd numbers.

2) Prove Theorem 1 without the assumption that $a$ is odd, or prove Theorem 2 without the extra assumptions that $(b-1)|(a-1)$ and $(y-1, b-1) \leq 2$. As it turns out, when $b = 3$ the first two even cases are quite easy, but the arguments do not seem to generalize to even numbers $a \geq 8$.

**Theorem 3.** *If $a = 4$ or 6, then the equation $x^a - x = y^3 - y$ has no integer solutions for which $x > 1$ and $y$ is a prime.*

In fact, the proof given below works just as well with the weaker hypothesis that $x$ and $y$ are relatively prime.

*Proof.* If $x^6 - x = y^3 - y$ and $x, y > 1$, note that $f(y) = y^3 - y$ is increasing, with $f(x^2) < x^6 - x$ and $f(x^2 + 1) > x^6 - x$. Hence $x^2 < y < x^2 + 1$, which is impossible if $y$ is an integer.

The proof for $a = 4$ starts in the same way. Suppose that $x^4 - x = y^3 - y$, where $y$ is prime. As above, it is easy to verify that $x^{4/3} < y < x^{4/3} + 1$. Also notice that

$$(y - x)(y^2 + xy + x^2 - 1) = y^3 - x^3 - y + x = x^4 - x^3,$$

from which we conclude $x^3|(y - x)(y^2 + xy + x^2 - 1)$. Because $x < y$ and $y$ is prime, it follows that $x$ is relatively prime to $y$, and hence $x^3$ is relatively prime to $(y - x)$. It follows that $x^3|(y^2 + xy + x^2 - 1)$. Now suppose, for the sake of deriving a contradiction, that $x \geq 8$, so that $x^{-1/3} \leq 1/2$. Then, because $y < x^{4/3} + 1$,

$$\frac{1}{x^3}(y^2 + xy + x^2 - 1) < x^{-1/3} + x^{-2/3} + x^{-1} + 2x^{-5/3} + x^{-2} < 1.$$

It is impossible for $(y^2 + xy + x^2 - 1)$ to be a multiple of $x^3$ and yet be less than $x^3$. So, by contradiction, we conclude that $x < 8$. But it is easy to verify that $y^3 - y = x^4 - x$ has no integer solution if $x = 2, 3, 4, 5, 6$, or 7. $\square$

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