Combinatorial Jenga

G4G15 Gift Exchange

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Introduction

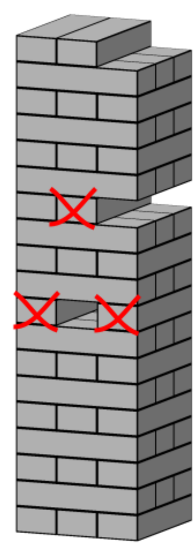
Jenga hardly needs an introduction. One of the most popular and best known tabletop games in the world, it was published by Leslie Scott in 1983 and has since sold more than 80 million copies.

Jenga is usually thought of as a dexterity game, in which coordination is the primary component of skill. While this is no doubt true, *Jenga* also has some interesting mathematical properties, which become apparent if we simply disregard the dexterity elements.

In particular, let's suppose that there are just two players, and they have become so skilled that they can execute any physically feasible move with perfect accuracy.[1] (We'll shortly give a precise definition of "feasible".) Then at some point during the game, one of the players will encounter a position in which no remaining move is feasible. At that point, any move at all will cause the tower to collapse.

We can simplify a bit by assuming the players will *only* make physically feasible moves, and that as soon as a position arises where no such move remains, they agree to end the game immediately and simply declare the player with the move to be the loser. With these assumptions, the winner is the player who makes the last move! This is known in game theory as a *normal-play combinatorial game*.

We'll call this idealized variant *Combinatorial Jenga*, and in this paper we'll give a perfect winning strategy. The strategy has essentially no application to actual *Jenga* (since even in games between players of great skill, the tower invariably collapses well before reaching its theoretical limit), but we have created a specially constructed, 3D printed set that makes it possible to play *Combinatorial Jenga* as a separate game.



Move Structure

Certain moves are prohibited by gravity: they will inevitably cause the tower to collapse, no matter how skilled the players are. The various kinds of prohibited moves can be seen in the figure on the right. If any layer is missing its center brick, then the two other bricks on the layer can never be removed. Likewise, if an end brick is missing, then the center brick on the same layer may never be removed, although it is still feasible to remove the brick on the other end of the layer.

By a **feasible move**, we mean any move that is not prohibited by one of these two constraints. With this definition, it is easily seen

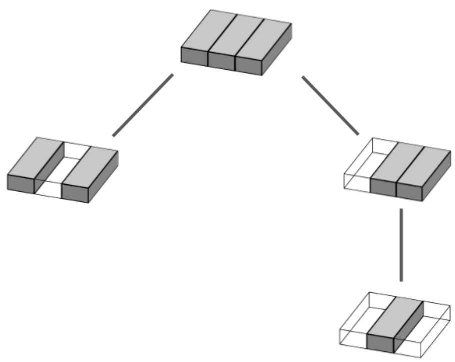
[1] We also have to assume that the blocks are competely smooth, perfectly arranged, free of any irregularities in size, and so forth.

that each layer is independent of all the others. Moreover, no player will ever move on a layer that is missing its center block, nor one that contains only a center block. We can therefore disregard such “stopped” layers from consideration, and an entire *Combinatorial Jenga* position reduces to three integer parameters:

- *m* = the number of playable complete layers (with a full complement of three bricks)
- *n* = the number of playable incomplete layers (with a center brick and one end brick)
- *k* = the number of bricks on the partial layer on top (always 0, 1, or 2)

Following standard *Jenga* rules, it is never permissible to play on the topmost complete layer, nor on the partial layer above it, and so they are *not* counted in the calculation of *m* and *n*.

Also note that *k* will always be 0, 1, or 2. As soon as a third brick is placed on top, a new playable complete layer is introduced (so that *m* increases by 1 and *k* resets to 0).



For brevity, we’ll write C for a playable complete layer and I for a playable incomplete layer, and we’ll denote a typical position by [C*m*,I*n*,+*k*]. For example, [C3,I4,+1] means three playable complete layers, four playable incomplete layers, and one brick on top. A standard *Jenga* set has 18 layers, but the topmost layer is initially unplayable, so the initial position is [C17,I0,+0].

Now every layer has the game tree shown at left; a C can be terminated immediately or replaced by an I, while from an I the only move is terminal. Thus the possible moves can be described as follows, where it is understood that no parameter may ever be decreased below 0:

- If *k* = 0 or 1, either:
  - Decrement *n* and increment *k*; or (take from an I)
  - Decrement *m* and increment *k*; or (take from the middle of a C)
  - Decrement *m*, increment *k*, and increment *n*. (take from the end of a C)
- If *k* = 2, either:
  - Decrement *n*, set *k* = 0, and increment *m*; or (take from an I)
  - Set *k* = 0 (only allowed if *m* > 1); or (take from the middle of a C)
  - Increment *n* and set *k* = 0 (only allowed if *m* > 1). (take from the end of a C)

For example, from [C3,I4,+1], one can move to [C3,I3,+2], [C2,I4,+2], or [C2,I5,+2]. Note that the moves for *k* = 2 are similar to the others, but with an extra increment given to *m*, corresponding to the introduction of a new playable complete layer.

The *P*-Positions

Following standard practice in combinatorial game theory, we call a position an *N*-position (*N*ext player wins) if a winning move exists; otherwise it’s a *P*-position (*P*revious player wins). Clearly every position is one or the other, and a perfect winning strategy may be found by classifying the *P*-positions. This we have done with the help of the *cgsuite* software.

[C1,I1,+0]	[C0,I0,+1]	[C0,I0,+2]
[C2,I0,+0]	[C2,I0,+1]	[C0,I1,+2]
[C2,I2,+0]		[C1,I2,+2]

The eight types of *P*-positions in *Combinatorial Jenga* are shown at left. All of the numbers are to be interpreted modulo 3: for example, [C*m*,I*n*,+*k*] is a *P*-position whenever $m \equiv n \equiv 1 \pmod 3$ and $k = 0$.

There is one exception to this classification: when *m* and *k* are both exactly 0. However, this exceptional situation can never arise during actual play, since any move that sets *k* to 0 necessarily also leaves $m \geq 1$. We leave it as an exercise to the reader to determine the *P*-positions when $m = k = 0$.

The Strategy

As with any combinatorial game, the winning strategy is:

- Always move to a *P*-position.
- If you can't, resign (or hope your opponent makes a mistake)!

The strategy is easier to remember if stated in a slightly different form:

- If there are no bricks on top ($k = 0$), play to leave C0,I0 or C2,I0;
- If there is one brick on top ($k = 1$), play to leave C0,I0 or C0,I1 or C1,I2;
- If there are two bricks on top ($k = 2$), play to leave C1,I1 or C2,I0 or C2,I2.

Memorize this strategy, and you can play perfect *Combinatorial Jenga*. Then all you have to do is develop flawless coordination, and you can master *Jenga* as well!

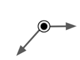
A Proof

Here's a simple visual proof that the strategy works. To prove that the classification of *P*-positions is correct, we must show that:

- From a given *N*-position, there is at least one move to a *P*-position;
- From a given *P*-position, *every* move is to an *N*-position.

The diagram on the right gives the proof. The box at row *m*, column *n* contains all the values of *k* for which [C*m*,I*n*,+*k*] is a *P*-position. From a position with $k = 0$ or 1, one can "move" west, north, or northeast on

C \ I	0	1	2
0	12	2	
1		0	2
2	01		0

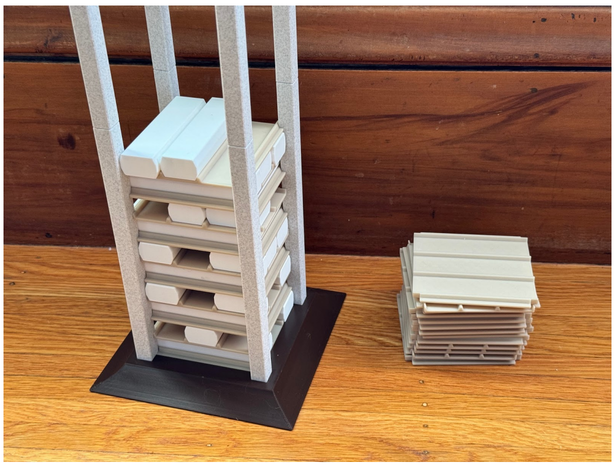
$k = 0$ or 1:  $k = 2$: 

the picture; from a position with $k = 2$, one can move east or southwest, or simply stay put. Moves that stray from the diagram “wrap around” to the other side.

The *P* and *N* recurrences are easily verified from the diagram. The base cases $m = 0$ and/or $n = 0$ must be checked separately, since in those cases some of the options are prohibited. (And as mentioned above, the specific case $m = 0$, $k = 0$ is not incorporated into the diagram, since it can never arise during actual play.)

A Combinatorial Jenga Set

In an actual game of *Jenga*, our strategy is only helpful if the players are able to execute every feasible move flawlessly. In the real world, this is of course never the case. Insofar as *Jenga* is concerned, the strategy is more of a mathematical curiosity than anything that might be useful in competition.



In order to more easily explore the mathematics of *Jenga*, we’ve created a specially constructed, 3D printed *Combinatorial Jenga* set. It uses a system of stabilizing plates to reduce the dexterity component, so that it is easy to execute any *feasible* move, whereas *infeasible* ones still cause the plates to collapse.

If you have a 3D printer and want to print your own copy, scan the QR code at the bottom of this page for a link to the print files (STLs).

That’s All!

Scan this QR code for more *Combinatorial Jenga* resources, including a rules sheet and a link to the 3D print files.

Exercises:

1. Is the starting position [C17,I0,+0] a *P*-position or an *N*-position?
2. If it’s an *N*-position, what is the winning move? If it’s a *P*-position, what is the winning response to each possible opening move?

Acknowledgement

We wish to thank the Lost Iguana Resort in Arenal, Costa Rica for the giant-sized Jenga set in their lobby, which provided the inspiration for this work.