A mathematical hangman game

L. Malato, F. Albuquerque
lia.malato@hotmail.com, fmapic@gmail.com

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Abstract

Everyone knows the hangman game! Well here is its mathematical counterpart. The goal of this game is to guess the right number. However, there is a catch... You are not allowed to guess the digits. How can you win then? You have to ask your opponent specific questions about the mathematical structure of the number you're trying to guess. Are you going to beat the game?

1 Introduction

The original hangman game is a two-player game, where player A secretly selects a random word and only displays how many letters that word has. Player B then has to guess the word letter by letter. But what if we didn't stop at letters?

It all started during a Sunday afternoon. The authors were having a hot drink at a local coffee shop when suddenly the first author had an idea to entertain their mathematical minds.

"What if we played a game of hangman, but with a little mathematical twist?"

The basic idea of the game is to guess a number instead of a word. In the original hangman game, the players guess letters in a way as to find a meaningful word. In a number variant of hangman, one might be drawn to guess the number digit-by-digit... which is something we will not do. Instead we will be playing with the fundamental properties of each number and how the secretly chosen number relates to other numbers. The goal is to guess the right number through well chosen questions.

2 Basic rules

In this game we have two players: player A and player B. Neither player needs to have any specific mathematical background. In what follows we will assume both players have high school mathematics knowledge.

To start the game, player A thinks of a number that player B will need to guess through a series of mathematical questions based on the structure of numbers. The questions must be made in a way that the answer can only be 'yes', 'no' or a number. Player A starts by drawing short lines (one line per digit) on a piece of paper. For example, if the number to be guessed is 490, then player A would write down

$$\_\_\_ \quad \_\_\_ \quad \_\_\_$$

After this, the game repeats the following pattern:

1. Player B asks a question.
2. Player A answers with a 'yes', a 'no' or a number, depending on the question asked.
3. Player B gives a guess.
4. In case player B is right, the game ends. In case player B is wrong, the game continues and player B loses a life.

Player B has 7 lives. Just like with the common hangman game, with each lost life player A draws a line as follows:

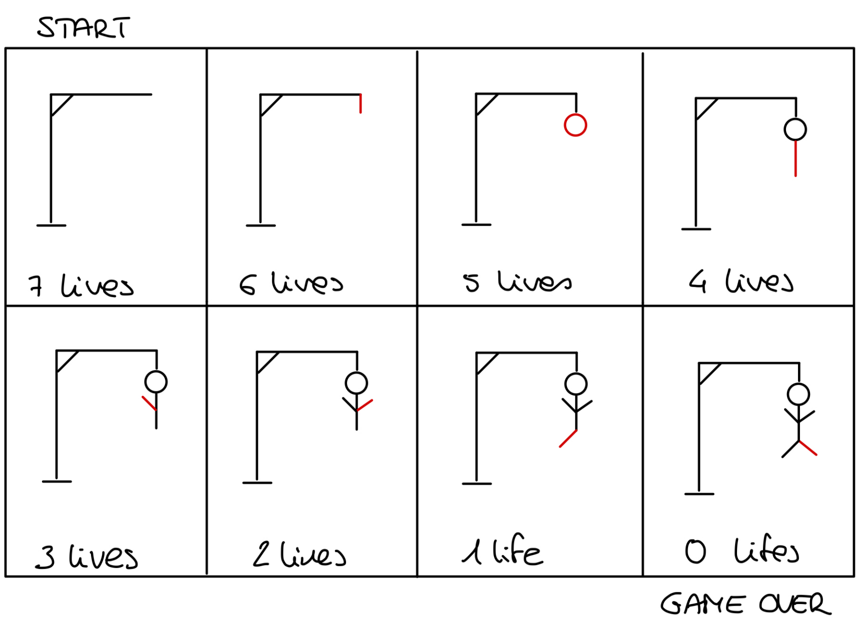

Figure 1: All possible drawings for the hangman game.

Bonus rule: If player B, at some point, gives a guess that is inconsistent with the previously gathered information, player B loses an extra life. This implies that it is possible to lose two lives in one guess!

3 An example

Consider two players: Saunders and Carl[1]. Saunders thinks of a number that Carl needs to guess. Saunders thinks of the number 476 and Carl will try to guess that number. Saunders starts off by writing 3 lines on a sheet of paper:

___ ___ ___

And so the game begins:

Carl: What is the lowest natural number above the square root of the number?
Saunders: 22 *(since* $\sqrt{476} \simeq 21.82$*).*
Carl: Is your number 400? *(Here Carl chooses any number below* $22^2 = 484$*)*
Saunders: No *(Carl loses their first life).*

Carl: How many distinct prime factors does the number have?
Saunders: 3 *(since* $476 = 2^2 \cdot 7 \cdot 17$*).*
Carl: Is your number 105? *(Carl tried* $3 \cdot 5 \cdot 7$*)*
Saunders: No *(Carl loses their second life).*

Carl: What is the greatest common divisor between your number and 30?
Saunders: 2 *(since* $30 = 2 \cdot 3 \cdot 5$ *and therefore* $\gcd(30, 476) = 2$*).*
Carl: Is your number 646? *(Carl tried* $2 \cdot 17 \cdot 19$*)*
Saunders: No and you just contradicted the answer to your first question.

Here Carl now loses two lives: one because his guess is incorrect and another one because his guess is greater than $22^2 = 484$*, an information obtained at the beginning of the game.*

Carl: What is the greatest common divisor between your number and 1001?
Saunders: 7 *(since* $1001 = 7 \cdot 11 \cdot 13$ *and therefore* $\gcd(1001, 476) = 7$*).*

From these four questions, Carl is able to conclude that the prime factorization of the secret number must be of the form $2 \cdot 7 \cdot p$*, where* p *is some prime greater than* 13*.*

Carl: Is the number 238? *(Carl tried* $2 \cdot 7 \cdot 17$*)*
Saunders: No *(Carl loses their fifth life).*

Carl now understands that one of the primes is repeated in the prime factorization.

Carl: Is any of the prime numbers in the prime factorization to the power of 2?
Saunders: Yes!
Carl: Is the number 476? *(Carl tried* $2^2 \cdot 7 \cdot 17$*)*
Saunders: Yes!

[1] The authors chose these names with great care as an homage to their favourite mathematicians: Saunders MacLane and Carl Friedrich Gauss.

In this game, Carl would have lost 5 lives and the hangman drawing would look as follows:

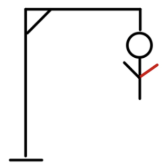

Figure 2: Final hangman drawing for Saunders' and Carl's game.

4 To ask or what not to ask: that is the question?

After this very exciting example, let's take a look at what the second player can and cannot ask.

4.1 Questions that are allowed

In short, any question with mathematical interest can be asked. In what follows, let $n = 490$ be the number to be guessed. Here are some examples.

- *Is the number a prime number?*
  Since 490 is not a prime number, the answer would be no.
- *How many (distinct) prime factors does the number have?*
  Since $490 = 2 \cdot 5 \cdot 7^2$, the answer would be 3.
- *What is the greatest natural number below the square root of the number?*
  Since $\sqrt{490} = 22.14$, the answer would be 22.
- *What is the lowest natural number above the square root of the number?*
  Since $\sqrt{490} = 22.14$, the answer would be 23.
- *What is the greatest common divisor between your number and the number* 2*?*
  Since $\gcd(490, 2) = 2$, the answer would be 2. This allows the player to know if the number is even or odd.
- *Is the number divisible by* 9*?*
  Since $\frac{490}{9} = 54.44$, the answer would be no.
- *What's the remainder of your number and* 11*?*
  Since $490 = 44 \cdot 11 + 6$, this means $490 \equiv 6 \ (mod\ 11)$, the answer would be 6.

Important rule: Questions about the greatest common divisor must follow a limitation in order for the game to not end after a small number of rounds. If the secret number has k digits, then we must use a number in the greatest common divisor that has at most $k+1$ distinct primes! For example, if we want to guess the number $n = 490$ (three digits), we could ask for $\gcd(n, 210)$ since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, but not for $\gcd(n, 2310)$ since $210 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.

4.2 Questions that are not allowed

In short, any question related to the digits of the number is not allowed. Moreover, any question that makes it too easy to guess the number is also not allowed. For example:

- *Are all the digits the same?*
- *Are all digits distinct?*
- *Does your number end with a* 0*?*

Questions like these three are discouraged. However, if it they can be formulated in terms of more interesting mathematical concepts, such as congruences, then it may be asked. For example, the third question *'Does your number end with a* 0*?'* could be reframed as *'Is the remainder of your number modulo* 10 *equal to* 0*?'*.

Although this basic set of rules has shown to be most productive in order to obtain an interesting game of mathematical hangman, this set is not closed. It is possible to adapt the rules based on the mathematical knowledge of each player.

5 What are the most interesting questions to ask?

As could be seen from the above, some questions are more useful than others. Some even allow to obtain multiple clues at once. Let's discuss this.

5.1 Strategy 1: Greatest common divisor

Thanks to the Fundamental Theorem of Arithmetic, we know that every natural number can be written as a product of prime numbers, and that this product is unique, up to the order of the factors. So, if one knows the right prime factors, one can guess the secret number easily.

Bearing this in mind, naively, we could focus on questions like:

- *"Is the number divisible by* 2*?"*
- *"Is the number divisible by* 11*?"*

This type of question allows us to check if a given prime (such as 2 or 11) belongs to the prime factorization of the number to be guessed. However, since player B only has 7 lives, this might not be optimal.

A powerful tool to help us in this task is the greatest common divisor between the number to be guessed and a number player B carefully chooses. For example, let $n = 490$ be the number to be guessed. One could ask the $\gcd(n, 2)$ which is 2. Although we now know the secret number is even, it is possible to formulate a better question. If we were to ask the $\gcd(n, 6)$, we would now know that 490 is divisible by 2 but not by 3, since $\gcd(n, 6) = 2$.

From this remark, we can deduce one of the most useful questions:

> *"What is the greatest common divisor between n and M?",*
> *where M is a carefully selected number.*

If player B chooses $M = 210$, since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, this question will tell the player whether or not the secret number is divisible by 2, 3, 5 or 7. This works very well, because we not only sieve out primes from the prime factorization, but we can also get confirmation that certain primes belong to the prime factorization.

After some calculations, player A can tell player B that the answer is 70. Now, player B knows that 3 is not a divisor of the number to be guessed but $2, 5$ and 7 are (since $70 = 2{\cdot}5{\cdot}7$).

When carefully choosing the number M, player B should choose M with a useful prime factorization. If one picks random numbers, we might exclude a prime factor twice, which is penalized by the game's bonus rule. For instance, in the previous question, we had already sieved out 3 as a prime factor. In the next question, if player B asks "What's the greatest common divisor between your number and 33?", we are not using the previous information that 3 is not a factor. So, our guesses will be inconsistent and we might lose two instead of one life.

5.2 Strategy 2: Root estimate

Another good strategy is to limit our search for numbers in terms of upper and lower bounds. A well-known result can be paraphrased as:

> *For every natural number m, there is always a prime factor that's less than or equal to $\sqrt{m}$. Any prime factor greater than $\sqrt{m}$ is unique.*

Asking for an estimate of the square root of the secret number n allows us to:

- Predict which primes will be good candidates to be divisors of n. For instance, for $n = 490$, we saw $\sqrt{490} \simeq 22.14$. This means that the *prime suspects* would be $2, 3, 5, 7, 11, 13, 17, 19$.

- Get an estimate for the number n. For instance, if we know that $\sqrt{490} > 22$, this means that $490 > 22^2 = 484$. So our guesses must only be numbers greater than 484.

Asking for an estimate of the square root, also allows us to get both a lower bound and an upper bound:

> *For every natural number m, if we know the greatest natural number less than $\sqrt{m}$, we can add one to that estimate and we get the lowest natural number greater than $\sqrt{m}$.*

In our example, since we know $\sqrt{490} \simeq 22.14 > 22$, we can also deduce $23 > \sqrt{490} > 22$. Which ultimately leads to $529 > 490 > 484$. So our guesses need to be only numbers greater than 484 but less than 529. With one question, we get two bounds.

This could be generalized to any root (cube root, fourth root...) but since the powers grow rapidly, this tends to be not very helpful sometimes.

An untested scenario worth checking would be if rational indices would be even more helpful. For instance, asking for an estimate of $n^{2/3}$.

6 What are the less interesting questions to ask?

In this section, we look at questions which do not really help us in our quest to find the secret number. These are questions that are allowed and interesting from a mathematical standpoint, but which are rarely useful. Here we follow the motto:

"Just because you can, does not mean you should."

For example, to ask whether or not a number is prime is not an ideal question. By the Prime Number Theorem, we know that primes get rarer and rarer as we go along the number line. Let $x \mapsto \pi(x)$ be the well-documented function which counts the number of primes less than x. Let's calculate the probability of randomly selecting a prime number:

- $\pi(10) = 4$ tells us that in the first ten numbers, there are four primes ($2, 3, 5$ and 7). Hence, we have a probability of $4/10 = 40\%$ of selecting a prime number less than 10.
- Since $\pi(100) = 25$, we have a probability of $25/100 = 25\%$ of selecting a prime number less than 100.
- Since $\pi(1000) = 168$, we have a probability of $168/1000 = 16.8\%$ of selecting a prime number less than 1000.
- Since $\pi(10000) = 1229$, we have a probability of $1229/10000 = 12.29\%$ of selecting a prime number less than 10000.

When playing the game with secret numbers with 3 or 4 digits, these odds are not enticing enough, and they only get worse as the number of digits grows. Also, since the number is chosen by the other player, this player could precisely avoid prime numbers altogether.

We could generalize this to other special number sets. One could ask questions like:

- *"Is it a triangular number?"*
- *"Is it a Catalan number?"*
- *"Is it a Fibonacci number?"*

All of these questions have a low chance of actually working, due to a similar phenomenon: These special numbers only get rarer and rarer as the number of digits of our secret number grow. So, one should stick to more "down-to-earth" properties of numbers (e.g. prime factorization).

7 Some considerations

The game's difficulty increases with the number of digits the player needs to guess. For this reason, we suggest to allow certain questions to be added to or removed from the base rules when guessing numbers with more or less digits. For example, when guessing a number with two digits, it should neither be allowed to use the greatest common divisor nor to ask for a lower or upper bound of the square root of the secret number. For example, if the number to guess is 72, then asking these three questions almost always allows the player to find the correct number too quickly.

Carl: What is the greatest natural number below the square root of your number?
Saunders: 8 *(since* $\sqrt{72} \simeq 8.49$*).*

From this question, Carl already knows that the number lies between 64 and 81 since Carl deduces $9 > \sqrt{n} > 8$ (refer to section 5.2).

Carl: Is your number 68?
Saunders: No.

If Carl now asks a well-formulated question related to the greatest common divisor, the game is nearly over.

Carl: What is the greatest common divisor between your number and 210?
Saunders: 6.

Now Carl knows that 2 and 3 must be in the prime factorization of n. Knowing that only three numbers between 64 and 81 are divisible by 6 (that is, $66, 72$ and 78) which makes it now easy to guess that the number is 72.

8 What's next?

The mathematical hangman game can be used as a tool to bring number theory closer to students that are dealing with structural properties of numbers for the first time. It can also be used as a tool to consolidate students understanding of basic number theory while doing so in a fun and competitive way and can be adapted to all ages.

This article outlines the first version of the game. However, both authors agree that there is much more to explore. If this game interests you and you would like to collaborate with its development or simply share some ideas, feel free to contact the authors!