# What the Liar Taught Achilles[1]

Gary Mar
GROUP FOR LOGIC AND FORMAL SEMANTICS
SUNY at Stony Brook, Stony Brook, New York 11794-3750

ABSTRACT. Zeno's paradoxes of motion and the semantic paradoxes of the Liar have long been thought to have metaphorical affinities. There are, in fact, isomorphisms between variations of Zeno's paradoxes and variations of the Liar paradox in infinite-valued logic. Representing these paradoxes in dynamical systems theory reveals fractal images and provides other geometric ways of visualizing and conceptualizing the paradoxes.

KEYWORDS: Zeno's paradox, Liar paradox, semantic paradox, dynamical systems, chaos, fractal, infinite regress, Lewis Carroll, Łukasiewicz, Sierpinski, Tarski.

In his classic "What the Tortoise Taught Achilles" [1895], Lewis Carroll borrowed characters from Zeno's paradoxes of motion and transported them into a dialogue about a paradox he had discovered in attempting to justify fundamental laws of logic. Carroll did not claim that there were any *formal* similarities between the infinite regresses in Zeno's paradoxes of motion and the infinite regress of logical justification. Instead, it is likely that Carroll's parable represents an attempt to express some difficulties that he intuitively felt but could not adequately explain.

In this paper we will show that there are, in fact, mathematically demonstrable isomorphisms between variations of Zeno's paradoxes and intriguing new variations of the paradox of the Liar (see Mar and Grim [1991]). These similarities can be visualized using the computer graphic tools of dynamical systems theory. The results of this paper support Wesley Salmon's observation that our current resolutions of Zeno's paradoxes often go hand in hand with our current mathematical tools.[2]

## I.

Zeno's paradox of motion known as the DICHOTOMY PARADOX comes in two forms. In the PROGRESSIVE FORM, Achilles is never able to complete the racecourse. If it is possible for Achilles to complete the racecourse, then he must first reach the halfway point. But before he can complete the racecourse, he must reach the halfway point of the remaining distance, and so on *ad*

[1] This article was published in *The Journal of Philosophical Logic*, 1992, vol. 28, pp. 29-46. In this revised version, some explanatory remarks have been added to the footnotes, including an intuitive account of fractal dimensions in footnote 4 and an update of Devaney's mathematical definition of chaos in footnote 9.

[2] The introduction to Wesley Salmon [1970].

*infinitum*. Achilles will never reach his final destination, so the argument goes, for to do so would require Achilles to traverse an infinite number of points in a finite amount of time. The REGRESSIVE FORM of the paradox shows, by a similar argument, that Achilles is never able to get started. Before he can reach the halfway point, he must reach the point halfway to the halfway point, and so on *ad infinitum*. Even for Achilles to get started at all would require him to traverse an infinite number of points in a finite amount of time.

Let's combine the Progressive and Regressive Dichotomies in a natural way. In this variation of Zeno's paradox, Achilles always runs *half* the distance to either the starting point or the ending point of the racecourse, whichever is *farther*. We call this the AMBIVALENT ACHILLES.

> AMBIVALENT ACHILLES: I run *half* the distance to the beginning of racecourse or the ending of the racecourse, whichever is *farther*.

The path of the AMBIVALENT ACHILLES is given by

$$x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n > 1/2 \\ (1+x_n)/2 & \text{if } x_n \leq 1/2 \end{cases}$$

and the path is attracted to two points on the racecourse, namely, the points 1/3 and 2/3. Achilles quickly oscillates between these two points in the limit. If Achilles were to reach one of these points, he would then be forever trapped in a cycle of period 2, oscillating back and forth between the two fixed points.

The path of the AMBIVALENT ACHILLES can be visualized graphically. Consider a time series diagram with $x_n$ representing the successive positions of Achilles on the unit interval. Given an initial starting value of $x_n = 0$ we have convergence toward a cycle of period two of the fixed-point attractors 1/3 and 2/3. An alternative way of visualizing this dynamical behavior in dynamical systems theory is in terms of a web diagram. A *web diagram* is a method of graphing the iterated values of a function f($x$). Beginning by drawing a line vertically from ($x_0$, 0) to ($x_n$, $x_{n+1}$), the web diagram next draws a line horizontally from ($x_n$, $x_{n+1}$) to ($x_{n+1}$, $x_{n+1}$) and then iterates the process by using $x_{n+1}$ for $x_n$. Figure 1 compares a time series graph with its corresponding web diagram. The fact that the AMBIVALENT ACHILLES approaches a cycle of period 2 is visually evident as the lines of the web diagram converge on a simple box whose corners intersect the $x = y$ line at the attractor points 1/3 and 2/3.

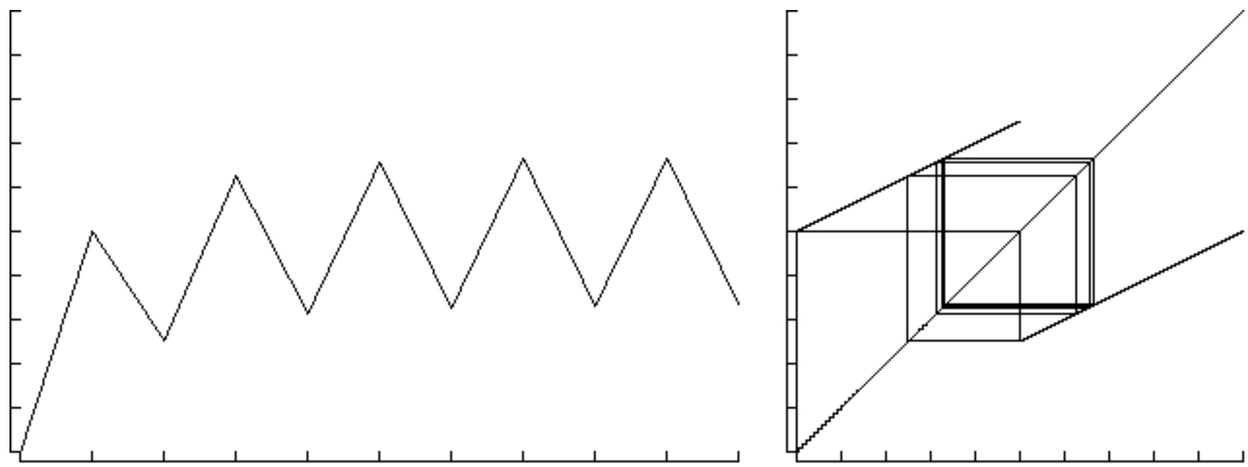

*Figure 1*. A times series graph and web diagram for the AMBIVALENT ACHILLES.

The behavior of the AMBIVALENT ACHILLES can be conveniently analyzed in binary arithmetic. We can represent running half way to the starting point by prefacing a '0' and right-shifting the binary string that represented Achilles previous position on the [0,1] interval. Similarly, we can represent running half way to the end point by prefacing the binary string with a '1' and right-shifting. It is now easy to see why the AMBIVALENT ACHILLES is drawn inexorably to the attractor points 1/3 and 2/3, which in binary notation are the strings 0.010101..... and 0.101010......, respectively. The successive positions of the AMBIVALENT ACHILLES can be seen as right-shifting and alternately prefacing of '0' and '1' to the binary representation of the initial starting point. The difference between the initial starting point and the attractor points, therefore, quickly diminishes as the binary string for the initial position is prefaced by increasingly long strings of alternating '0's and '1's.[3]

[3] It is intriguing to note that Conway's "surreal" numbers can be represented as a sequence of 1's and 0's and modeled on Zeno's paradoxes. Each number is represented by a finite or transfinite sequence of 1's and 0's. Each repeated initial sequence of 1's advance one unit forward. Hence, 1 := 1; 2 := 11, 3 := 111, etc. However, once there is a change from 1 to 0 or from 0 to 1, you go half your current unit in the opposite direction. Thus, for example, 111 is 3, but 110 is 3/2, 11 is 2, but 10 is 1/2. See Conway [1976], Knuth [1974], and Shulman [1995].

That Zeno's runners rehearse self-similar patterns at decreasing scales suggests the *fractal character* of Zeno's paradoxes. This fractal character becomes visually evident when we generalize the one-dimensional AMBIVALENT ACHILLES to two dimensions. We call this two-dimensional Achilles the TRIVALENT ACHILLES. In this new variation Achilles will have the *three* end points of a triangle, rather than just *two* end points of a line, toward which to run. We can imagine Achilles beginning somewhere in the middle of a triangular field. Achilles trifurcates, and each of his three counterparts runs halfway toward the three goal points. Then each of these counterparts trifurcates, and this process is repeated.

The limit points of the TRIVALENT ACHILLES form the famous SIERPINSKI FRACTAL. The Sierpinski fractal is generated by the iterative procedure of dividing an equilateral triangle in four equal triangles and removing the middle fourths. Figure 2 shows a variation of the Sierpinski fractal generated from an isosceles right triangle. Fractals derive their name from the fact that they can have fractional dimensions. The *Hausdorff dimension* for the Sierpinski fractal, for example, is log3/log2, which is approximately 1.58.[4]

To generate a computer image of the Sierpinski fractal, we plot a random sampling of all such paths in a process which has been dubbed by Michael Barnsley [1988] as the *chaos game*.

TRIVALENT ACHILLES: I run *halfway* to one of the three goal points chosen at random.

A *deterministic* way of obtaining the Sierpinski fractal (intuitively obtained by playing backwards a "movie" of one of the TRIVALENT ACHILLES runners) is discussed by Manfred Schroeder [1991]. It proceeds as follows:

ESCAPIST ACHILLES: I run *twice* the distance away from the nearest point (along a straight line from that point).

---

[4] Fractal dimensions are a generalization of integer dimensions. Suppose we divide the sides of a square in half to obtain four smaller copies. So when the reduction factor $r = 2$, the number of similar members $m = 4$. Intuitively, the dimension of the square is the power $d$ such that $m = r^d$ so the dimension $d$ of the square is 2 since $4 = 2^2$. Notice if we subdivide the side of the square into thirds, then the reduction factor $r = 3$, $m = 9$, and again the dimension $d = 2$ since $9 = 3^2$. Dividing the side of a cube by 2 results in 8 smaller cubes so $m = 8$, $r = 2$, and so the dimension $d$ of the cube is 3 since $8 = 2^3$. Now when the side of the Sierpinski triangle is divided by $r = 2$, we obtain only $m = 3$ copies of the Sierpinski triangle, and so, generalizing the above idea, the *fractal* or *Hausdorff* dimension $d$ satisfies the equation $3 = 2^d$. Thus, the *fractal* or *Hausdorff dimension* of the Sierpinski trangle $d = \log 3/\log 2 \approx 1.58$. For discussion see Schroeder [1991], pp. 16-17.

The set of all points that do *not* eventually escape from the triangle forms the Sierpinski triangle.

The reason why the ESCAPIST ACHILLES generates the Sierpinski triangle is easily seen when analyzed in binary arithmetic. Consider a square whose corners are (0,0), (0,1), (1,0), and (1,1). Let's say that the first three points are the goal posts for the TRIVALENT ACHILLES. Now doubling the distance from the nearest point is equivalent to left-shifting the binary strings. Consider, for example, ($x_0$, $y_0$), where $x_0$ = .0011 and $y_0$ = .0101. This point will *not* be in the Sierpinski fractal because there is a '1' in both strings in the fourth position. We can verify this by tracing the path of the ESCAPIST ACHILLES. We begin at the point (.0011, .0101) or (3/16, 5/16). We then double the distance from (0,0), the closest of the goal posts, to arrive at (.011, .101) or (3/8, 5/8). We then double the distance from (0,1) (which are the leading values of the binary expansion) to arrive at (.11, .01) or (3/4, 1/4). Doubling the distance from (1,0), the nearest goal point, we arrive at (.1, .1) or (1/2, 1/2). Doubling the distance from (0,0) once again, the ESCAPIST ACHILLES finally escapes the Sierpinski triangle arriving at the point (1,1).

The pairs of binary strings that are points in the Sierpinski triangle are those where both values cannot simultaneously be greater than or equal to 1/2. In binary notation, these will be precisely those pairs of strings that do not have '1's in the same position in their respective strings.

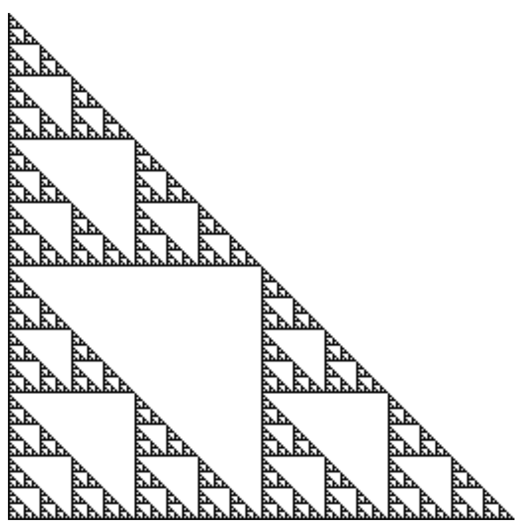

*Figure 2.* The TRIVALENT and ESCAPIST ACHILLES generate the Sierpinski triangle.

The above characterization of the points in the Sierpinski fractal makes clear an intriguing connection between Zeno's paradoxes of motion and propositional logic. The points *not* in the Sierpinski fractal are precisely those that have the constant value 0 for a binary bit-wise conjunction. The following simple QBasic program for plotting points not in the Sierpinski triangle can verify this:

```
Screen 9
For p = 0 to 255
For q = 0 to 255 – p
If p and q then Pset (p, q)
Next p
Next q
```

Suppose the propositional letters $p$ and $q$ have the following final columns in a truth table: $/p/$ = <1, 1, 0, 0> and $/q/$ = <1, 0, 1, 0>. Then $/(p \wedge q)/$ = <1, 0, 0, 0>. Given x = $/p/$ and y = $/q/$, the plot at the point $(x, y)$ is $/(p \wedge q)/$. The point $(x,y)$ is *not* plotted only if $/(p \wedge q)/$ = <0,0,0,0>. This will be true if the bit-wise conjunction for /p/ and /q/ is *not* true, i.e., if the Sheffer Stroke of $p$ and $q$, $(p|q)$, is true. If we now consider not merely finite truth assignments, but *infinitary* ones, we generate the Sierpinski fractal. The successive approximations of the Sierpinski fractal therefore represent the non-tautologies. In the limit, therefore, the Sierpinski triangle created by the negative space among the plotted points can be thought of as a picture of the set of tautologies of propositional logic.[5]

## II.

To make the structural identity of a variation of Zeno's paradox and paradoxes of logic precise, we will set forth an infinite-valued Łukasiewiczian logic with a self-reference operator based on ideas discussed by Rescher [1969] and van Fraassen [1972]. We call the language

[5] See St. Denis and Grim [1997].

DIALOGUE (for <u>D</u>ynamical <u>I</u>terative <u>A</u>lgorithmic <u>L</u>anguage <u>O</u>ffering <u>G</u>enuinely <u>U</u>nstable <u>E</u>valuations). The syntax for DIALOGUE is given by first specifying an *alphabet* of symbols:

| | |
|---|---|
| $p$, $q$, $r$ (with or without subscripts) | propositional variables |
| $\sim$ , $\rightarrow$ | propositional connectives |
| ) , ( | parentheses |
| / / | the propositional value operator |
| $\boldsymbol{t}$ , $\boldsymbol{f}$ | truth-value constants |
| V | 2-placed multi-valued truth predicate |
| $\theta$ | the self-referential propositional operator |
| + , ÷ | the addition sign and the division sign |

We complete the specification of the syntax by giving a set of *grammatical rules* defining the set of *well-formed formulas* (wffs) and the set of *value terms*.

(G1) Any propositional variable is a wff.

(G2) $\theta$ is a wff.

(G3) If $\varphi$ and $\psi$ are wffs, then so are $\sim\varphi$ and $(\varphi \rightarrow \psi)$.

(G4) The truth-values $\boldsymbol{t}$ and $\boldsymbol{f}$ are value terms.

(G5) If $\alpha$ and $\beta$ are value terms so are $\alpha + \beta$ and $\alpha \div \beta$.

(G6) If $\varphi$ is a wff, then $/\varphi/$, the *value* of the wff $\varphi$, is a value term.

(G7) If $\varphi$ is a wff and $\alpha$ is a value term, then $V\alpha\varphi$ is a wff.

We shall say that a wff in which $\theta$ occurs is *self-referential*; otherwise, we shall call the wff *normal*.

The semantics for DIALOGUE is given by assigning to all normal wffs a real value in the [0,1] interval and assigning to self-referential sentences an iterative evaluative algorithm.

(S1) For each propositional variable $p$, we assign $p$ a value $/p/ \in [0,1]$.

(S2) $/\theta/$ is assigned a value $x_0 \in [0,1]$.

(S3) For normal wffs $\varphi$ and $\psi$, where $/\varphi/$ and $/\psi/$ are the values of $\varphi$ and $\psi$, respectively, we have:

(A) $/\sim\varphi/ = 1 - /\varphi/$.

(B) $/(\varphi \rightarrow \psi)/ = \text{MIN}\{1, 1 - /\varphi/ + /\psi/\}$.

(C) $/V\alpha\varphi/ = 1 - \text{ABS}(/\alpha/ - /\varphi/)$.

(S4) The truth-values ***t*** and ***f*** are assigned the values $/t/ = 1$ and $/f/ = 0$, respectively.

(S5) If $\alpha$ and $\beta$ are value terms, where $/\alpha/$ and $/\beta/$, then $/\alpha + \beta/ = /\alpha/ + /\beta/$, and $/\alpha \div \beta/ = /\alpha/ \div /\beta/$, where $/\beta/ \neq 0$ in which case the expression is undefined.

(S6) If $\chi$ is a self-referential sentence, then $/\chi/$ is an iterative evaluative algorithm defined as follows:

(A) If $\chi = \sim \theta$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - x_n$ , where $x_0 = /\theta/$.

(B) If $\chi = (\theta \rightarrow \psi)$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - \text{ABS}(\text{MIN}\{1, 1 - x_n + /\psi/\} - x_n)$;
if $\chi = (\varphi \rightarrow \theta)$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - \text{ABS}(\text{MIN}\{1, 1 - /\varphi/ + x_n\} - x_n)$;
if $\chi = (\theta \rightarrow \theta)$, then $/\chi/$ is the algorithm $x_{n+1} = 1 - \text{ABS}(\text{MIN}\{1, 1 - x_n + x_n\} - x_n)$;
where $x_0 = /\theta/$.

(C) If $\varphi = \theta$, then $/V\alpha\theta/$ is the algorithm $x_{n+1} = 1 - \text{ABS}(/\alpha/ - x_n)$, where $x_0 = /\theta/$.

This completes the semantics for DIALOGUE. A few comments are in order.

First, the infinite-valued rules for normal sentences in (S3) are *faithful* to classical logic: when the values of the sentences are restricted to the classical truth-values, we obtain the classical truth tables. Intuitively, the *negation* of *p* is true to the extent that *p* is *un*true, i.e., to the extent the value of *p differs* from the value of 1 or complete truth. The rule for the conditional is what makes the system characteristically Łukasiewiczian. Given the definitions:

$$(\varphi \vee \psi) := ((\varphi \rightarrow \psi) \rightarrow \psi)$$

$$(\varphi \wedge \psi) := \sim(\sim \varphi \vee \sim \psi)$$

we obtain Łukasiewicz's *Boolean* evaluation rules:

$$/(\varphi \wedge \psi)/ = \text{MIN}\{/\varphi/, /\psi/\} \ ,$$

and

$$/(\varphi \vee \psi)/ = \text{MAX}\{/\varphi/, /\psi/\} \ .$$

Given the classical equivalence $(\varphi \leftrightarrow \psi) := ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$, we derive the *biconditional rule*:

$$/(\varphi \leftrightarrow \psi)/ = 1 - \text{ABS}(/\varphi/ - /\psi/) \ .$$

This rule states that the *biconditional* is *true* to the extent that its constituents do *not differ* in truth-value.

Secondly, the value of the proposition asserting that the proposition $p$ has the value $\alpha$, $V\alpha p$, is given by the schema $/V\alpha p/ = 1 - \text{ABS}(\alpha - /p/)$. This schema is a generalization of the Tarskian (T) schema. Using the biconditional rule, we can state *Tarskian* (T) *schema* by $/T'p'/ = 1 - \text{ABS}(\boldsymbol{t} - /p/)$ , where $\boldsymbol{t}$ is the value of 1 or complete truth. Replacing the constant $\boldsymbol{t}$ with a parameter $\alpha$ (which ranges over the [0,1] interval) and replacing the bivalent truth predicate 'T' with a multi-valued relation '$V\alpha p$' (which is to be read '$\alpha$ is the truth-value of the sentence $p$'), we obtain Rescher's [1969] schema for his parametric-operator development of many-valued logics.[6] Intuitively, Rescher's schema states that the sentence $V\alpha p$ is true to the extent to which the value of $p$ does *not differ* from $\alpha$.[7]

Thirdly, DIALOGUE contains its own truth predicate and a form of self-reference but avoids the inconsistency of semantically closed languages by assigning iterative semantic algorithms to self-referential sentences rather than univocal truth-values. The semantic paradoxes have, in a way, been a trap for logicians who, in their attempts to *solve* the paradoxes, have tended to view the patterns of paradox as simpler and more predictable than they actually are. Even in the sophisticated work of Barwise and Etchemendy on the Liar [1987], the cyclical regularity of the semantical paradoxes has been obvious but their incalculable complexity has remained hidden. Here, instead of searching for simple patterns of semantic *stability* (as in Gupta [1982] and Herzberger [1982]), in *DIALOGUE* we will exhibit infinitely complex and chaotic patterns of semantic *in*stability, which have gone virtually unexplored.

## III.

The above infinite-valued Łukasiewiczian logic can be used to obtain generalizations and variations on the classical paradox of the Liar. Recall that the CLASSICAL LIAR is a sentence that asserts it own falsity:

---

[6] Rescher [1969], p. 81.

[7] The Tarskian (T) schema was stated above in terms of sentences; Rescher states his $V\mathbf{v}p$ schema in terms of propositions. For present purposes, we set aside the philosophical controversy as to what should properly be regarded as the bearers of truth. See Church [1956], p. 27, footnote 72.

$\boxed{\text{The boxed sentence is false.}}$

According to the Tarskian (T) schema, a sentence stating that a sentence *p* is true has the same truth-value as *p* itself. Hence,

(1) 'The boxed sentence is false' is true if and only if the boxed sentence is false .

But since it is empirically true that

(2) 'The boxed sentence is false' is identical to the boxed sentence ,

we may infer from (1) and (2) by Leibniz's Law that

(3) The boxed sentence is true if and only if the boxed sentence is false .

The assumption that the boxed sentence is true leads to the conclusion that it is false, and the assumption that it is false leads to the conclusion that it is true. The semantic behavior of the CLASSICAL LIAR can therefore be represented as an infinite oscillation between the classical truth-values true and false.

Intuitively, the CHAOTIC LIAR is a sentence that self-referentially states that it has the value of falsehood. In *DIALOGUE* we can represent this sentence by $\mathrm{V}f\,\theta$. The semantic algorithm for this sentence of *DIALOGUE* is given by the iterative algorithm $x_{n+1} = 1 - \mathrm{ABS}(0 - x_n)$. Given an initial estimated value of $x_0$, the successive estimated values for the continuous valued CLASSICAL LIAR will be an alternating cycle of period 2 between the values $x_0$ and $1 - x_0$. The single exception is the fixed-point value of 1/2. Figure 3 shows a web diagram for the CLASSICAL LIAR within an infinite-valued logic with the periodic values 1/3 and 2/3, reminiscent of the AMBIVALENT ACHILLES.

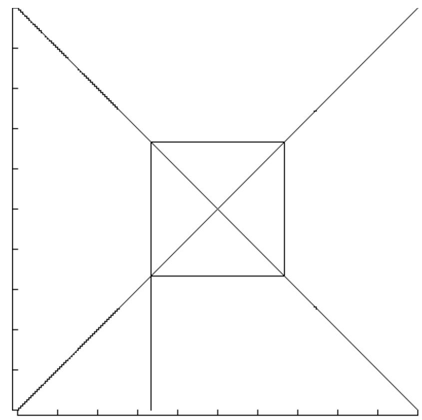

*Figure 3.* Web diagram for the CLASSICAL LIAR in an infinite-valued context.

Other infinite-valued variations of Liar-like sentences are possible. *DIALOGUE* allows us to evaluate sentences that do not attribute to themselves a *particular* truth-value α but rather attribute to themselves a truth-value expressed as a *function* of its self-referential values. To obtain the algorithm for evaluating such sentences, we successively replace the α in the Vαθ schema with the sequentially estimated values of the self-referential wff. Consider, for example, the following pair of Liar-like sentences:

CAUTIOUS TRUTH-TELLER: This sentence is *half* as true as it is estimated to be true.

CONTRADICTORY LIAR: This sentence is *as true as the contradiction* consisting of the conjunction of itself and its negation.

This CAUTIOUS TRUTH-TELLER is represented by V/θ/÷2 θ, where the numeral '2' abbreviates '/***t***/ + /***t***/'. The semantic algorithm for the CAUTIOUS TRUTH-TELLER is therefore:

$$x_{n+1} = 1 - \text{ABS}(x_n \div 2 - x_n) \ .$$

The CONTRADICTORY LIAR, on the other hand, can be represented by V/($p$∧~$p$)/θ. Using Łukasiewicz's rule for conjunction, we obtain the following semantic algorithm:

$$x_{n+1} = 1 - \text{ABS}(\text{MIN}\{x_n, 1 - x_n\} - x_n) \ .$$

The semantic behaviors of these Liar-like sentences can be made visually perspicuous using web diagrams. The semantics for the continuous-valued CLASSICAL LIAR appears in a web diagram as a nested series of simple boxes. Given an initial value 1/3, for example, the CLASSICAL LIAR will oscillate monotonously in a cycle of period 2 between 1/3 and 2/3. The behaviors of the CAUTIOUS TRUTH-TELLER and the CONTRADICTORY LIAR, on the other hand, are diametrically opposed. The CAUTIOUS TRUTH-TELLER yields a *fixed-point attractor*: no matter what the initial value, the successively estimated values are inevitably drawn toward the fixed point of 2/3. The CONTRADICTORY LIAR, in contrast, yields a *fixed-point repellor*: for any values other than the fixed point 2/3, the successively revised estimates for the CONTRADICTORY LIAR are repelled away from 2/3 until the values settle on the oscillation between 1 and 0, characteristic of the CLASSICAL LIAR.

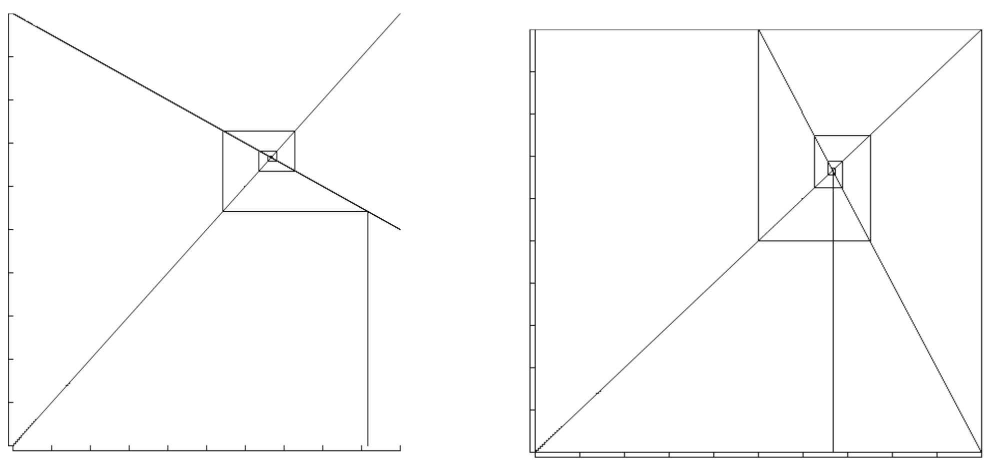

*Figure 4.* The CAUTIOUS TRUTH-TELLER and the CONTRADICTORY LIAR exhibit opposite semantic behaviors in terms of fixed-point attractors and repellors.

The semantic behavior of the CLASSICAL LIAR, though identical to the CONTRADICTORY LIAR on classical values, diverges from the CLASSICAL LIAR on the range of values between 0 and 1. Sentences like the CAUTIOUS TRUTH-TELLER and the CONTRADICTORY LIAR, on the other hand, have the opposite semantic behavior in terms of being attractors and repellors around the same fixed point. Infinite-valued logic can, therefore, reveal intriguing new patterns of paradox that have remained hidden in a classical bivalent setting.[8]

## IV.

Consider next an intriguing variation of Zeno's paradox, which we call the SISYPHUS. Imagine Sisyphus is pushing a boulder up a hill. If Sisyphus is less that halfway up the hill, he is able to push the boulder to a point *twice* as far up the hill as he is from the bottom. Once Sisyphus passes the halfway point, however, his fortunes reverse. Sisyphus slips down the hill until he is at a point that is *twice* as far from the bottom as he had left to reach the top of the hill. This Sisyphean task continues on *ad infinitum*.

> SISYPHEAN ACHILLES: If I'm less than halfway up the hill, I'll double my progress. However, if I'm more than halfway up the hill, I will slip to a point that is twice that distance I had left.

The successive points in SISYPHUS's journey is given by the algorithm:

$$x_{n+1} = \begin{cases} 2^* x_n & \text{if } x_n < 1/2 \\ 2^*(1 - x_n) & \text{if } x_n \geq ½ \end{cases}$$

In binary notation the successive values of the Sisyphean algorithm for an initial value $x_o$ can be obtained by the following operation on binary strings:

(1) if the leading digit is '0', shift all the digits to the left,

and

(2) if the leading digit is '1', take the complement and then shift all the digits to the left.

---

[8] Consider the CONTINGENT LIAR based an infinite-valued generalization of the paradox due to Kleene and Rosser [1935], discussed by Curry [1941], and also known as Löb's paradox (see Barwise and Etchemendy [1987], footnote 14). The CONTINGENT LIAR is the sentence: *This sentence is as true as the conditional: if this sentence* (i.e., the CONTINGENT LIAR itself) *is true then q*. Here *q* is some contingent sentence. Using the infinite-valued rule for the Łukasiewiczian conditional, the sequence of estimated values for the Contingent Liar is $x_{n+1} = 1 - \text{ABS}(\text{ MIN}\{1 - x_n + /q/, 1\} - x_n )$. The CONTINGENT LIAR exhibits the fixed-point semantic behavior of the TRUTH-TELLER (*This sentence is true*) on the interval [0, /q/] and exhibits chaotic semantic behavior on the interval [/*q*/,1].

If you start with any irrational initial value, then the orbits of the algorithm will *appear* to be completely *random* even though the rule for generating the value is completely *deterministic*. Practically speaking, unless we know the initial value with infinite precision, the successive values of this algorithm are unpredictable. The algorithm for the Sisyphean Achilles is a paradigmatic example of what is known as *deterministic chaos* (see Devaney [1989]).

`Consider next an intriguing, infinite-valued generalization of the CLASSICAL LIAR, which asserts of itself, not that it is simply false, but that it is *true* to the *extent* that it is *false*:

| |
|---|
| The boxed sentence is as true as it is false. |

This is perhaps the most natural generalization of the CLASSICAL LIAR into an infinite-valued context. This sentence has been dubbed the CHAOTIC LIAR for reasons that will soon become apparent.

What the CHAOTIC LIAR asserts of itself is that it is false; hence, $S(x_n) = 1 - \mathrm{ABS}(0 - x_n)$. Since $x_n \geq 0$, we have that $S(x_n) = 1 - x_n$. Hence, the CHAOTIC LIAR can be expressed by V/~*p*/*p* and its semantic algorithm will be given by:

$$x_{n+1} = 1 - \mathrm{ABS}((1 - x_n) - x_n)\ .$$

The CHAOTIC LIAR derives it name from the fact that its algorithm is *chaotic* in a precise mathematical sense.[9]

As a chaotic function, the algorithm for the CHAOTIC LIAR, will exhibit:

---

[9] There are stronger and weaker definitions of chaos. Devaney's [1989] definition of chaos is as follows. A function $f: I \rightarrow I$ is *chaotic* on a set $I$ if all three of the following conditions hold:
(i) f has *sensitive dependence* on initial conditions: there exists points arbitrarily close to $x$ which eventually separate from $x$ by at least δ under iteration of $f$, i.e., $\exists\delta > 0\ \forall x \in I\ \forall$ neighborhood N of $x$ $\exists y \in \mathrm{N}\ \exists n \geq 0\ |f^n(x) - f^n(y)| > \delta$ (here '$f^n(x)$' represents the $n$th iteration of the function $f$);
(ii) $f$ is *topologically transitive*: $f$ has points which eventually move under iteration from one arbitrarily small neighborhood to any other, i.e., ∀ opens sets $U, V \subset I\ \exists k > 0\ \mathrm{f}^k(U) \cap V \neq \phi$;
(iii) the *periodic points are dense* on $I$: there is a periodic point between any two periodic points in the interval $I$, where a point $x$ is *periodic* if $\exists n\ f^n(x) = x$.
Devaney [1992], p. 119 notes that J. Banks, *et. al.* in "On Devaney's Definition of Chaos," *Amer. Math. Monthly*, **99**, (1992), 332 – 334 show that sensitive dependence follows from the density of periodic points and topological transitivity.

- *unpredictability*: the sensitive dependence on initial conditions of chaotic functions entails that prediction will fail if the initial conditions are not known with infinite accuracy;
- *infinitely many periodic cycles*: since the CHAOTIC LIAR has a cycle of period three it will have cycles of all other periods according to Sarkovskii's ordering (see Devaney [1989], pp. 60-62);
- *fractal images of the semantics of paradox*: *fractals* or sets with a fractional Hausdorff dimension (see Peitgen and Saupe [1988], pp. 28-29) are characterized by self-affinity at increasing powers of magnification. Within the patterns of paradox for DUALIST forms of the Liar, for example, there are infinitely complex fractal patterns (see Mar and Grim [1991]).

It happens that this semantic algorithm for the CHAOTIC LIAR is mathematically identical to the algorithm for the successive positions of the Sisyphean Achilles given above. The Sisyphean Achilles therefore inherits all the mathematical properties of the CHAOTIC LIAR. The web diagram for the CHAOTIC LIAR makes it clear that the complexity of its semantic behavior far surpasses the monotonous regularity of the CLASSICAL LIAR:

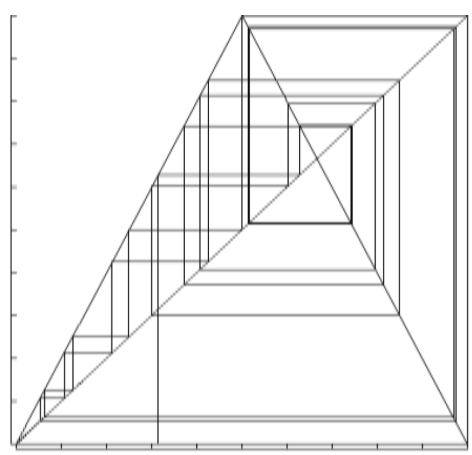

*Figure 5.* The web diagram for the CHAOTIC LIAR and the SISYPHEAN ACHILLES.

The variation of Zeno's paradox that generated the Sierpinski fractal can be analyzed separately into the behaviors of its $x$ and $y$ components. The function for the iterated values of each coordinate is given by the *Baker function*:

$$x_{n+1} = \begin{cases} 2^* x_n & \text{if } x_n < 1/2 \\ 2^* x_n - 1 & \text{if } x_n \geq 1/2 \end{cases}$$

We can obtain the algorithm for the coordinates of the CHAOTIC LIAR or Sisyphean Achilles by simply reversing the slope of the graph for values greater than 1/2. When we do so, we again obtain the algorithm for the SISYPHEAN ACHILLES or CHAOTIC LIAR expressed as the chaotic tent function.[10]

The Baker function was used to generate the Sierpinski fractal: given an ordered pair $(x_0, y_0)$, we check to see if the successive $x_n$ and $y_n$ values as computed by the Baker function, both exceed the threshold of 1/2. If so, then $(x_0, y_0)$ is *not* in the Sierpinski fractal. Similarly, the algorithm for the Chaotic Liar can be used to generate a new fractal, which we call the Sisyphean fractal. Given $(x_0, y_0)$, we check to see if the successive $x_n$ and $y_n$ values as computed by the Chaotic Tent function, both exceed the threshold of 1/2. If so, then $(x_0, y_0)$ is *not* in the Sisyphean fractal.

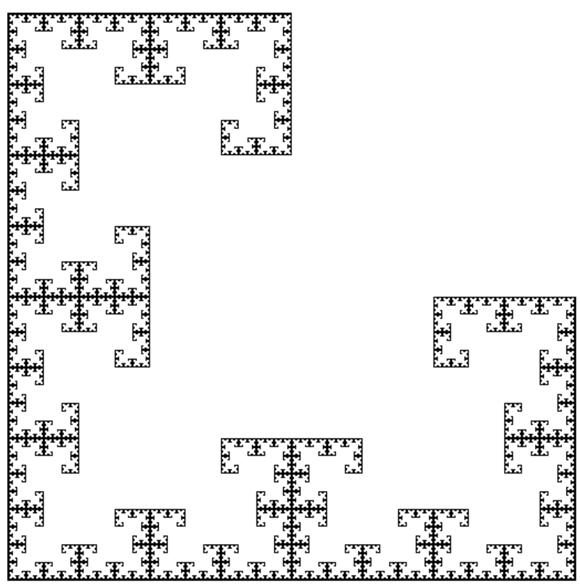

*Figure 6.* The Sisyphean Fractal is generated by the SISYPHEAN ACHILLES and the CHAOTIC LIAR.

[10] The tent function is mentioned in Robert May's [1976] groundbreaking paper as a "mathematical curiosity." Here, however, the tent function appears as perhaps the simplest generalization of the CLASSICAL LIAR that yields chaotic semantic behavior.

The Sisyphean fractal can be obtained, like the Sierpinski fractal, by an iterative geometric construction. In this construction we remove the quadrant where both x and $y$ are greater than 1/2. The resulting L-shaped figure is folded on top of the square quadrant where $x$ and $y$ are both less than 1/2, which is now considered to be the new unit square. We again remove the quadrant where both x and $y$ are greater than 1/2. The L-shaped figure is folded on top of the new unit square and this procedure is repeated.

An analysis of the binary arithmetic of the algorithm for the CHAOTIC LIAR reveals its intimate connection with *Gray codes*. Gray codes (invented by Frank Gray in 1872) is a way of symbolizing numbers in a positional notation so that when the numbers are in counting order, any adjacent pair will differ in their digits in at most one position. There are different Gray coding schemes, but the most familiar is *binary reflected* Gray codes. It is generated by reflection in the following way: beginning with 0, 1, the next numbers are obtained by taking the mirror image of the digits and prefixing 1. This procedure is iterated, to obtain the values:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 7 | 100 | 14 | 1001 |
| 1 | 1 | 8 | 1100 | 15 | 1000 |
| 2 | 11 | 9 | 1101 | 16 | 11000 |
| 3 | 10 | 10 | 1111 | 17 | 11001 |
| 4 | 110 | 11 | 1110 | 18 | 11011 |
| 5 | 111 | 12 | 1010 | 19 | 11010 |
| 6 | 101 | 13 | 1011 | 20 | 11110 |

To convert a binary number to its reflected Gray equivalent, we start with the digit at the right and consider each digit in turn:

(1) if the next digit to the left is 0, let the former digit alone;

(2) if the next digit to the left is 1, change the former digit. (The digit at the extreme left is assumed to be 0.)

For example, 41 in binary is 101001 and is assigned the Gray code of 111101. This procedure is equivalent to performing the following operations in binary strings: (1) right-shifting the binary string, and (2) taking the bit-wise *exclusive disjunction* of the two binary strings.

Given the second procedure above, we can obtain a simple QBasic program for the Sisyphean fractal modeled on the previous program for the Sierpinski fractal. Simply replace the binary strings with their binary reflected Gray Codes equivalents and alter the range of $q$:

Screen 9

For $p$ = 0 to 255

For $q$ = 0 to 255

G$p$ = $p$ XOR INT($p$/2): REM The Gray code for $p$

G$q$ = $q$ XOR INT($q$/2): REM The Gray code for $q$

If G$p$ and G$q$ then Pset ($p$, $q$)

Next $p$

Next $q$

For our last example, consider a TRIPLIST version of the Liar paradox (such variations were discussed by the medieval master of paradox John Buridan (see Scott [1966] and more recently by Tyler Burge [1982] and Brian Skyrms [1982]):

SOCRATES: What Plato says is true.

PLATO: What Socrates says is false.

CHRYSIPPUS: It is not the case that both Socrates and Plato speak truly.

What Socrates says is that what Plato says is *true*, but Plato says that what Socrates says is *false*. Hence, Socrates' statement is true to the extent that it is false. Similarly, what Plato says is true to the extent that it is false. Hence, both Socrates' and Plato's statements are modeled by the evaluative algorithmic sequences of the CHAOTIC LIAR. Chrysippus statement is true to the extent that not both Socrates and Plato speak truly.

We can represent the history of bivalent semantic evaluations of the TRIPLIST LIAR as an expanding binary expansion. Each place in the expansion represents a triple of values for a bivalent semantic evaluation. The entire binary string can be seen as a way of encoding the *semantic history* of one of the speakers. Now let each ordered pair ($x$, $y$) of binary strings expressed in Gray codes represent a possible semantic history of Socrates' and Plato's statements. We associate with each ordered pair ($x$, $y$) a value $z$, the value of the *bit-wise binary conjunction* of the pair of binary strings. The value $z$ represents the value of Chrysippus' evaluation of Socrates' and Plato's statements considered as semantic histories.

When the value $z$ is expressed in terms of Gray codes and is mapped as the height above the point ($x$, $y$), we obtain a tetrahedral Sisyphean fractal. Figure 6 is an approximation of the Sisyphean fractal for the first four iterations.[11] We have rotated the cube so that the origin (0,0,0) is located in the upper right-hand corner of the square base.

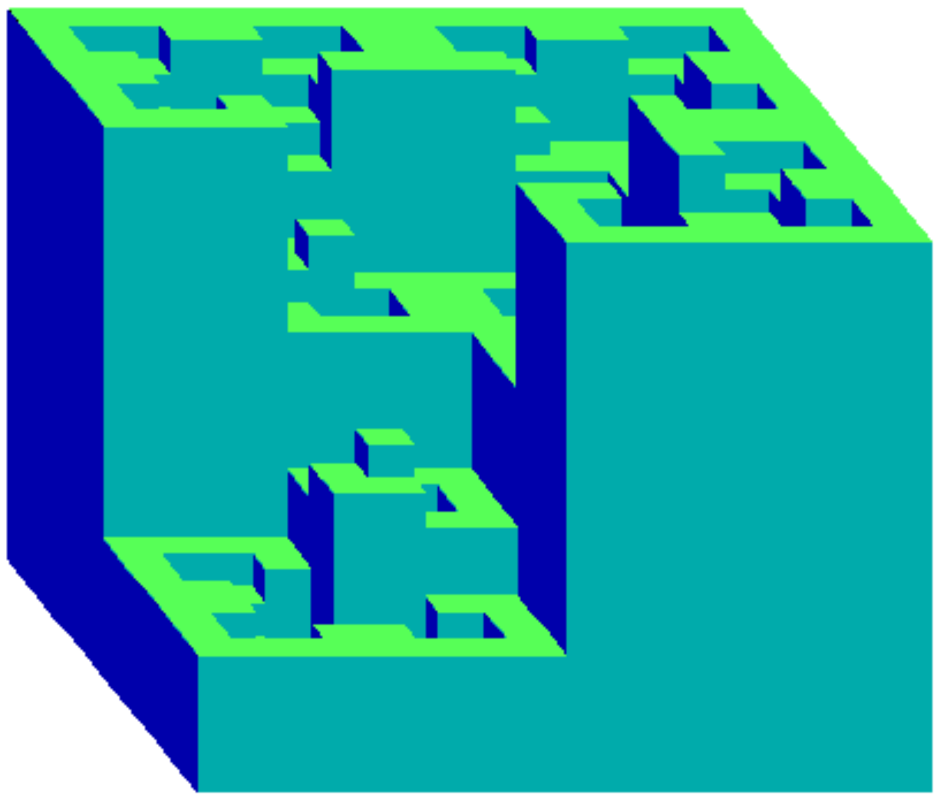

*Figure 7.* A TRIPLIST LIAR represented as a three-dimensional Sisyphean fractal.

V.

The paradoxes of Zeno and the paradoxes of logic have long been thought to have some metaphorical affinities. We have shown that the affinity between the two variations of Zeno's paradoxes of motion and the paradox of the Liar can be strengthened to mathematical identity. In retrospect, the intuitive equivalence between Zeno's paradoxes of motion and the semantical paradoxes of the Liar may now seem obvious. The isomorphism is due to the fact that both the

[11] We are indebted to Robert Rothenberg for programming assistance.

paradoxes of motion and the paradoxes of semantics involve infinite-regresses that have a fractal character. This fractal character can now be precisely characterized using the mathematics of dynamical systems theory. Commenting on the Protean power of Zeno's paradoxes, Wesley Salmon observed

> Each age, from Aristotle on down, seems to find in the paradoxes difficulties that are roughly commensurate with the mathematical, logical, and philosophical resources then available. When more powerful tools emerge, philosophers seem willing to acknowledge deeper difficulties that would have proved insurmountable for more primitive methods. We may have resolutions which are appropriate to our present level of understanding, but they may appear quite inadequate when we have advanced further. The paradoxes do, after all, go to the very heart of space, time, and motion, and these are profoundly difficult concepts.

This paper can be seen as a confirmation of Salmon's observation. Using the mathematics of dynamical systems theory, the intuitively felt but unproved underlying structural similarities between the paradoxes of Zeno and the paradoxes of the logic can now be mathematically proven.

## REFERENCES

Barnsley, Michael
[1988] *Fractals Everywhere*, Academic Press: San Diego, CA.
Barwise, Jon, and John Etchemendy
[1987] *The Liar*, Oxford University Press: New York.
Burge, Tyler
[1982] The Liar Paradox: Tangles and Chains, *Philosophical Studies* **41**, 353-366. Reprinted in Martin [1984], 83-117.
Carroll, Lewis (Charles Dodgson)
[1895] What the Tortoise Said to Achilles, *Mind*, New Series **4**, 278-280.
Church, Alonzo
[1956] *Introduction to Mathematical Logic, vol. 1*, Princeton University Press: Princeton, NJ.
Conway, John Horton
[1976] *On Numbers and Games*, Academic Press: San Diego, CA.
Curry, Haskell B.
[1941] The paradox of Kleene and Rosser, *Trans. Am. Math. Soc.* **50**, 454-516.
Devaney, Robert L.
[1989] *An Introduction to Chaotic Dynamical Systems, 2nd edition,* Addison-Wesley: Menlo Park, CA.
[1992] *A First Course in Chaotic Dynamical Systems: Theory and Experiment,* Addison-Wesley: Menlo Park, CA.

van Fraassen, Bas

[1972] Inference and Self-Reference, in *Semantics of Natural Language*, ed. by Davidson and Harman, D. Reidel: Dordrecht-Holland, 695-708.

Gardner, Martin

[1972], "The Binary Gray Code," *Scientific American*, **277** (August), 106 ff.

Gupta, Anil

[1982] Truth and paradox, *J. of Phil. Logic* **11**, 1-60.

Herzberger, Hans

[1982] Notes on naive semantics, *J. of Phil. Logic* **11**, 61-102.

Kleene, Stephen C.

[1936] λ-definability and recursiveness, *Duke Math. J.* **2**, 340-353.

Kleene, Stephen C. and Barkley J. Rosser

[1935] The inconsistency of certain formal logics, *Ann. Math.* **36**, 630-636.

Knuth, Donald E.

[1974] *Surreal Numbers*, Addison-Wesley: Menlo Park, CA.

Mar, Gary and Patrick Grim

[1991] Pattern and chaos: new images in the semantics of paradox, *Noûs* **25**, 659-693.

Martin, Robert L.

[1984] *Recent Essays on Truth and the Liar Paradox*, Oxford University Press: New York.

May, Robert

[1976] Simple mathematical models with very complicated dynamics, *Nature* **261**, 459-467.

Peitgen, Heiz-Otto and Dietmar Saupe (eds.)

[1988] *The Science of Fractal Images*, Springer-Verlag: New York.

Rescher, Nicholas

[1969] *Multi-valued Logic*, McGraw-Hill: New York.

Salmon, Wesley

[1970] *Zeno's Paradoxes*, Bobbs-Merrill: Indianapolis, IN.

Schroeder, Manfred

[1991] *Fractals, Chaos, Power Laws*, W. H. Freeman: New York.

Scott, Kermit

[1966] *John Buridan: Sophisms on Meaning and Truth*, Appleton-Century-Crofts: New York.

Shulman, Polly

[1995] Infinity Plus One, and the Surreal Numbers, *Discover* **16**, 96-105.

Skryms, Brian

[1982] Intensional Aspects of Semantical Self-Reference, in Martin [1984], 119-131.

St. Denis, Paul and Patrick Grim

[1997] Fractal Images of Formal Systems, *J. of Phil. Logic* **26**, 181-222.

GARY MAR
GROUP FOR LOGIC AND FORMAL SEMANTICS
Stony Brook University
Stony Brook, New York, USA 11974-3750
*email:* Gary.Mar@stonybrook.edu