Database of Common Nets of Polyhedra

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Abstract

Polyhedra can be unfolded to form nets. Some nets, named common nets, can be folded into multiple polyhedra. This has been extensively studied for some types of polyhedra (usually with regular faces) but barely touched for more complex shapes. In this paper we introduce a database with examples of common nets.

1 Introduction

While researching the most efficient nets to allow self-folding of micro structures we stumbled upon some nets that were able to fold to more than one polyhedra. This revealed to be a very interesting problem. In fact, Demaine et al. proved that every convex polyhedron can be unfolded and refolded to a different convex polyhedron [1].

Different researchers have named this type of phenomenon as "common unfolding", "common development" or "ambiguous unfolding", here we use the denomination of "common net" for a net that can be folded onto several polyhedra. One of the simplest examples is given in Figure 1.

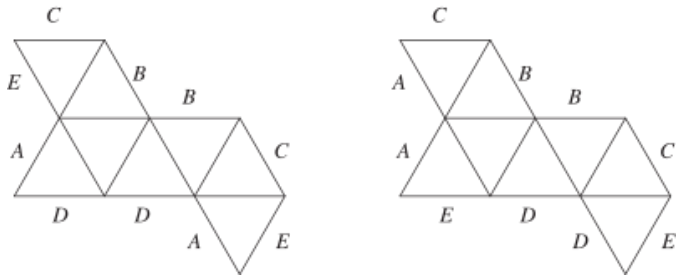

Figure 1. Common net of both an octahedron and a Tritetrahedron [2].

In this context we can have two types of common nets, strict edge unfoldings and free unfoldings. Strict edge unfoldings refers to common nets where the different polyhedra that can be folded use the same folds, that is, to fold one polyhedra from the net of another there is no need to make new folds. Free unfoldings refer to the opposite case, when we can create as many folds as needed to enable the folding of different polyhedra. We also add the concept of multiplicity, which refers to the number of common nets for the same set of polyhedra.

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In this paper we present a comprehensive database of nets that can be folded to more than one polyhedron. This problem has already been studied extensively for some kinds of polyhedron but still has a lot of open questions. To complement this paper a Wikipedia page has been created where new cases can be added as they are found [3].

2 Classes

Regular Polyhedra

Finding a common net between two regular polyhedra has been one of the most sought after problems. Open problem 25.31 in Geometric Folding Algorithm by Rourke and Demaine reads:

> *"Can any Platonic solid be cut open and unfolded to a polygon that may be refolded to a different Platonic solid? For example, may a cube be so dissected to a tetrahedron?"* [4]

This problem has been partially solved by Shirakawa et al. with a net that is conjectured to fold to a tetrahedron and a cube with infinite iterations.

Table 1 lists nets of regular polyhedra that can also be folded onto other polyhedra. All cases represent free unfolding common nets.

Table 1. Common nets of regular polyhedra. *Fractal net. $^{+}$20 polyhedra foldable from the Latin cross (seven tetrahedra, three pentahedra, four hexahedra, and six octahedra).

Multiplicity	Polyhedra 1	Polyhedra 2	Reference
1	Tetrahedron*	Cube*	[5]
1	Tetrahedron	Cuboid (1 × 1 × 1.232)	[4], [6]
87	Tetrahedron	Jonhson Solid J17	[7]
37	Tetrahedron	Jonhson Solid J84	[7]
2	Cube	Tetramonohedron	[8]
9	Cube	1 × 1 × 7 and 1 × 3 × 3 Cuboids	[9]
1	Cube	Octahedron (non-regular)	[5]
1	Cube	20 Polyhedra^{+}	[4]
1	Octahedron	Tetramonohedron	[4]
1	Octahedron	Tetramonohedron	[8]
1	Octahedron	Tritetrahedron	[2]
1	Icosahedron	Tetramonohedron	[8]

Cuboids

Common nets of cuboids have been deeply researched, mainly by Uehara and coworkers. To the moment, common nets of up to three cuboids have been found. It has, however, been proven that there exist infinitely many examples of nets that can be folded into more than one polyhedra

[10]. Table 2 shows the different common nets of cuboids found to the moment. With the exception of the marked ones, all the nets present an strict orthogonal folding despite still being considered free unfoldings.

Table 2. Common nets of Cuboids. *Non-orthogonal foldings

Area	Multiplicity	Cuboid 1	Cuboid 2	Cuboid 3	Reference
22	6495	$1 \times 1 \times 5$	$1 \times 2 \times 3$		[11]
22	3	$1 \times 1 \times 5$	$1 \times 2 \times 3$	$0 \times 1 \times 11$	[12]
28	1	$1 \times 2 \times 4$	$\sqrt{2} \times \sqrt{2} \times 3\sqrt{2}$*		[13]
30	30	$1 \times 1 \times 7$	$1 \times 3 \times 3$	$\sqrt{5} \times \sqrt{5} \times \sqrt{5}$*	[9]
30	1080	$1 \times 1 \times 7$	$1 \times 3 \times 3$		[9]
34	11291	$1 \times 1 \times 8$	$1 \times 2 \times 5$		[11]
38	2334	$1 \times 1 \times 9$	$1 \times 3 \times 4$		[11]
46	568	$1 \times 1 \times 11$	$1 \times 3 \times 5$		[11]
46	92	$1 \times 2 \times 7$	$1 \times 3 \times 5$		[11]
54	1735	$1 \times 1 \times 13$	$3 \times 3 \times 3$		[11]
54	1806	$1 \times 1 \times 13$	$1 \times 3 \times 6$		[11]
54	387	$1 \times 3 \times 6$	$3 \times 3 \times 3$		[11]
58	37	$1 \times 1 \times 14$	$1 \times 4 \times 5$		[11]
62	5	$1 \times 3 \times 7$	$2 \times 3 \times 5$		[11]
64	50	$2 \times 2 \times 7$	$1 \times 2 \times 10$		[11]
64	6	$2 \times 2 \times 7$	$2 \times 4 \times 4$		[11]
70	3	$1 \times 1 \times 17$	$1 \times 5 \times 5$		[11]
70	11	$1 \times 2 \times 11$	$1 \times 3 \times 8$		[11]
88	218	$2 \times 2 \times 10$	$1 \times 4 \times 8$		[11]
88	86	$2 \times 2 \times 10$	$2 \times 4 \times 6$		[11]
160	1	$4 \times 4 \times 8$	$\sqrt{10} \times 2\sqrt{10} \times 2\sqrt{10}$*		[12]
532		$7 \times 8 \times 14$	$2 \times 4 \times 43$	$2 \times 13 \times 16$	[10]
1792		$7 \times 8 \times 56$	$7 \times 14 \times 38$	$2 \times 13 \times 58$	[10]

Polycubes

Maybe the first cases of common nets of polycubes found was the work by George Miller, with a later contribution of Donald Knuth, that culminated in the Cubigami puzzle [14]. It's composed of a net that can fold to all 7 tree-like tetracubes. All possible common nets up to pentacubes

were found. Table 3 lists the known common nets of polycubes. All the nets follow strict orthogonal folding despite still being considered free unfoldings.

Table 3. Common nets of Polycubes.

Area	Multiplicity	Polyhedra	Reference
14	29026	All tricubes	[15]
14		All tricubes	[11]
18	6495	All tree-like tetracubes	[14]
22		23 pentacubes	[16]
22	3	22 tree-like pentacubes	[16]
22	1	Non-planar pentacubes	[16]

Deltahedra

Deltahedra pose a class with many developments to be made. To our knowledge, there has been no extensive search for common nets of deltahedra.

Table 4. Common nets of deltahedra. *Not all faces are equilateral triangles.

Area	Multiplicity	Polyhedra	Reference
8	1	Both 8 face deltahedra	[2]
10	4	7-vertex deltahedra	[17]
8*	1	Both 8 face deltahedra (non regular)*	[18]

3 Conclusions

This paper presents a comprehensive list of common nets found to the moment. A Wikipedia page [3] has been created as a repository for new cases found in the future, we invite everyone to contribute to this collaborative page. The knowledge of a wider range of common nets can help us answer some open problems but also raise new ones.

4 References

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