The Dezign-8 Puzzle[1]

Lyman Hurd[2]

Abstract This paper describes some of the intriguing properties of the Dezign-8 puzzle published by Kadon Enterprises. Sixty-four tiles are arranged in an 8x8 grid matching edges. All patterns formed this way have the property that the number of simple closed loops always equals the number of connected components. An upper bound on the number of components is derived and the various degrees of symmetry possible. are described

Keywords: puzzle, tiling.

Introduction

Created by Bill Biggs in 1959, Dezign-8[1] pictured in Figure 1, has 64 tiles representing the various ways a path can emerge from one, two, three or four sides of a square.

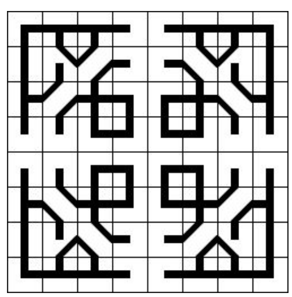

Figure 1: The Dezign-8 Puzzle

The solution in Figure 1 has eight connected components and eight “loops” (by which we mean connected components in the complement not including the outside.) The fact that these two counts are the same is not a coincidence as further shown below.

For the purposes of this paper, the types of tile are assigned names. All of the tiles are mirror symmetric except for the LEFT and RIGHT tiles which are each other’s mirror images.

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[1] Dezign-8 is a trademark of Kadon Enterprises, Inc. ©2000.

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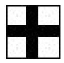
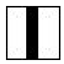
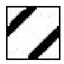
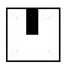
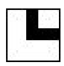
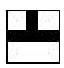
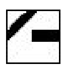
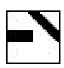
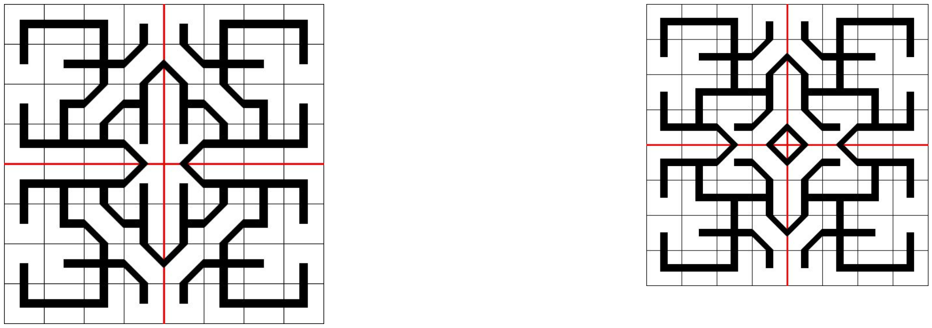
Tile	*Name*	*Count*	*Rotations*
	PLUS	4	1
	STRAIGHT	4	2
	DIAGONAL	8	2
	END	16	4
	CORNER	16	4
	TEE	12	4
	RIGHT	2	4
	LEFT	2	4
		64	

Table 1: Distribution of tiles.

Loops = Components

Figure 2 shows solutions with one component and one loop and two components and two loops. Other figures show solutions with varying numbers of loops and components but in every case the two are equal (e.g., Figure 1, 8 components, 8 loops, Figure 4, 11 components, 11 loops).



One loop, one component. Two loops, two components.

Figure 3: More sample solutions.

The property that the number of loops equals the number of components has to be a property not only of these kinds of tiles, but of their frequency. For example, Figure 2 illustrates two different solutions with other sets of tiles violating this equality.

32 Components 0 Loops 1 Component 49 Loops

Figure 3: Counterexamples with different tile sets.

By Euler's Polyhedral Formula[3] all polyhedra (equivalently connected graphs drawn on a sphere) satisfies the following relationship among vertices. edges and faces:

$$V - E + F = 2$$

The plane can be considered a sphere with one point punctured, or equivalently we can consider the entire region outside the graph as comprising one face which leads to the equation for the plane:

$$V - E + F = 1$$

And finally, we note that for a connected graph component (C=1) and for a graph with more than one component one can add an edge and subtract a component without affecting V or F hence:

$$V - E + F - C = 0$$

This means that to show that loops = components , $F = C$, it suffices to show that $V = E$.

To show the relationship, one associates each solution with a graph by adding vertices to some of the tiles as illustrated in Table 2. On those tiles with vertices, each line from the vertex to the edge of the table represents half a graph edge since two such segments are required to join one vertex to another.

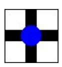
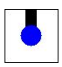
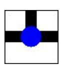
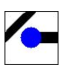
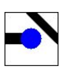
Tile	*Name*	*Count*	*Vertices*	*Edges*
	PLUS	4	1	2
	END	16	1	1/2
	TEE	12	1	3/2
	RIGHT	2	1	1/2
	LEFT	2	1	1/2
		Weighted Sum	36	36

Table 2: Tiles with vertices added.

Figure 3 shows a solution marked with its associated graph.

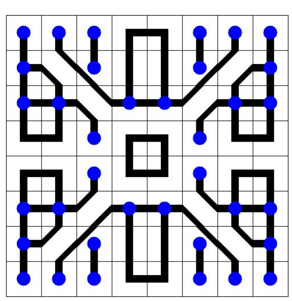


Figure 4: A solution with its associated graph.

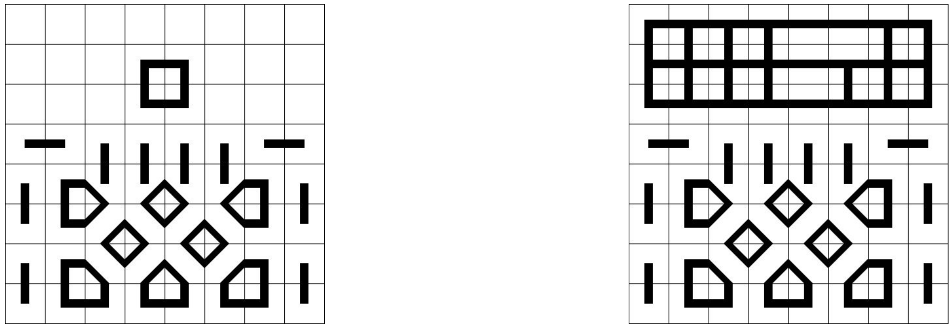
Note that the central square in Figure 4 has no vertices whatsoever. The CORNER and DIAGONAL tiles as well as the diagonal portion of the LEFT and RIGHT tiles can form closed loops, but in each case it is a simple loop contributing one loop and one component simultaneously and therefore having no effect on the difference: $Loops - Components$.

Note that with the current set of tiles, adding up the total number of edges is, as required equal to the total number of vertices. Using the labeling in Table 1 and collecting terms one reaches Equation 1 which gives necessary and sufficient conditions for a combination of these tiles to satisfy the Euler property:

Equation 1: $N_{TEE} + 2\,N_{PLUS} = N_{END} + N_{LEFT} + N_{RIGHT}$

Maximal Solutions

A solution will be called "maximal" if it exhibits as many components (or loops) as possible for a given set of tiles. The first question that arises is what this maximal number is. The most efficient way to form a connected component is to connect two END tiles together. Alternatively one can join four corner tiles. For the purposes of this enumeration the corners provided by CORNER tiles are topologically equivalent to the diagonal lines of the DIAGONAL, RIGHT and LEFT tiles. Setting aside for the moment the PLUS, TEE and STRAIGHT tiles and trying to form as many components as possible from the remaining tiles, one can form 19 connected components as illustrated on the left of Figure 3. While this is an upper bound, it only is achievable if the tiles we did not use can be incorporated into a full solution. It is apparent that on their own there is no way to form the remaining PLUS. TEE, STRAIGHT tiles into an additional component. Such as extension is illustrated on the right of Figure 3.



(a) 19 Components with CORNER, DIAGONAL, END tiles. (b) Solution extended using all tiles.

Figure 5: Maximal number of components.

Summarizing, an upper bound on the number of components achievable with a tile set is given by:

$$(N_{CORNER} + 2N_{DIAGONAL} + N_{LEFT} + N_{RIGHT})/4 + (N_{END} + N_{LEFT} + N_{RIGHT})/2$$

simplified to the following formula for an upper bound, U for the number of components:

Equation 2: $\frac{1}{4}(N_{CORNER} + 2N_{DIAGONAL} + 3N_{LEFT} + 3N_{RIGHT} + 2N_{END}) = U$

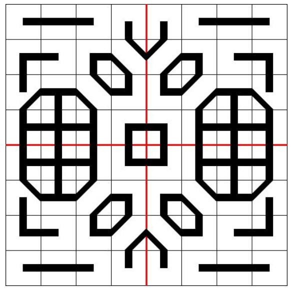
Combining Equation 1 and Equation 2 one can derive a formula that only depends on pieces with “corners” (whether straight as in CORNER pieces or slanted as in DIAGONAL).

Equation 3: $\frac{1}{4}(N_{CORNER} + 2N_{DIAGONAL} + N_{LEFT} + N_{RIGHT} + 2N_{TEE} + 4N_{PLUS}) = U$

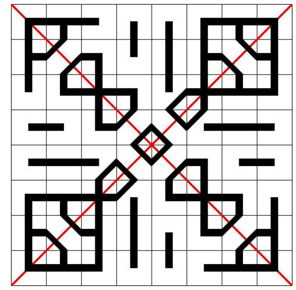
What this bound implies is that in a solution with the maximum number of components and loops, every corner has to form the corner of its own loop. This constraint restricts the form of such a solution and should make searching for such solutions much faster.

Symmetries

As has been noted above, solutions can be left-right and top-bottom symmetrical or can be symmetrical in both diagonals. Figure 6 shows examples of each type.



Orthogonal Symmetry



Diagonal Symmetry

Figure 6: Different types of symmetry.

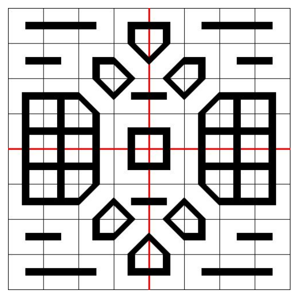


Figure 7: Daniel Austin's orthogonally symmetric maximal solution.

Question 1: Figure 7 shows an orthogonally symmetric maximal solution discovered in 2015 by Daniel Austin. What is the largest number of components that can be achieved for a diagonally symmetric solution?

Question 2: Being maximally symmetric introduces a number of constraints on a solution. In searching for an orthogonally symmetric solution, the author worked independently only to discover that he had rediscovered this solution. Is this maximal orthogonally symmetric solution unique (apart from a trivial 90 degree rotation)?

Dihedral Symmetries

Left-right and diagonal symmetries cannot be achieved simultaneously because having a diagonal and an orthogonal axis of symmetry implies that the solution is rotationally symmetric and this is not possible with the default set of tiles.

As illustrated in Figure 8, a solution with all eight dihedral symmetries would require the presence of a multiple of eight STRAIGHT tiles and a multiple of four RIGHT and LEFT tiles, whereas the original set has four of the former and two each of the latter.

Each STRAIGHT piece forces seven more. Each LEFT or RIGHT piece forces three more.

Figure 8: Tiles forced by dihedral symmetry.

However, by altering the default set of tiles by making the adjustments shown in Table 2, a set of tiles can be arranged that satisfies Equation 1, and therefore maintains the property that *Loops* = *Components* while allowing a fully symmetrical solution seen in Figure 9.

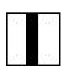
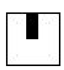
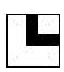
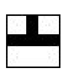
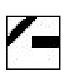
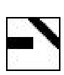
Tile	*Name*	*Count*	*Delta*
	STRAIGHT	8	+4
	END	8	-8
	CORNER	20	+4
	TEE	8	-4
	RIGHT	4	+2
	LEFT	4	+2

Table 2: Distribution of new tile set.

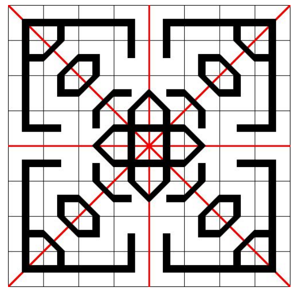


Figure 9: An alternate tile set admitting all eight dihedral symmetries.

Question 3: How many components/loops can be formed with this new set of tiles? Note that the upper bound equation yields 19 in this case as well, but is this achievable?

Enumerating Solutions

When searching for positions, it is also noted that the DIAGONAL and PLUS are interchangeable in any pattern and in any pattern the DIAGONAL pieces can be rotated 90 degrees without affecting any other pieces.

In the orthogonally symmetric case, the pattern is determined by one quadrant which contains on PLUS and two DIAGONALS. The DIAGONALS can be oriented in any of four ways and the PLUS can take any of the three positions giving twelve distinct positions by permuting these pieces.
The same argument applies to the TEE and LEFT/RIGHT pieces. For the orthogonally case the tiles used comprise one LEFT (or RIGHT and three TEE tiles yielding three possibilities as the asymmetric piece can take any of the four positions (orientation is fixed).Combined with the observations above, when searching for solutions, by arranging the DIAGONAL pieces are PLUS pieces and the LEFT/RIGHT and TEE pieces, every solution of this simplified puzzle can be rearranged to form a total of 48 different solutions.

In the general case where we do not enforce symmetry constraints and instead allow the LEFT, RIGHT and TEE pieces to be permuted, the PLUS and DIAGONAL pieces to be permuted and the DIAGONAL pieces to be oriented symmetry, each solution can be rearranged in:

$$\binom{16}{4}\binom{12}{4}2^8 = 1,820 \times 495 \times 256 = 230,630,400$$

different ways.

Acknowledgements

Thanks to Kate Jones and Kadon Enterprises for fruitful discussions as well as supporting the use of their puzzle as the topic of this paper. Their website is a treasure trove of mathematically oriented puzzles.

References

[1] Kadon Enterprises, Inc., Tilings and Designs, Page 2 of 4, www.gamepuzzles.com/tiling2.htm#D8.

[2] Malkevitch, Joseph. "American Mathematical Society." Euler's Polyhedral Formula, American Mathematical Society, Dec. 2004, www.ams.org/publicoutreach/feature-column/fcarc-eulers-formula.