Almost orthogonal polyhedra in general, and this one in particular

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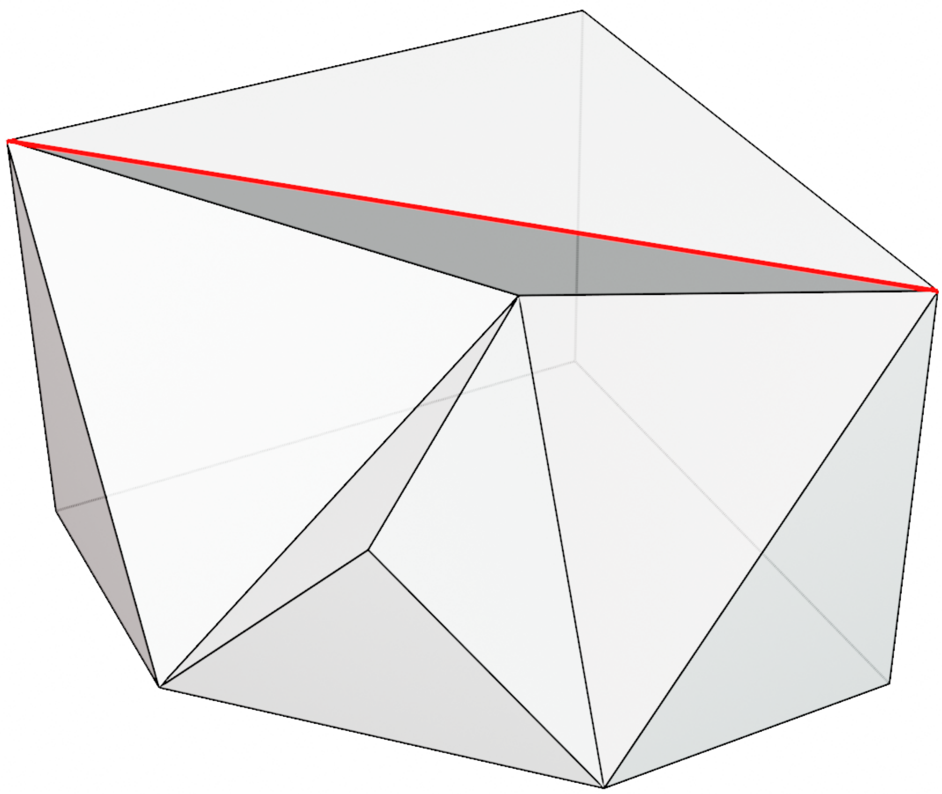


Figure 1

If everything has gone to plan, you will find among your gift exchange delights a smallish white plastic non-convex polyhedron that has twelve faces, and looks like Figure 1.

The intention of this booklet is to answer some of the questions you may have about this object. In particular, it aims to answer the three primary questions: *What?*, *How?*, and *Why?*.

If you have other questions, you are cordially invited to buttonhole me, or to email me at robin.houston@gmail.com. I may not know the answers, but I expect I shall enjoy thinking about your questions.

What is it?

It is an *almost orthogonal* polyhedron: a polyhedron whose adjacent faces are orthogonal to each other, *except on one edge*. On this special edge, drawn in red in Figure 1, the faces meet at 45°.

It is the simplest such polyhedron that is known to exist, or at any rate it is the simplest that *I* know.

If the special edge is permitted to have some other dihedral angle, not necessarily 45°, then I know a simpler one, illustrated in Figure 2 below. It has eight triangular faces. Its two largest faces meet at an angle of arccos(-1/3) ≈ 109.5° on the blue edge, and all the other edges have a dihedral angle of 90° or 270°.

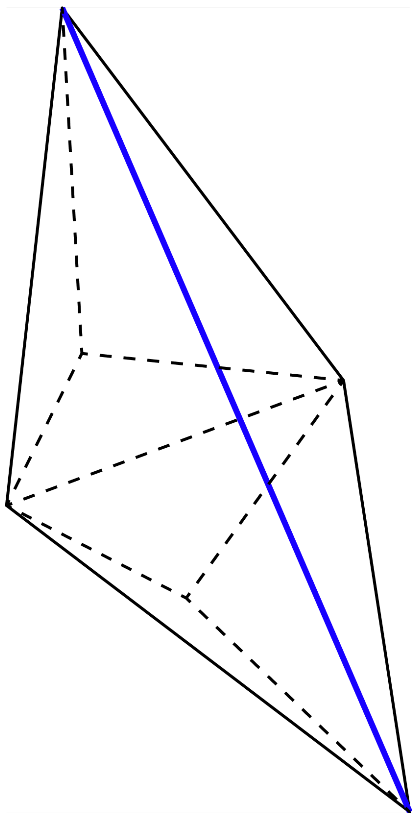


Figure 2. An almost orthogonal octahedron

How did I construct it?

I began with the hexahedron shown in Figure 3. All its dihedral angles are right, with two exceptions: the red edge has a dihedral angle of 135°, and the blue edge has a dihedral angle of arccos(-1/3) ≈ 109.5°. The red and blue edges are orthogonal to each other.

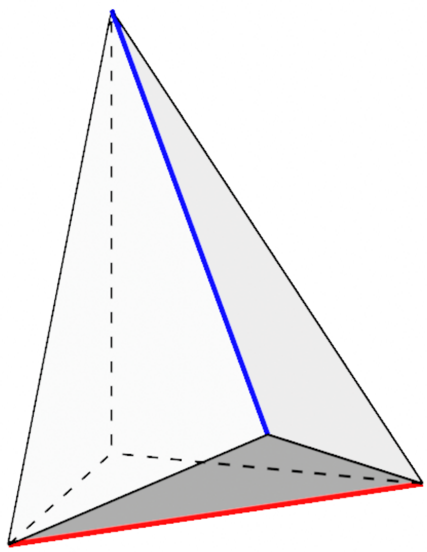


Figure 3. This hexahedron is due to Sydler, of which more below.

Take three copies of this hexahedron. Glue two of them together, so that their red edges coincide and their blue edges are collinear and share an endpoint: you will obtain the octahedron of Figure 2.

Scale down your octahedron by a factor of 2, so that its blue edge is the same length as the blue edge of your remaining hexahedron.

Overlay the two objects, the scaled-down octahedron and the remaining hexahedron, so their blue edges coincide and the faces that meet at those edges are coplanar. You will notice that, in this arrangement, the interior of the octahedron is a strict subset of the interior of the hexahedron. Subtract the octahedron from the hexahedron: this has the effect of removing the edge whose dihedral angle is arccos(-1/3). The result is shown in Figure 4: an almost orthogonal decahedron whose non-right dihedral angle is 135° (marked in red on the diagram).

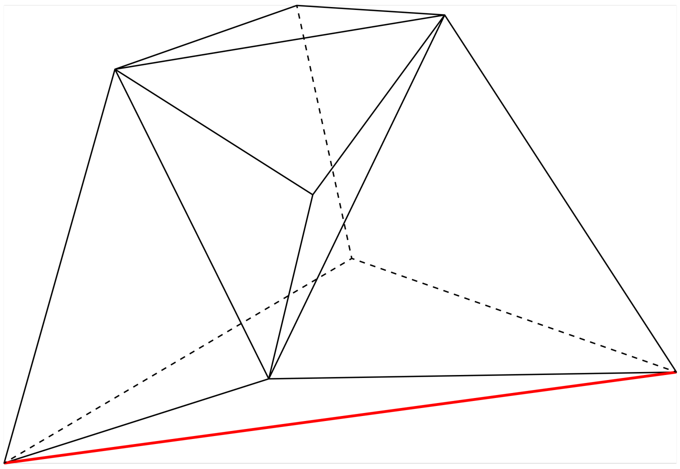


Figure 4. An almost orthogonal decahedron
whose non-right dihedral angle is 135°.

This construction is not easy to visualise from diagrams alone. I have a set of 3D-printed models that may help: I can show them to you if you ask me.

The final step is to take the complement of this decahedron with respect to a suitably-aligned box, as shown in Figure 5.

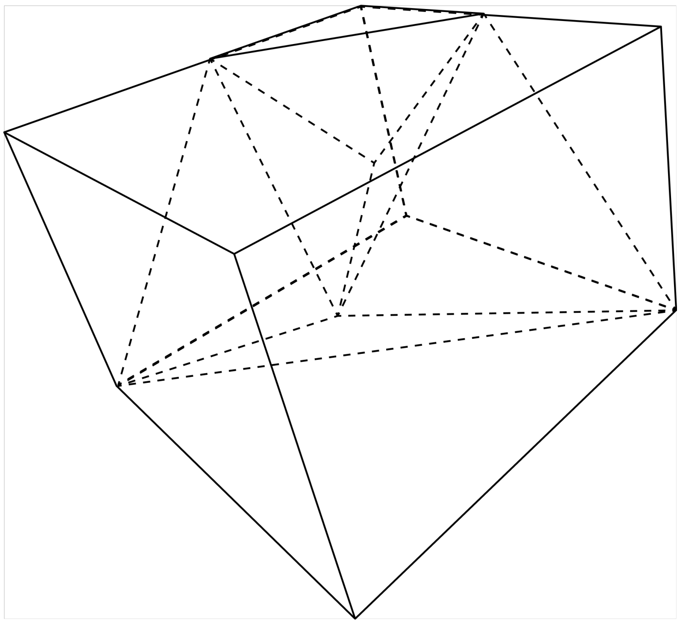


Figure 5

The complement is the almost-orthogonal dodecahedron of Figure 1.

There are other ways to obtain a 45° almost-orthogonal polyhedron starting from the 135° almost-orthogonal polyhedron of Figure 4. For example, we could attach an isosceles right-triangular prism as shown in Figure 6.

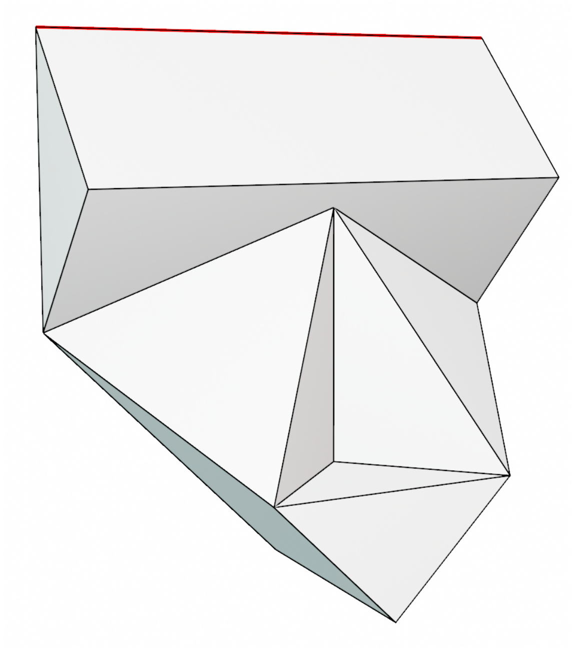


Figure 6

Or we could attach it as shown in Figure 7, which would make an imposing statue.

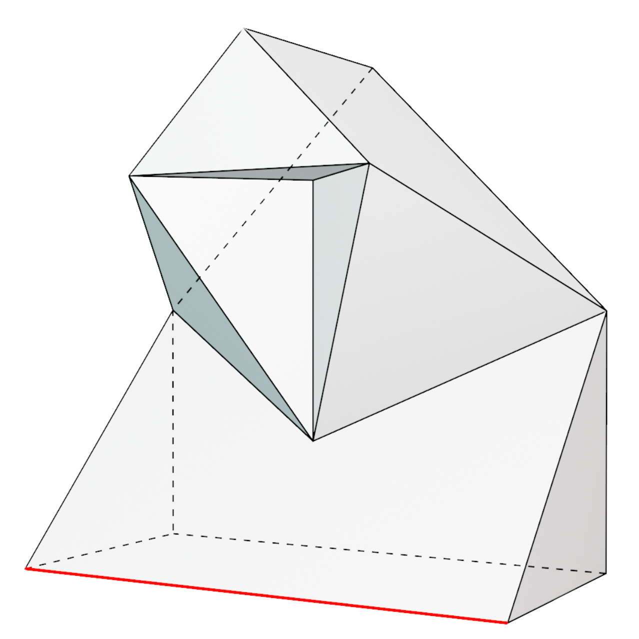


Figure 7

In both cases, the back face is pentagonal, combining a rectangular face of the prism and a triangular face of the decahedron.

It is interesting to deconstruct the decahedron of Figure 4 into three pieces: an isosceles right-triangular prism, and two pieces that are mirror images of each other and have rotational symmetry of order 3. These pieces have seven faces, and we will refer to their shape as the *fundamental heptahedron*. This convex polyhedron has three edges whose dihedral angles are not right, drawn in red in Figure 8. These are mutually orthogonal, and each has a dihedral angle of 135°.

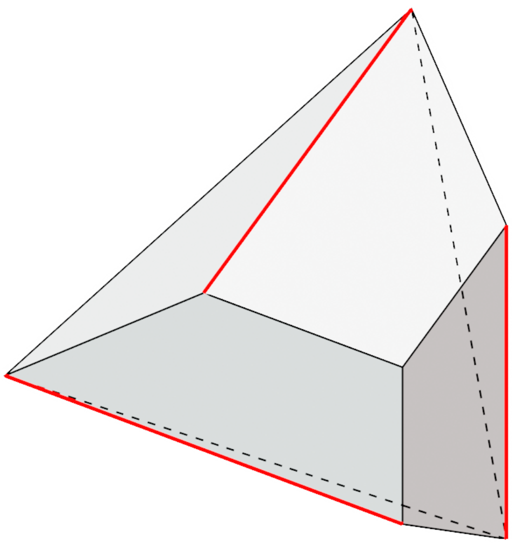
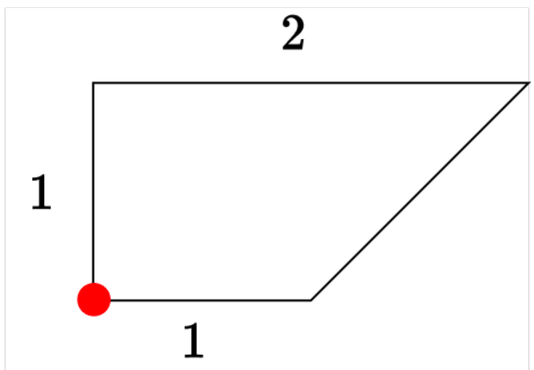


Figure 8. The fundamental heptahedron



The fundamental heptahedron is the convex hull of three mutually orthogonal copies of this quadrilateral, coinciding at the marked point

Eight of these – four in each orientation – can be assembled into Jessen's orthogonal icosahedron[1], illustrated in Figure 9.

[1] Børge Jessen. Orthogonal Icosahedra, Nordisk Matematisk Tidskrift 15, no. 2/3 (1967) pp 90–96

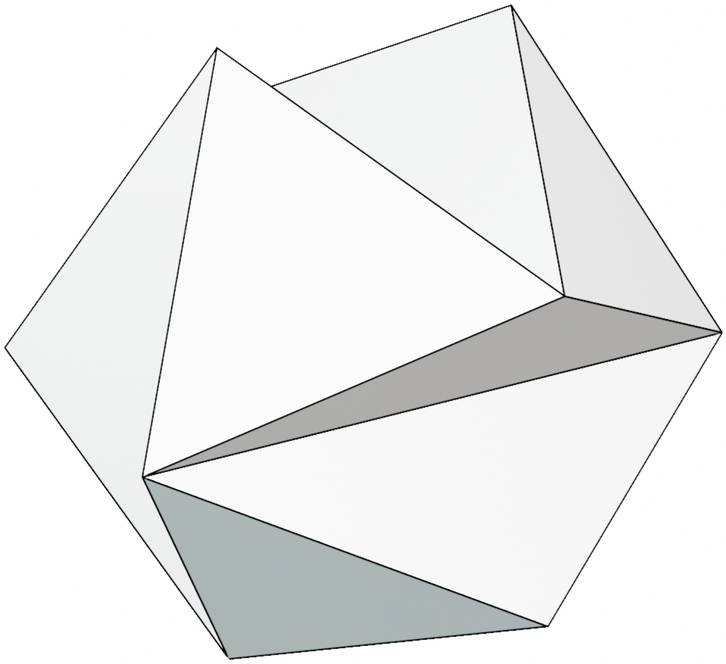
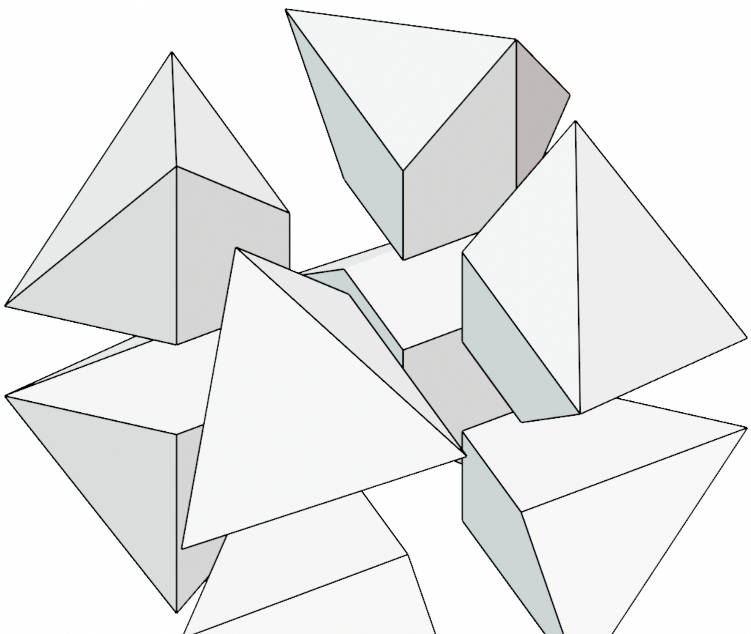


Figure 9. Jessen's orthogonal icosahedron. It may be dissected into eight copies of the fundamental heptahedron, four in each orientation.



Why?

In 1900, David Hilbert proposed a list of 23 then-unsolved mathematical problems that he regarded as important. These problems exerted a strong influence on mathematics well into the second half of the 20th century.

Hilbert's Third Problem asked whether, given two polyhedra of the same volume, it is always possible to cut one of them into a finite number of polyhedral pieces and reassemble those pieces into the other.

It was the first of Hilbert's problems to be solved. Max Dehn showed that, in addition to volume, there is another quantity, determined by the polyhedron's edges – now known as the *Dehn Invariant* – that always stays the same even when a polyhedron is cut into a finite number of polyhedral pieces and reassembled. Since a cube and a regular tetrahedron of the same volume have different Dehn invariants, it is impossible to cut up a cube and reassemble the pieces into a regular tetrahedron, or vice versa.

Dehn's rapid triumph suggested a more difficult question: is that the only obstruction? If two polyhedra have the same volume *and the same Dehn invariant*, then is it always possible to cut one of them into a finite number of polyhedral pieces and reassemble those pieces into the other?

This harder question was eventually answered in the affirmative in 1965, by Jean-Pierre Sydler[2], a Swiss librarian who had studied the problem as a PhD student in the 1950s, and afterwards continued to work on it in his spare time.

For one part of his argument (in Chapitre 1), Sydler needed to construct a family of polyhedra with the property that we're interested in here:

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[2] Sydler, J.-P.. "Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions.." Commentarii mathematici Helvetici 40 (1965/66): 43–80.

adjacent faces are orthogonal to each other, except for one pair of adjacent faces that are at 45° to each other.

The hexahedron of Figure 3 belongs to a family of hexahedra that Sydler constructs in Chapitre 1, and which play a key role in his construction: they are *almost* almost-orthogonal, i.e. they have *two* edges whose dihedral angles are not right. Those two non-right dihedral angles α and β are related by the equation

$$(3 + \cos 2\alpha)(1 - \cos \beta) = 4$$

where $90° < \alpha, \beta < 180°$; letting $\alpha = 135°$ gives $\cos \beta = -1/3$.

Although Sydler's construction is effective, he does not dwell on the particulars of the polyhedra that result. Much more recently (in 2017) Matthias Goerner[3] went through Sydler's construction step-by-step and created a 3D model of the resulting polyhedron, which proved to be hilariously complicated:

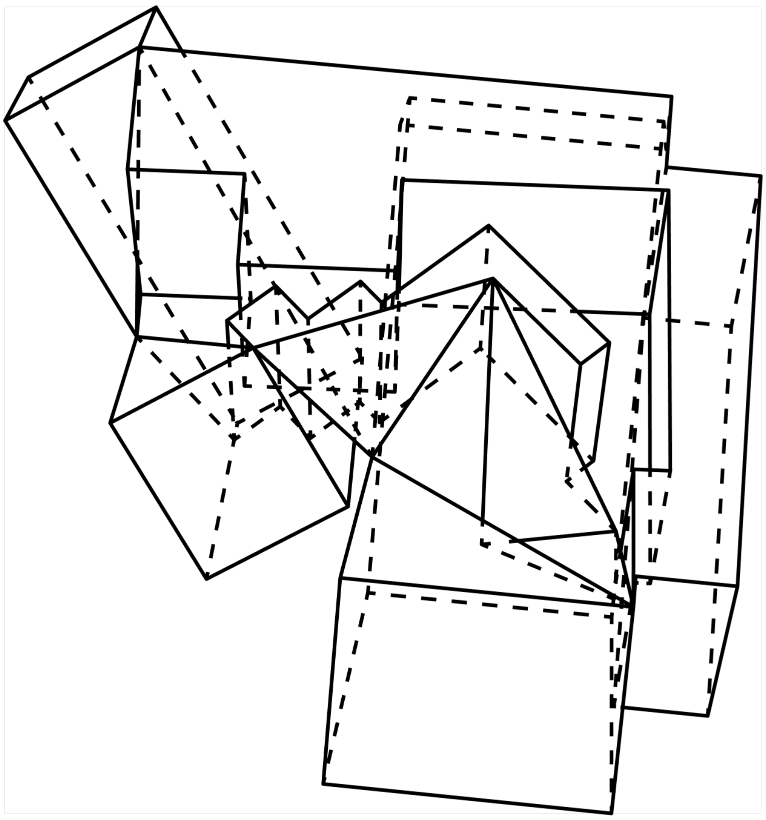

Figure 10. You are not expected to understand this diagram, but to marvel at its complexity.

[3] https://www.unhyperbolic.org/sydler.html

In July of 2022, Henry Segerman published a video[4] demonstrating a 3D print of Goerner's model, which inspired me to look for a simpler polyhedron with the same property. You've seen various results of that effort: my favourite is shown in Figure 1; Figures 6 and 7 show related alternatives.

It follows from work of Jessen[5] that there exist almost-orthogonal polyhedra whose non-right dihedral angle is any *algebraic angle*, i.e. any angle whose cosine (or equivalently sine) is an algebraic number.

However, Jessen's proof is not effective: it does not yield an actual construction for any such angle. So we are left with an interesting geometric challenge: for any given algebraic angle α, to construct an almost-orthogonal polyhedron whose remaining dihedral angle is α.

Might there even be a general method to construct, for *any* algebraic angle, an almost-orthogonal polyhedron with two faces meeting at that angle?

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[4] https://www.youtube.com/watch?v=tH6vLXMaCwQ

[5] Børge Jessen, The algebra of polyhedra and Sydler's theorem, Math Scand 22 (1968) pp 241–256