Color Contrast Contortions

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Last year (2023) I was reading Josef Albers's classic book *Interaction of Color*, and got to wondering about questions of color contrast: what you perceive when different colors are adjacent. A standard way of studying color contrasts is by juxtaposing squares of different colors. Doing so presents a plethora of possible problems. I went down one particular path. It's likely other directions lead to problems that are equally interesting, if not more so. I'll content myself here with describing what I wound up pondering in playing around with arrangements of colored squares.

The main thing I settled on was to produce arrangements that contain all possible color contrasts—i.e., for any pair of colors, there should be a pair of adjacent squares of those two colors. This condition is vacuous, of course, if there's just one color, and trivial if there are two:

Figure 1. For a two-color contrast, a simple domino suffices.

It's a little less trivial with three colors: putting Red, Blue, and Green in a row only gives two of the three color contrasts; to get the third you need one more square:

Figure 2. For a three-color contrast, four squares are needed.

It's worth pointing out that you actually can get all three contrasts with just three squares if you have a Green square that straddles the Red-Blue pair:

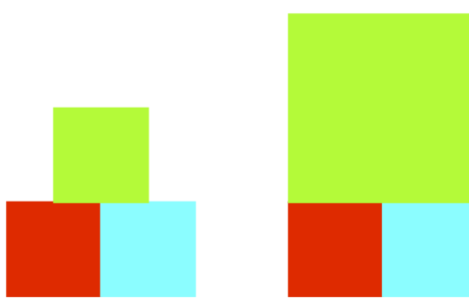

Figure 3. These arrangements of squares also "solve" the problem, but they're not what we have in mind.

But I hope you'll agree it's fair to disallow such shenanigans (though, of course, allowing them may be one of the interesting paths not traveled). In other words, it seems natural to pose the problem in terms of coloring the squares of a *polyomino*.

Similarly, it seems fair to disallow arrangements that have adjacent squares of the same color, so the three-color arrangement below on the left is OK, but not the one on the right:

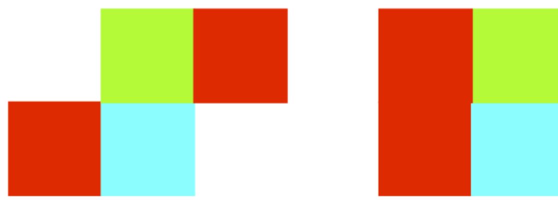

Figure 4. The arrangement on the left looks fine. Not so much for the one on the right.

With four colors things begin to get a little interesting. Here, for example, is a string of squares with all six color contrasts for Red, Blue, Green and Yellow:

Figure 5. An arrangement with all 6 contrasts among 4 colors. But Blue and Green are next to each other twice.

But note, there is a duplication here of the Blue-Green contrast. I decided to disallow duplications—i.e., I decided to look for arrangements of colored squares that contain all possible color contrast once *and only once* (or, more succinctly, *exactly* once). This became my central criterion.

The "exactly once" criterion for k colors requires there be exactly $\binom{k}{2}$ adjacencies of squares. That's a significant constraint. For one thing, it puts an absolute upper bound on the number of squares that can participate in a complete set of color contrasts, namely $k(k-1)$, which is what you get with $\binom{k}{2}$ disconnected dominoes, each domino being a pair of contrasting colors. If you insist that your arrangements be connected—that is, if you want your arrangement to be an honest-to-god polyomino (which, by convention, is defined to be rookwise connected)—the upper bound (for $k \geq 3$) is less, and this raises the first interesting question:

As a function of k, what is the largest number of squares in a "color-contrast polyomino"—i.e., a connected arrangement in which each color contrast among k colors occurs exactly once?

And, of course, the same question can be posed with "smallest" instead of "largest."

But wait, there's more: Even if you establish proper upper and lower bounds, it's not clear what happens with numbers in between. Could it be, for example, that there is a color-contrast polyomino with $k = 8$ colors of size 22 and one of size 24 but not of size 23? It seems doubtful but who knows?

Another direction I decided to explore was the possibility of "equi-color" color-contrast polyominoes—i.e., arrangements that use each color the same number of times. In this case it's not possible to get connected arrangements with $k = 3$ or $k = 4$ colors, but it is possible with $k = 5$, 6, and 7 (and it's always trivially possible if you don't require connected solutions, simply using dominoes).

Here are some arrangements with five colors, using two squares of each color:

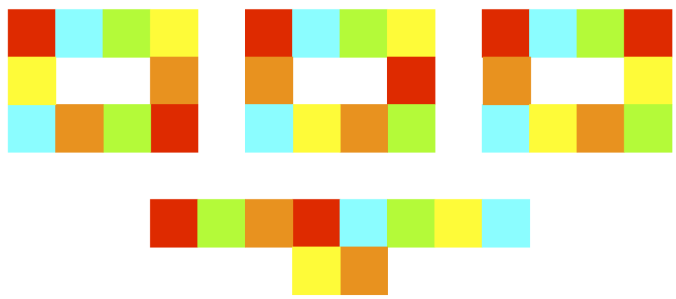

Figure 6. Four arrangements of five colors, with two copies of each color. Are the top three the "same" or "different"?

The top three arrangements here may look similar, but they are really quite different. In the one at left, there is a pair of squares of the same color—red—that are separated by four squares (both clockwise and counterclockwise); in the one at right each colored square is within three squares of its twin; and neither of these is true for the middle one. Consequently it's not possible to turn one arrangement into another by simply permuting the colors, or by advancing each square's color some number of squares clockwise (which you can think of essentially as turning the rectangle into a circle, rotating the circle, and then turning it back into a rectangle).

The colors can also be permuted in the fourth arrangement in Figure 6, of course, but there are other simple actions one can take to produce arrangements that differ from but are arguably the "same" as the fourth arrangement:

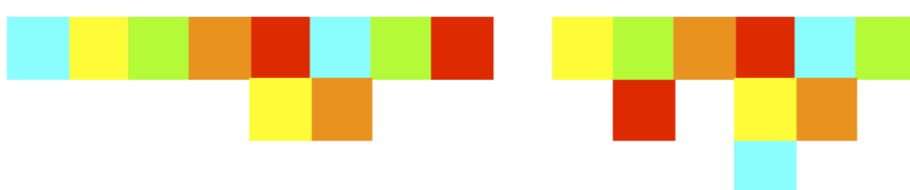

Figure 7. Two more arrangements of five colors with two copies of each color. Do they really differ from what's in Figure 6?

The arrangement at left can be thought of as swapping the red square at the left end of the fourth arrangement in Figure 6 and the yellow/blue domino from the right end, both of which are attached to green squares. The one at right moves the red square back below the leftmost green square and the leftmost blue square below the other yellow square. The reader can probably see many other simple rearrangements of such ilk, raising the question of how to say, mathematically, when two arrangements are "essentially the same" and when they're "distinctly different." For example, the two 9-color arrangements with four copies of each color in Figure 8 below are pretty clearly the same, but what's a succinct way to describe the steps that turn one into the other?

Figure 8. How would you describe, mathematically, the steps that turn the top arrangement into the bottom (in a way that might apply in other cases)?

It's worth noting that whenever the number of colors k is of the form $k = 4h+1$, then one can create rectangular arrangements using $2h = (k-1)/2$ copies of each color, such as the ones in Figures 8 and 6 for $k = 9$ and 5, respectively.

Clearly the number of copies of each color in any equi-color color-contrast polyomino must increase with the number of colors: With 9 colors, for example, there must be at least two squares of each color, since each color must appear next to the other eight colors but no square has more than four neighbors. In fact, since there are always squares on the outer edge that have fewer than four neighbors, the minimum with 9 colors is necessarily at least three copies of each color. But is there a three-copy arrangement with 9 colors? The reader is invited to find out.

All questions with regard to the existence of color-contrast polyominoes with various properties such as minimality or maximality of the numbers of squares are of NP complexity: If an arrangement exists, it is straightforward to verify, but proving arrangements (satisfying some additional constraints) don't exist can be a challenge.

It's easy to experiment forming color-contrast polyominoes with squares cut from colored construction paper. (Indeed, Albers recommends using colored paper rather than pigment and paint for the experiments he has in mind, because it "permits a repeated use of precisely the same color without the slightest change in tone, light, or surface quality.") But the "straightforward" process of verifying that some arrangement of colored squares is a color-contrast polyomino can be a bit of a pain. Even with just five colors I often found I was either omitting a contrast or duplicating one (and sometimes both), so anytime I had what I thought was a solution, I had to systematically check my work, which is kind of tedious; if you don't believe me, I urge you to make sure that the rectangles in Figure 8 really do exhibit all 36 color contrasts among the nine colors once and only once—don't just take my word for it!

It occurred to me that the verification process could be done at a glance if the colored squares all had notches cut into their four sides, and there were a batch of two-color diamonds to fill the holes created when squares are set adjacent:

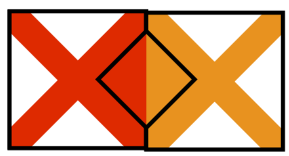

Figure 9. Filling the hole in adjacent notched squares with a two-colored diamond makes it easy to verify you have a legitimate color-contrast polyomino.

In this way, given a complete collection of $\binom{k}{2}$ diamonds and a set of notched squares, it's easy to see that you've used all the diamonds (and all the notched squares, if that's important to you), and any duplicate color contrasts will stand out as unfilled potholes in the arrangement. The final page here is a complete set of 15 diamonds for the color contrasts on six colors, with two notched squares of each color; the reader is invited to print it out (stiff paper works best, the stiffer the better) and play with it.

I showed this idea to my friend Loren Larson, who came back the next day with a beautiful version using different woods to represent the colors. Figure 10 shows an arrangement that uses all ten diamonds for five different woods. Loren's version extends to six colors with an additional five diamonds and two more notched squares, and to seven colors with a total of 21 diamonds and three notched squares for each type of wood.

Figure 10. Loren Larson's wooden version of the Color-Contrast Puzzle with five types of wood.

In addition to being a tactile pleasure, the rigidity of wooden pieces solves a problem that the reader may notice when working with paper, namely that even relatively stiff paper is hard to keep in place; wood does a much better job of staying put.

Loren, who is an accomplished mathematician in addition to being an expert woodworker, also answered the question asked earlier about the maximum number of squares in a color-contrast polyomino, showing the maximum to be $\binom{k}{2}+1$. Indeed, he showed that any polyomino with N squares and A adjacencies satisfies the equation

$$N = A + 1 - (I + C)$$

where I counts the number of "Interior" vertices of the polyomino and C counts the number of "Cavities"—i.e., regions that are completely surrounded by squares of the polyomino—with definitions of I and C suggested by the example in Figure 11, which has $N = 24$ squares, $A = \binom{8}{2} = 28$ adjacencies, $I = 4$ interior points, and $C = 1$ cavity. (The white "holes" in the Figure 11 polyomino are not cavities, because they are not completely surrounded by colored squares. Note, you can think of an interior point as a kind of cavity, because it's completely surrounded by the four squares for which it's a common vertex.) Since $I + C$ is necessarily non-negative, $\binom{k}{2} + 1$ is an upper bound on the number of squares in a k-color color-contrast polyomino. Loren (and I) then found various ways to achieve the upper bound.

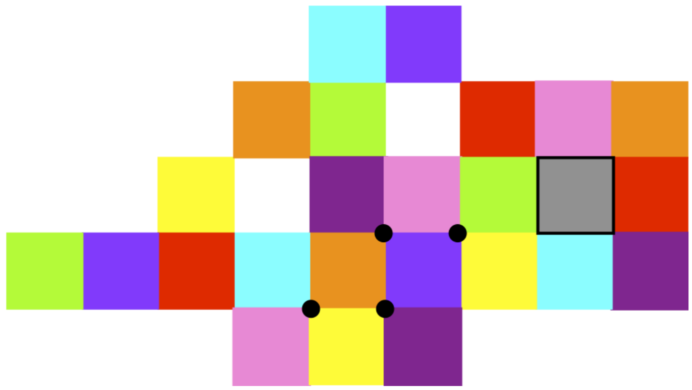

Figure 11. An 8-color color-contrast polyomino with $N = 24$ squares, $I = 4$ interior vertices (black dots) and $C = 1$ Cavity (gray square).

One way to achieve the upper bound, if k is odd, is to follow an Euler circuit on the complete graph on k vertices, with each vertex correponding to one of the k colors, producing a long string of colored squares whose colors correspond to the sequence of vertices visited in the circuit, as indicated for $k = 5$ in Figure 12. This approach adapts to even values of k (as indicated for $k = 6$ in Figure 12), by deleting one edge per vertex of the complete graph, producing a string of squares for an Euler circuit on the reduced graph (whose vertices now all have even degree), and then studding that string on top and/or bottom with squares to account for the missing contrasts, which correspond to the edges that were removed from the complete graph.

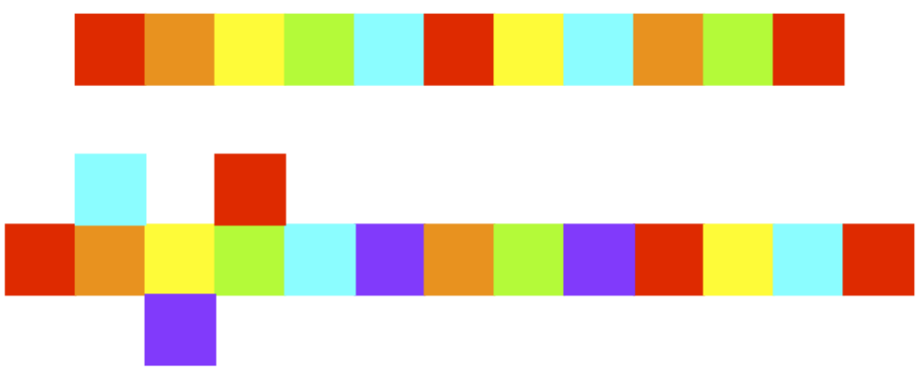

Figure 12. Maximal color-contrast polyominoes for 5 and 6 colors, respectively.

The "minimal" version of the problem seems to be a good deal more difficult. It's not hard to show that the minimum number of squares for $k = 2, 3, 4, 5,$ and 6 colors are $2, 4, 6, 9,$ and 12. A little more work gives the value 15 for $k = 7$. (Loren's formula implies that a polyomino with 14 squares and $\binom{7}{2} = 21$ adjacencies would have to have 8 interior points and/or cavities; colored or not, no such polyomino exists.)

These values led me to the sequence A278299 in the Online Encyclopedia of Integer Sequences (oeis.org), which was contributed in 2016 by Alec Jones and Peter Kagey. Their sequence, which continues with $19, 24, 30$, and 34 for $k = 8, 9, 10$, and 11, is defined to be the smallest number of squares in a polyomino that contains each color contrast at least once, rather than exactly once, so for now at least it's only a lower bound for our sequence. It's plausible, perhaps even likely, that the lower bound is sharp—Jones's and Kagey's OEIS entry includes an example with 10 colors that includes duplicate color contrasts (though Loren noticed an easy way to move two of its squares so as to eliminate the duplicates)—but it's also plausible that our no-duplicate constraint is strict enough to necessitate some additional squares.

Regarding the question of determining whether the squares of a given polyomino can be colored so that each color contrast occurs, either at least once or exactly once, Lily Chung at MIT pointed out a connection with "complete" and "exact" colorings of graphs—i.e., determining when the vertices of a graph can be colored so that for each color pair there is "at least" or "exactly" one edge connecting vertices of those two colors (and, as usual, there are no edges connecting vertices of the same color). The graphs here simply amount to using the centers of a polyomino's squares as the vertices, with edges connecting the centers of adjacent squares—what we might call "grid graphs."

According to the Wikipedia article on exact coloring, Keith Edwards has shown that the problem is NP-complete in general, even when restricted to graphs that are trees, but is solvable in polynomial time on graphs of bounded degree. It's unclear (to me at least) what happens with grid graphs, which are certainly of bounded degree (no vertex has more than four neighbors) but are not trees (except when $I + C = 0$).

The difficulty of determining whether a polyomino can be "exactly" colored (i.e., turned into a color-contrast polyomino) is nicely illustrated by an example that Loren discovered: Each of the polyominoes in Figure 13 has 15 squares and 21 adjacencies, so each is a possible color-contrast polyomino on $k = 7$ colors. Two of them actually can be exactly colored, while one cannot, and of the two that can, one can be exactly colored in just one way (aside from permuting the colors) while the other can be exactly colored in two different ways. The reader is invited to figure out which is which.

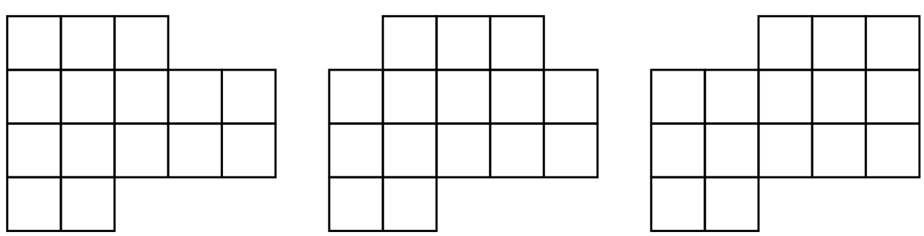

Figure 13. How many different ways can each of these be turned into a color-contrast polyomino?

The status of the equi-colored color-contrast problem is similarly uncertain. Loren's equation tells us something about the number of interior points and cavities: if m copies of k colors are used, we must have

$$m = \frac{k-1}{2} + \frac{1-I-C}{k}$$

which in particular implies (for $k > 2$) that there will be at least one interior point or cavity (and, for odd k, a positive number congruent to 1 mod k). On the other hand, we must have $k - 1 \leq 4m$. In fact, since polyominoes always have "corner" squares with only two neighbors, we must have $k - 1 \leq 4m - 2$. Together these tell us

$$\left\lceil \frac{k+1}{4} \right\rceil \leq m \leq \left\lfloor \frac{k+1}{2} + \frac{1}{k} \right\rfloor$$

In general this only gives a range of possible values for m. But for $k = 8$, it pins things down precisely: We must have $m = 3$ copies of each color, and any equi-color color-contrast polyomino on 8 colors must have $I + C = 5$ interior points and/or cavities. The reader is invited to see if such color-contrast polyominoes actually exist.