The Great Beetle Escape

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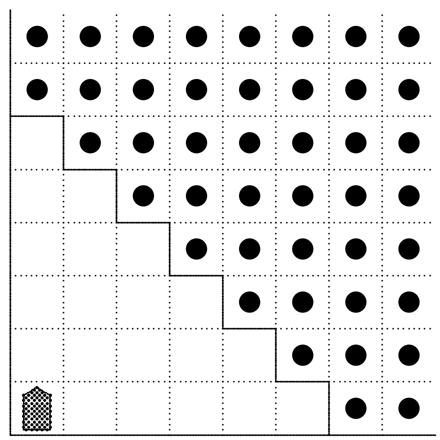
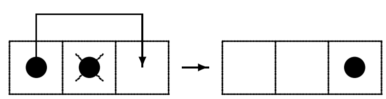
An evil wizard casts a spell on the beetles and transports them to a world which consists of the first quadrant of the plane divided into unit squares. There is a black magic zone at the southwest corner, consisting of 21 squares separated from the remaining squares by a zigzag fence. There are no beetles inside the zone, but there is one in each square outside. A black tower stands at the southwest corner square. If any beetle can reach the black tower, the spell will be broken and all surviving beetles will be released. In that case, the Great Escape is successful.



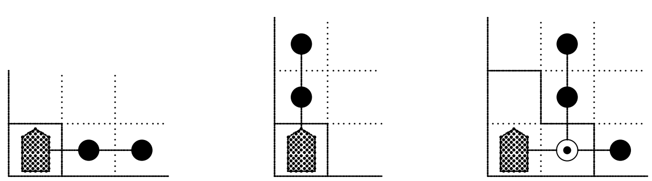
Let us first describe the movement of a beetle. It can only hop over another beetle in an adjacent square in the same row or column, landing on a square beyond which must be vacant at the time. The beetle being hopped over is crushed into the ground, perishes and vanishes. The fence does not hinder hopping.



Section A. Initial Attempts.

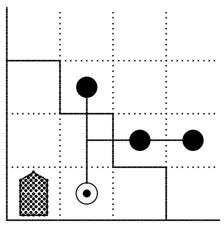
The beetles hold a council of war. The task on hand seems daunting. So they decide to work on simpler cases first, to gain some insight on the problem.

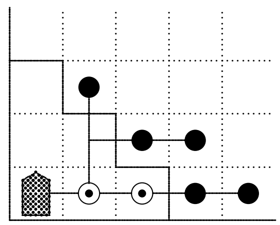
Case 1. The zone has only 1 square.
The diagram below on the left or in the middle shows that two beetles are sufficient to get one of them to the black tower.



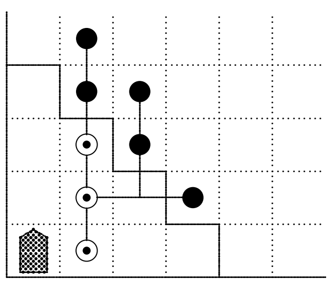
Case 2. The zone has 3 squares.
Copying the solution in the diagram above in the middle will put one beetle in the zone. Copying the solution in the diagram above on the left will get a third beetle to the black tower. Thus three beetles are required, as shown in the diagram above on the right.

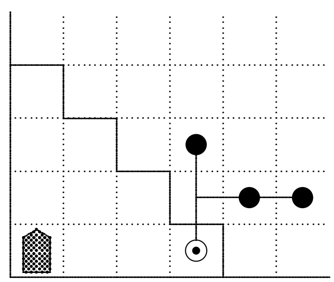
Case 3. The zone has 6 squares.
The first objective is to get a beetle next to the black tower. This can be accomplished using the solution to Case 2, as shown in the diagram below on the left. The next objective is to get a beetle next to the first beetle. This can be accomplished using the solution to Case 1, as shown in the diagram below on the right. Altogether, 5 beetles are involved.



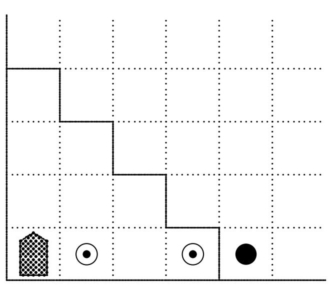


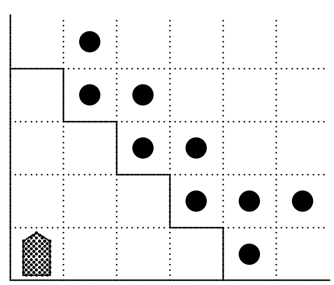
Case 4. The zone has 10 squares.
In the diagram below on the left, the solution to Case 3 is used to put a beetle inside the zone. In the diagram below on the right, the solution to Case 2 is used to put another beetle inside the zone.



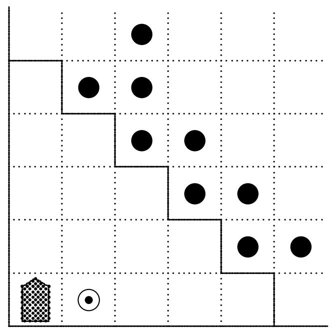


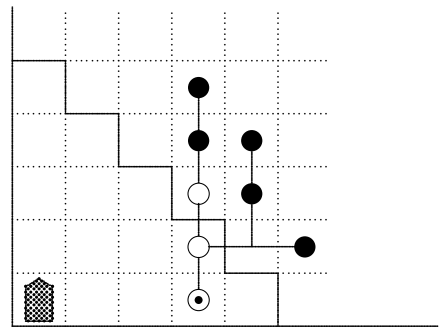
These two beetles provide the stepping stones for another beetle to reach the black tower, as shown in the diagram below on the left. Altogether, 9 beetles are involved. Their starting positions are shown in the diagram below on the right.



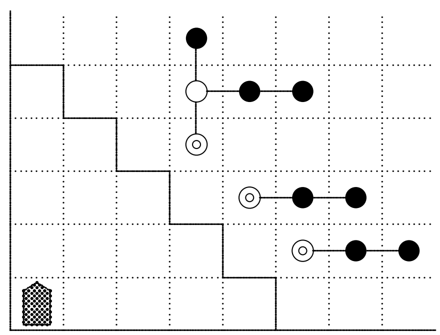


Case 5. The zone has 15 squares.
In the diagram below on the left, the solution to Case 4 is used to put a beetle inside the zone. In the diagram below on the right, the solution to Case 3 is used to put another beetle inside the zone.

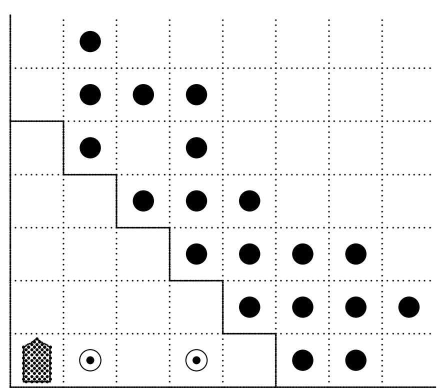




There are three beetles which are involved in both sides of the diagram above. They are marked by double circles in the diagram below. They may be duplicated by using the solutions to Cases 1 and 2.



There are now two beetles inside the zone. A third may be added on the bottom row just outside the zone. Together, these three beetles provide the stepping stones for another beetle to reach the black tower. Altogether, 19 beetles are involved. Their starting positions are shown in the diagram below.



Section B: Interlude.

The beetles have now come to Case 6, which is the actual task on hand. However, following the same strategy is getting much harder as the number of "overlapping" beetles becomes too large to manage. The council of war decides to take a break, and consider a theoretical question.

Clearly, if the zone gets larger and larger, there will come a point when the Great Escape is doomed to failure. Would that start with Case 6? Not daring to face the truth right away, the beetles decided to postpone it and consider the next case.

Case 7. The zone has 28 squares.

Suppose the Great Escape succeeds. All beetles that are not involved can be ignored, so that there will only be a single beetle in the black tower at the end. Assign it the value 1. It gets to this position by hopping over another beetle. Assign x to the crushed beetle and y to the hopping beetle before making its move. After the move, the beetle in the new position replaces the other two. It is desired that $x+y=1$, so that the total value of the beetles remains constant. The closer a beetle is to the black tower, the more valuable it is. Hence $1>x>y$. Suppose a beetle with value z crushes the one of value y and becomes one with value x. Then $y+z=x$. Take $y=x^2$ with $x+y=x^2+x=1$ and $z=x^3$. Indeed, $y+z=x^3+x^2=x(x^2+x)=x$.

Clearly, the value of a beetle is determined by its location. So values may be assigned directly to the squares themselves. These are shown in the diagram below. The extension to the squares not shown is obvious.

x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}
x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
1	x	x^2	x^3	x^4	x^5	x^6	x^7

The total value of the squares outside the fence is

$$S=8x^7+9x^8+10x^9+11x^{10}+\cdots.$$

Then

$$xS=8x^8+9x^9+10x^{10}+\cdots.$$

Subtracting this equation from the preceding one, we have

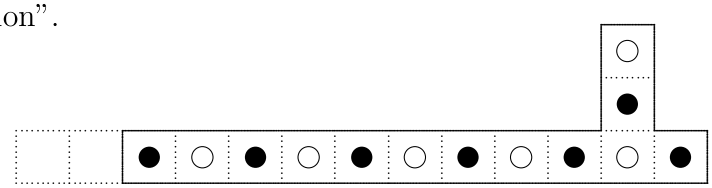
$$(1-x)S=8x^7+x^8+x^9+x^{10}+\cdots=7x^7+\frac{x^7}{1-x}.$$

Recall that $x^2+x=1$, so that $1-x=x^2$. Hence $x^2S=7x^7+x^5$ and $S=7x^5+x^3=37x-22<1$ since $x=\frac{\sqrt{5}-1}{2}<0.62$. Thus there are not enough beetles to reach the black tower.

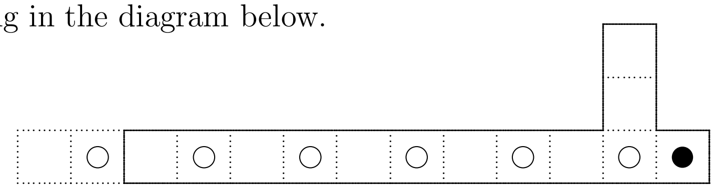
Section C: The Final Assault.

A similar calculation for Case 6 shows there are just enough beetles for the task, though that in itself does not guarantee the success of the Great Escape. An actual algorithm must be developed.

In the solution to Case 2, the beetles' formation resembles the capital letter T. They now stretch one end of the T-bar into a long "handle", as shown in the diagram below. This is called a "generalized T-formation".

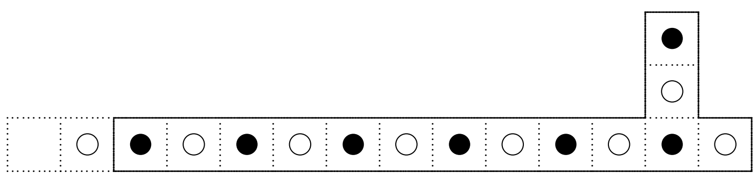


In operation, all beetles marked by white circles hop over their neighbors to the left or immediately beneath, resulting in the diagram below.

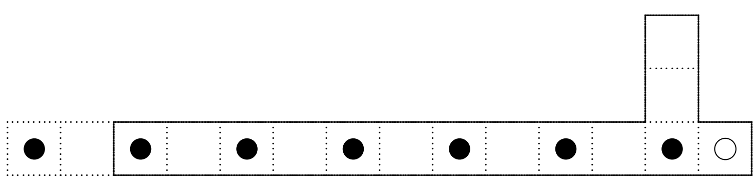


The lone beetle marked by a dark circle can now make a sequence of hops and end up two squares beyond the handle.

Suppose there is a beetle immediately beyond the handle, as shown in the diagram below. It apparently blocks the eventual projection of a beetle two squares beyond the handle.



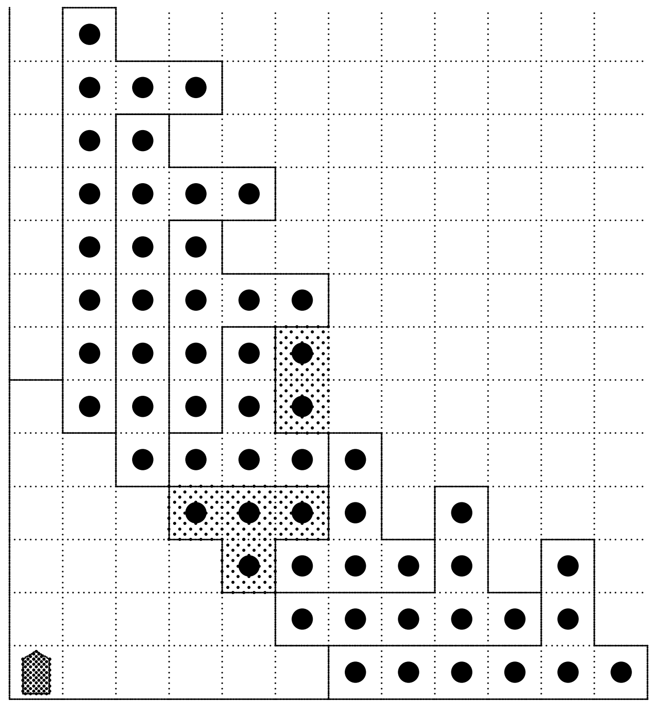
This time, all beetles marked by dark circles hop over their neighbors to the left or immediately beneath, resulting in the diagram below.



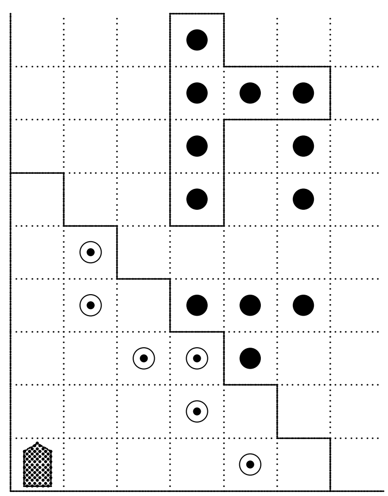
The lone beetle marked by a white circle can now make a sequence of hops and replace the "blocking" beetle.

The advantage of a generalized T-formation is that the length of its handle can be stretched arbitrarily. This is critical in avoiding "overlapping" beetles.

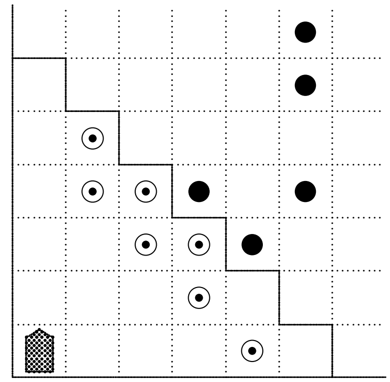
After much work, the beetles come up with a battle plan involving 50 beetles in seven generalized T-formations, plus 6 individual beetles for a total of 56, as shown in the diagram below.



The diagram below shows the result after all but one of the generalized T-formations have projected their beetles into the zone.



We use (i, j) to denote the square in the ith column from the left and the j row from the bottom. Thus the black tower is on (1,1). Now the beetle on (5,4) hops over the beetle on (4,4) and lands inside the zone on (3,4). The beetle on (4,4) is then replaced by the one projected from the remaining generalized T-formation, as shown in the diagram below.



The following chart shows the last 11 hops of this successful Great Escape.

1. (3,4) over (3,3) to (3,2)
2. (4,2) over (3,2) to (2,2)
3. (2,5) over (2,4) to (2,3)
4. (2,3) over (2,2) to (2,1)
5. (4,4) over (4,3) to (4,2)
6. (6,7) over (6,6) to (6,5)
7. (6,5) over (6,4) to (6,3)
8. (6,3) over (5,3) to (4,3)
9. (4,3) over (4,2) to (4,1)
10. (5,1) over (4,1) to (3,1)
11. (3,1) over (2,1) to (1,1)